Quantum vacuum definition for spin-$\frac{1}{2}$ fields in Robertson-Walker metrics

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We study the quantum vacuum definition for spin-$\frac{1}{2}$ fields in a Robertson-Walker universe using the coincidence of a local property (singularity structure of the DeWitt-Schwinger kernel) and a global property (energy minimization) and we obtain two kinds of vacua: strong and weak (which coincide with the energy minimization vacuum). The density of particles created during the expansion of the Universe between two weak vacua is found to be finite. However, we prove that the energy-momentum tensor vacuum expectation value is nonrenormalizable.

I. INTRODUCTION

In quantum field theory in curved spacetime, the problem of the definition of a quantum state and the derivation of a renormalizable energy-momentum tensor vacuum expectation value (VEV) to be used as the source of the semiclassical Einstein equation have not yet found a satisfactory solution.[1,2]

The first point has been studied by several authors.[3–9] In a series of papers, Castagnino *et al.*[10–13] proposed a criterion for the definition of a vacuum state based on the coincidence of a local property [singular structure of the kernel $G_1 = \langle \{\psi(x),\psi(x')\}\rangle$] and a global property (energy minimization), leading to the definition of two kinds of vacua: weak and strong. For scalar fields, they showed that the density of particles created during the expansion of the Universe between two weak vacua is finite and the energy-momentum tensor VEV is renormalizable.[10,11,13] In the case of massive spin-1 fields, the results are similar for transverse-polarization modes.[12]

In order to study the generality of the above results, this work is devoted to the study of vacuum-state definition and energy-momentum tensor VEV renormalization for the case of spin-$\frac{1}{2}$ fields in Robertson-Walker metrics with a spatially flat section. We prove that the density of particles created is finite through the computation of the Bogoliubov transformation coefficients between the basis defined by weak vacuum in the considered times. We also show that the energy-momentum VEV is nonrenormalizable. This result is in agreement with the one obtained by Najmi and Ottewill in Ref. 8. They studied the vacuum state definition for spin-$\frac{1}{2}$ fields using the energy-minimization criterion and proved that the energy-momentum tensor VEV is nonrenormalizable because the singular structure of the commutation function $\langle [\psi(x),\psi(x')]\rangle$ does not have the Hadamard form.

In Sec. II, a brief Dirac-equation generalization to curved spacetime is presented and the adiabatic solution in the Robertson-Walker metric is found.

In Sec. III the spinorial field is quantized and the density of particles created between two times is calculated as a function of Bogoliubov transformation coefficients. Energy-minimization Cauchy data are also calculated.

In Sec. IV the cases in which it is possible to define weak and strong vacua are determined and the density of particles created between two weak vacua is computed.

In Sec. V the energy-momentum tensor renormalization is studied, and finally in Sec. VI the conclusions are presented.

II. DIRAC EQUATION: FORMALISM

In this section we shall briefly develop the formalism of the Dirac equation in curved spacetime. First, we introduce at each spacetime point X a set of locally inertial coordinates ξ_X^α. In terms of ξ_X^α the metric at X is simply $\eta_{\alpha\beta}$. In any noninertial coordinate system, the metric is related to $\eta_{\alpha\beta}$ by

$$g_{\mu\nu}(x) = V^\alpha{}_\mu(x) V^\beta{}_\nu(x) \eta_{\alpha\beta} , \tag{2.1}$$

where

$$V^\alpha{}_\mu(x) \equiv \left[\frac{\partial \xi_X^\alpha(x)}{\partial x^\mu} \right]_{x=X} \tag{2.2}$$

is a set of four vector fields called vierbein.[14] Note that the label α refers to the local inertial frame, associated with the normal coordinates ξ_X^α, while μ is associated with the general coordinate system x^μ. We adopt the convention that labels from the beginning of the greek alphabet refer to the former, and those from the end refer to the latter.

The covariant derivative of a spinor field ψ is given by

$$\nabla_\mu \psi = (\partial_\mu + \sigma_\mu)\psi , \tag{2.3}$$

where σ_μ is the spinorial affine connection

$$\sigma_\mu = \tfrac{1}{2} \Sigma^{\alpha\beta} V_\alpha{}^\nu V_{\beta\nu;\mu} \tag{2.4}$$

with

$$\Sigma^{\alpha\beta} = \tfrac{1}{4}[\gamma^\alpha, \gamma^\beta] . \tag{2.5}$$

$\{\gamma^\alpha\}$ denotes a set of constant Dirac matrices satisfying the usual anticommutation relations

$$\{\gamma^\alpha, \gamma^\beta\} = -2\eta^{\alpha\beta} . \tag{2.6}$$

Then, the Lagrangian density for a spin-$\frac{1}{2}$ field in curved spacetime is given by

$$\mathscr{L}(x)=\sqrt{-g}\,\{\tfrac{1}{2}[\bar{\psi}\Gamma^{\mu}\nabla_{\mu}\psi-(\nabla_{\mu}\bar{\psi})\Gamma^{\mu}\psi]-m\bar{\psi}\psi\}\ , \tag{2.7}$$

where

$$\Gamma^{\mu}=V_{\alpha}{}^{\mu}\gamma^{\alpha} \tag{2.8}$$

are the curved-spacetime analogs of Dirac matrices, and satisfy the anticommutation relations

$$\{\Gamma^{\mu},\Gamma^{\nu}\}=-2g^{\mu\nu}\ . \tag{2.9}$$

$\bar{\psi}$ is the Dirac adjoint field, defined by

$$\bar{\psi}=\psi^{\dagger}\beta \tag{2.10}$$

with

$$\beta=-i\gamma^{0}\ . \tag{2.11}$$

Variation of the action with respect to $\bar{\psi}$ yields the covariant Dirac equation

$$(\Gamma^{\mu}\nabla_{\mu}-m)\psi(x)=0\ . \tag{2.12}$$

The internal product between two Dirac equation solutions is defined by

$$(\psi_1,\psi_2)=i\int_{\Sigma}\bar{\psi}_1\Gamma_{\mu}\psi_2 d\sigma^{\mu} \tag{2.13}$$

which is Hermitian and does not depend on the Cauchy surface Σ where the integration is carried out.[15]

The energy-momentum tensor for the spin-$\frac{1}{2}$ field is defined by

$$T_{\mu\nu}=\frac{2}{\sqrt{-g}}\frac{\delta S}{\delta g^{\mu\nu}}=\frac{V_{\alpha\mu}}{\det V}\frac{\delta S}{\delta V_{\alpha}{}^{\nu}}$$
$$=-\tfrac{1}{2}[-\bar{\psi}\Gamma_{(\mu}\nabla_{\nu)}\psi+(\nabla_{(\mu}\bar{\psi})\Gamma_{\nu)}\psi]\ . \tag{2.14}$$

We shall use the following representation for the Dirac matrices:

$$\gamma^{0}=\gamma_{0}=i\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix},\quad \gamma_{j}=-\gamma^{j}=i\begin{bmatrix}0 & \sigma_j\\ -\sigma_j & 0\end{bmatrix}, \tag{2.15}$$

where σ_j are the Pauli matrices

$$\sigma_1=\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix},\quad \sigma_2=\begin{bmatrix}0 & -i\\ i & 0\end{bmatrix},\quad \sigma_3=\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}. \tag{2.16}$$

In this work we shall deal with a spatially flat Robertson-Walker universe, characterized by the spacetime interval

$$ds^2=dt^2-a^2(t)(dx^2+dy^2+dz^2)\ . \tag{2.17}$$

Choosing the vierbein given by

$$V^{0}{}_{0}=1,\quad V^{i}{}_{i}=a\ , \tag{2.18}$$

and the other elements equal to zero, we obtain that generalized Dirac matrices (2.8), and the spinorial affine connections (2.4) take the form

$$\Gamma^{0}=\gamma^{0},\quad \Gamma^{i}=a^{-1}(t)\gamma^{i}\ , \tag{2.19a}$$

$$\sigma_0=0,\quad \sigma_i=\tfrac{1}{2}H\Gamma^{0}\Gamma_{i}\ , \tag{2.19b}$$

where $H=\dot{a}/a$ is the Hubble constant. Inserting (2.19) in Eq. (2.12), this can be written as

$$(\Gamma^{i}\partial_{i}+\tfrac{3}{2}H\Gamma^{0}-m)\psi(x)=0\ . \tag{2.20}$$

To solve it, we shall follow Ref. 16, using the separation of variables given by

$$\psi(x)=\frac{1}{(2\pi a)^{3/2}}f_{\mathbf{k}}(t)e^{i\mathbf{k}\cdot\mathbf{x}}\ , \tag{2.21}$$

where $f_{\mathbf{k}}(t)$ is a four-component column matrix.

Substituting (2.21) into (2.20), we obtain the following differential equation for $f_{\mathbf{k}}(t)$:

$$\gamma^{0}\dot{f}_{\mathbf{k}}(t)+i\frac{k_i}{a}\gamma^{i}f_{\mathbf{k}}(t)-mf_{\mathbf{k}}(t)=0\ . \tag{2.22}$$

A priori, $f_{\mathbf{k}}(t)$ could be any set of four functions of time satisfying (2.22). However, it is well known that every solution of the Dirac equation is also a solution of the Klein-Gordon equation with a D'Alambert operator defined by $\Box=-\nabla_i\nabla^i+\frac{1}{4}R$, where R is the curvature scalar.[15] Because of the particular representation of Dirac matrices we have chosen, when $\Box$ is applied to $f_{\mathbf{k}}(t)$, two independent differential equations arise: one must be satisfied by the first two components of $f_{\mathbf{k}}(t)$ and the other by the remaining two. Then, without loss of generality we can propose for $f_{\mathbf{k}}(t)$ a two-independent-variable function

$$f_{\mathbf{k}}(t)=\begin{bmatrix}A_{\mathbf{k}}\exp\left[i\int\Omega_k dt\right]\\ B_{\mathbf{k}}\exp\left[i\int\Lambda_k dt\right]\end{bmatrix}, \tag{2.23}$$

where $A_{\mathbf{k}}$ and $B_{\mathbf{k}}$ are two-component column matrices, depending only on the momentum $\mathbf{k}$, while Ω_k and Λ_k are arbitrary complex functions of time, depending on the momentum modulus.

So, replacing (2.23) into (2.22) we obtain a lineal algebraic homogeneous equation system:

$$\begin{bmatrix}\Omega_k+m & -\frac{k_i}{a}\sigma^{i}\\ -\frac{k_i}{a}\sigma^{i} & \Lambda_k-m\end{bmatrix}f_{\mathbf{k}}(t)=0\ . \tag{2.24}$$

The nontrivial solution is given by

$$(\Omega_k+m)(\Lambda_k-m)=k^2/a^2\ . \tag{2.25}$$

Inserting this condition into (2.24), we obtain two independent solutions. As a complete base of Dirac equation solutions must consist of four spinors; two more spinors must be found. We shall obtain them using the following property of the Dirac equation: If $f_{\mathbf{k}}(t)$ is a solution of equation (2.22), $\gamma_2 f^{*}_{-\mathbf{k}}(t)$ is too. The demonstration follows immediately from conjugating Eq. (2.22) and using anticommutation rules (2.6).

It is interesting to note that even if in the Dirac equation (2.22), the electric charge does not appear explicitly (we do not consider the interaction with the electromagnetic field), we can call the former property charge conjugation[15,16] because in spatially flat Robertson-Walker metric, it satisfies the usual properties which define it in flat spacetime,[17]

However, if we consider more general metrics, where it is not possible to use separation of variables, the preceding demonstration fails to identify this property with charge conjugation.

The solution base obtained can be written as

$$\psi_{\mathbf{k}}^{1,2}=\left[\frac{\Omega_k^*+m}{\Omega_k^*+\Lambda_k}\right]^{1/2}\begin{bmatrix}\mathbb{1}\\ \dfrac{k_\alpha\sigma^\alpha}{a(\Omega_k^*+m)}\end{bmatrix}\frac{\exp\left[-i\int\lambda_k dt\right]}{(2\pi a)^{3/2}}e^{-i\mathbf{k}\cdot\mathbf{x}}\,, \tag{2.26a}$$

$$\psi_{\mathbf{k}}^{3,4}=\left[\frac{\Omega_k^*+m}{\Omega_k^*+\Lambda_k}\right]^{1/2}\begin{bmatrix}\dfrac{-k_\alpha\sigma^\alpha}{a(\Omega_k^*+m)}\\ \mathbb{1}\end{bmatrix}\frac{\exp\left[i\int\lambda_k dt\right]}{(2\pi a)^{3/2}}e^{-i\mathbf{k}\cdot\mathbf{x}}\,, \tag{2.26b}$$

where $\lambda_k=\mathrm{Re}\Lambda_k$. These spinors have been normalized according to the internal product (2.13).

However, spinors (2.26) are a formal solution; in order that they be solutions of the Dirac equation, the functions Ω_k and Λ_k must satisfy the following differential equations:

$$i\dot{\Omega}_k-\Omega_k{}^2+iH(\Omega_k+m)+w_k{}^2=0\,, \tag{2.27a}$$

$$i\dot{\Lambda}_k-\Lambda_k{}^2+iH(\Lambda_k-m)+w_k{}^2=0\,, \tag{2.27b}$$

where

$$w_k{}^2=\frac{k^2}{a^2}+m^2\,. \tag{2.27c}$$

In fact, only one of them is necessary, because the other function can be obtained using expression (2.25).

It can easily be seen that Eqs. (2.27) admit a simple solution only in some particular cases, as when we consider a massless field or a static universe. Nevertheless, it is possible to develop a solution of (2.27) in a power series of the metric and its derivatives, i.e., the adiabatic solution, which up to second order results in

$$\Omega_k^{(2)}=\pm w_k\left[1\pm i\frac{H}{w_k}\left[\pm\frac{m}{2w_k}+\frac{m^2}{2w_k{}^2}\right]\mp\frac{R}{6}\frac{1}{w_k{}^2}\left[\frac{m}{4w_k}\pm\frac{m^2}{4w_k{}^2}\right]\mp\frac{H^2}{w_k{}^2}\left[\pm\frac{1}{8}\frac{m^2}{w_k{}^2}-\frac{1}{2}\frac{m^3}{w_k{}^3}\mp\frac{5}{8}\frac{m^4}{w_k{}^4}\right]+\cdots\right], \tag{2.29a}$$

$$\Lambda_k^{(2)}=\pm w_k\left[1\pm i\frac{H}{w_k}\left[\mp\frac{m}{2w_k}+\frac{m^2}{2w_k{}^2}\right]\mp\frac{R}{6}\frac{1}{w_k{}^2}\left[-\frac{m}{4w_k}\pm\frac{m^2}{4w_k{}^2}\right]\mp\frac{H^2}{w_k{}^2}\left[\pm\frac{1}{8}\frac{m^2}{w_k{}^2}+\frac{1}{2}\frac{m^3}{w_k{}^3}\mp\frac{5}{8}\frac{m^4}{w_k{}^4}\right]+\cdots\right]. \tag{2.29b}$$

Note that in the flat-spacetime limit, spinors (2.26) reduce to the usual base,[17] as the solution of Eqs. (2.27) have the limit $\Omega_k=\Lambda_k=w_k$, so that we can identify $\psi_{\mathbf{k}}^{1,2}$ with spinors characterizing particles with both spin projections and $\psi_{\mathbf{k}}^{3,4}$ with the corresponding antiparticle ones.

In this way, associating

$$u_{\mathbf{k},+}=\psi_{\mathbf{k}}^1,\quad v_{\mathbf{k},+}=\psi_{\mathbf{k}}^3\,,$$
$$u_{\mathbf{k},-}=\psi_{\mathbf{k}}^2,\quad v_{\mathbf{k},-}=\psi_{\mathbf{k}}^4\,, \tag{2.30}$$

we can expand the field as

$$\phi(x,t)=\sum_{\pm s}\sum_{\mathbf{k}}[b_{\mathbf{k},s}u_{\mathbf{k},s}(\mathbf{x},t)+d_{\mathbf{k},s}^\dagger v_{\mathbf{k},s}(\mathbf{x},t)]\,,$$
$$\bar{\phi}(x,t)=\sum_{\pm s}\sum_{\mathbf{k}}[b_{\mathbf{k},s}^\dagger\bar{u}_{\mathbf{k},s}(x,t)+d_{\mathbf{k},s}\bar{v}_{\mathbf{k},s}(\mathbf{x},t)]\,, \tag{2.31}$$

where the similarity with the flat-spacetime Dirac field is clear.

Finally it is convenient to state the charge-conjugation property mathematically,[16] as we shall use it later. We define

$$C=\gamma_2=\begin{bmatrix}0&0&0&1\\0&0&-1&0\\0&-1&0&0\\1&0&0&0\end{bmatrix},\quad C^2=1\,. \tag{2.32}$$

If we consider a matrix $\psi_{\mathbf{k}}$, whose columns are the spinors (2.26) ordered as

$$\psi_{\mathbf{k}}=(\psi_{\mathbf{k}}^1\ \ \psi_{\mathbf{k}}^2\ \ \psi_{\mathbf{k}}^3\ \ \psi_{\mathbf{k}}^4)\,, \tag{2.33}$$

it can be shown that the following property is valid:

$$C\psi_{\mathbf{k}}C=\psi^{*}_{-\mathbf{k}} \ . \tag{2.34}$$

If it is developed for each column, the transformation of the spinors under charge conjugation is obtained:

$$C\psi^{1}_{\mathbf{k}}=\psi^{4*}_{-\mathbf{k}}, \quad C\psi^{2}_{\mathbf{k}}=\psi^{3*}_{-\mathbf{k}} \ . \tag{2.35}$$

These relations are similar to those obtained in flat-spacetime case.

III. QUANTIZATION OF THE DIRAC FIELD

Until now we have made a purely classic study of the Dirac equation. However, as our main interest is the particle creation effect due to the expansion of the universe, it is necessary to quantize the Dirac field. To do that, we transform into operators the coefficients in expansion (2.31) and adopt the usual anticommutation rules:

$$\{ b_{\mathbf{k},s}, b^{\dagger}_{\mathbf{k},s'} \} = \delta_{ss'}\delta(\mathbf{k}-\mathbf{k}') \ , \qquad \{ d_{\mathbf{k},s}, d^{\dagger}_{\mathbf{k}',s'} \} = \delta_{ss'}\delta(\mathbf{k}-\mathbf{k}') \ , \tag{3.1}$$

and all other combination of operators anticommute, $b^{\dagger}_{\mathbf{k},s}$ and $d^{\dagger}_{\mathbf{k},s}$ are particle and antiparticle creation operators and $b_{\mathbf{k},s}$ and $d_{\mathbf{k},s}$ are the corresponding annihilation operators.

As in an expanding Universe, there is no invariance under time translations, the creation and annihilation operators are time dependent.

The vacuum state associated to a given spinor base is defined by

$$b^{\tau}_{\mathbf{k}}\,|\,0\rangle_{\tau}=0, \quad d^{\tau}_{\mathbf{k}}\,|\,0\rangle_{\tau}=0, \quad \forall \mathbf{k} \tag{3.2}$$

and the corresponding Fock space can be constructed as in Minkowski spacetime.

If we define the vacuum state at time $\tau_0\,|\,0\rangle_{\tau_0}$, it will not be annihilated by annihilation operators defined at time τ:

$$b^{\tau}_{\mathbf{k}}\,|\,0\rangle_{\tau_0}\neq 0, \quad d^{\tau}_{\mathbf{k}}\,|\,0\rangle_{\tau_0}\neq 0 \ . \tag{3.3}$$

In the general case in which we choose a base $\phi^{\tau}_{\mathbf{k}}$ satisfying the Cauchy data $\Pi^{\tau}_{\mathbf{k}}$ on a surface Σ^{τ}, it is possible to calculate the density of particles created in the interval (τ_0,τ) as a function of the Universe evolution $a(t)$, its derivatives, and the Cauchy data $\Pi^{\tau}_{\mathbf{k}}$. This is achieved by expanding the Dirac field in the bases $\phi^{\tau}_{\mathbf{k}}$ and $\phi^{\tau_0}_{\mathbf{k}}$, defined by the Cauchy data $\Pi^{\tau}_{\mathbf{k}}$ and $\Pi^{\tau_0}_{\mathbf{k}}$, respectively, with the corresponding creation and annihilation particle operators associated to each Cauchy surface:

$$\phi_{\mathbf{k}}=\phi^{\tau_0}_{1\mathbf{k}}a^{\tau_0}_{1\mathbf{k}}+\phi^{\tau_0}_{2\mathbf{k}}a^{\tau_0}_{2\mathbf{k}}+\phi^{\tau_0}_{3\mathbf{k}}a^{\tau_0\dagger}_{3(-\mathbf{k})}+\phi^{\tau_0}_{4\mathbf{k}}a^{\tau_0\dagger}_{4(-\mathbf{k})} \ , \qquad \phi_{\mathbf{k}}=\phi^{\tau}_{1\mathbf{k}}a^{\tau}_{1\mathbf{k}}+\phi^{\tau}_{2\mathbf{k}}a^{\tau}_{2\mathbf{k}}+\phi^{\tau}_{3\mathbf{k}}a^{\tau\dagger}_{3(-\mathbf{k})}+\phi^{\tau}_{4\mathbf{k}}a^{\tau\dagger}_{4(-\mathbf{k})} \ . \tag{3.4}$$

In order to simplify the notation, we introduce the column operator

$$a_{\mathbf{k}}=\begin{bmatrix} a_{1\mathbf{k}} \\ a_{2\mathbf{k}} \\ a^{\dagger}_{3(-\mathbf{k})} \\ a^{\dagger}_{4(-\mathbf{k})} \end{bmatrix} \tag{3.5}$$

and the matrix

$$\phi^{\tau(\tau_0)}_{\mathbf{k}}=(\phi^{\tau(\tau_0)}_{1\mathbf{k}}\ \phi^{\tau(\tau_0)}_{2\mathbf{k}}\ \phi^{\tau(\tau_0)}_{3\mathbf{k}}\ \phi^{\tau(\tau_0)}_{4\mathbf{k}}) \ , \tag{3.6}$$

so that (3.4) can be written as

$$\phi_{\mathbf{k}}=\phi^{\tau_0}_{\mathbf{k}}a^{\tau_0}_{\mathbf{k}}, \quad \phi_{\mathbf{k}}=\phi^{\tau}_{\mathbf{k}}a^{\tau}_{\mathbf{k}} \ . \tag{3.7}$$

On the other hand, the base $\phi^{\tau}_{\mathbf{k}}$ can be expressed as a linear combination of spinors (2.26):

$$\phi^{\tau}_{\mathbf{k}}(\mathbf{x},t)=\psi_{\mathbf{k}}(\mathbf{x},t)A^{\tau}_{\mathbf{k}} \ , \tag{3.8}$$

where $A^{\tau}_{\mathbf{k}}$ is a matrix constituted by all the coefficients of the transformation. It can be calculated in terms of Cauchy data $\Pi^{\tau}_{\mathbf{k}}$ by inverting expression (3.8):

$$A^{\tau}_{\mathbf{k}}=\psi^{\dagger}_{\mathbf{k}}(\mathbf{x},\tau)\Pi^{\tau}_{\mathbf{k}} \ . \tag{3.9}$$

As $\phi^{\tau}_{\mathbf{k}}$ and $\phi^{\tau_0}_{\mathbf{k}}$ are two complete solution bases of the Dirac equation, there is a "Bogoliubov" transformation which relates them:

$$\phi^{\tau}_{\mathbf{k}}=\phi^{\tau_0}_{\mathbf{k}}\alpha_{\mathbf{k}}(\tau_0,\tau) \ . \tag{3.10}$$

The matrix transformation $\alpha_{\mathbf{k}}(\tau_0,\tau)$ can be calculated using (3.8), (3.9), and (3.10):

$$\alpha_{\mathbf{k}}(\tau_0,\tau)=\Pi^{\tau_0\dagger}_{\mathbf{k}}\psi_{\mathbf{k}}(\mathbf{x},\tau_0)\psi^{\dagger}_{\mathbf{k}}(\mathbf{x},\tau)\Pi^{\tau}_{\mathbf{k}} \ . \tag{3.11}$$

We see that matrix $\alpha_{\mathbf{k}}(\tau_0,\tau)$ depends on the Cauchy data in Σ^{τ_0} and Σ^{τ} and on the evolution $a(t)$ and its derivatives evaluated in τ_0 and τ.

The matrix $\alpha_{\mathbf{k}}(\tau_0,\tau)$ has two very important properties: (a) It is unitary as it transforms an orthonormal base in another orthonormal base

$$\alpha^{\dagger}_{\mathbf{k}}=\alpha_{\mathbf{k}}^{-1} \ ; \tag{3.12}$$

(b) the charge conjugation fixes some relations between the coefficients of the matrix $\alpha_{\mathbf{k}}$. It can be proved from (3.10) and (2.33) that

$$C\alpha_{\mathbf{k}}C=\alpha^{*}_{-\mathbf{k}} \ . \tag{3.13}$$

Developing this expression, we obtain the following relations that reduce the number of independent elements of $\alpha_{\mathbf{k}}$:

$$\begin{aligned} &\alpha_{41\mathbf{k}}=\alpha^{*}_{14(-\mathbf{k})}, \quad \alpha_{42\mathbf{k}}=-\alpha^{*}_{13(-\mathbf{k})} \ , \\ &\alpha_{31\mathbf{k}}=-\alpha^{*}_{24(-\mathbf{k})}, \quad \alpha_{32\mathbf{k}}=\alpha^{*}_{23(-\mathbf{k})} \ , \\ &\alpha_{21\mathbf{k}}=-\alpha^{*}_{34(-\mathbf{k})}, \quad \alpha_{22\mathbf{k}}=\alpha^{*}_{33(-\mathbf{k})} \ , \\ &\alpha_{11\mathbf{k}}=\alpha^{*}_{44(-\mathbf{k})}, \quad \alpha_{42\mathbf{k}}=-\alpha^{*}_{43(-\mathbf{k})} \ . \end{aligned} \tag{3.13}$$

Property (a) is valid for any metric and property (b) follows when it is possible to define the charge-conjugation operation with property (2.33).

Now, we can deduce the relation between the creation and annihilation at times τ_0 and τ, replacing the Bogoliubov transformation (3.10) in (3.7)

$$a^{\tau}_{\mathbf{k}}=\alpha^{\dagger}_{\mathbf{k}}(\tau_0,\tau)a^{\tau_0}_{\mathbf{k}} \ . \tag{3.14}$$

Using this relation, we can compute the density of particles and antiparticles created by an expanding Universe in the interval (τ_0,τ). If the vacuum state is defined with the Cauchy data in τ_0, then

$$\begin{aligned} {}_{\tau_0}\langle 0 | N^{\tau}_{1\mathbf{k}} | 0 \rangle_{\tau_0} &= {}_{\tau_0}\langle 0 | a^{\tau\dagger}_{1\mathbf{k}} a^{\tau}_{1\mathbf{k}} | 0 \rangle_{\tau_0} \\ &= |\alpha_{31\mathbf{k}}|^2 + |\alpha_{41\mathbf{k}}|^2 , \\ {}_{\tau_0}\langle 0 | N^{\tau}_{2\mathbf{k}} | 0 \rangle_{\tau_0} &= |\alpha_{32\mathbf{k}}|^2 + |\alpha_{42\mathbf{k}}|^2 , \\ {}_{\tau_0}\langle 0 | N^{\tau}_{3\mathbf{k}} | 0 \rangle_{\tau_0} &= |\alpha_{42\mathbf{k}}|^2 + |\alpha_{32\mathbf{k}}|^2 , \\ {}_{\tau_0}\langle 0 | N^{\tau}_{4\mathbf{k}} | 0 \rangle_{\tau_0} &= |\alpha_{41\mathbf{k}}|^2 + |\alpha_{31\mathbf{k}}|^2 , \end{aligned} \tag{3.15}$$

where relations (3.13) have been used for the last two.

From (3.15), we see that the creation of particles is due to the matrix elements not in the diagonal blocks, i.e., those which mix particles and antiparticles. Note also that

$$\begin{aligned} {}_{\tau_0}\langle 0 | N^{\tau}_{1\mathbf{k}} | 0 \rangle_{\tau_0} &= {}_{\tau_0}\langle 0 | N^{\tau}_{4\mathbf{k}} | 0 \rangle_{\tau_0} , \\ {}_{\tau_0}\langle 0 | N^{\tau}_{2\mathbf{k}} | 0 \rangle_{\tau_0} &= {}_{\tau_0}\langle 0 | N^{\tau}_{3\mathbf{k}} | 0 \rangle_{\tau_0} . \end{aligned} \tag{3.16}$$

So, particle-antiparticle pairs with opposite spin projections are created.

In order to compute explicitly the density of particles created (3.15), we must know the Cauchy data which define the vacuum state and we need to determine the spinor base (2.26), taking some solution of system (2.27). As it is a generic base, we can choose the adiabatic one, i.e., obtained from replacing the series (2.29) into (2.26).

In respect to Cauchy data, we can take the adiabatic development up to a given order; so particle creation begins in the next adiabatic order. There are also other criteria to define the Cauchy data. In this work, following several authors,[3–8,10–13] we shall consider those which minimize the energy vacuum expectation value defined by

$$\langle 0 | E | 0 \rangle = \int_{\Sigma} \langle 0 | T_{\mu\nu} | 0 \rangle^{\text{ren}} \eta^{\mu} d\Sigma^{\nu} , \tag{3.17}$$

where $d\Sigma^{\nu}$ is the surface element of the Cauchy surface Σ, η^{μ} is a unitary vector normal to Σ and $\langle 0 | T_{\mu\nu} | 0 \rangle^{\text{ren}}$ means the energy-momentum tensor VEV is renormalized.

We shall now find the vacuum state which minimize the energy VEV. The T_{00} of energy momentum tensor (2.14) is

$$T_{00} = i k_i \bar{\psi} \Gamma^i \psi - m \bar{\psi} \psi . \tag{3.18}$$

Replacing (2.31) and (3.1) in (3.18) we obtain

$$\langle 0 | T_{00} | 0 \rangle = \sum_{\pm s} \sum_{\mathbf{k}} (- i k_i \bar{v}_{\mathbf{k},s} \Gamma^i v_{\mathbf{k},s} - m \bar{v}_{\mathbf{k},s} v_{\mathbf{k},s}) . \tag{3.19}$$

Note that only antiparticles contribute to the VEV.

Actually, we are searching the state that minimizes $\langle 0 | T_{00} | 0 \rangle^{\text{ren}}$. However, the quantity that must be subtracted from $\langle 0 | T_{00} | 0 \rangle$ to renormalize it is a local quantity, independent of $| 0 \rangle$. So, $\langle 0 | T_{00} | 0 \rangle$ and $\langle 0 | T_{00} | 0 \rangle^{\text{ren}}$ will have the same minimum.[10] The expression that must be minimized results from replacing spinors (2.26) into (3.19):

$$\langle 0 | T_{00} | 0 \rangle = \frac{2}{a^3} \sum_{\mathbf{k}} \frac{w_k{}^2 + \Omega_k \Lambda_k^*}{\Lambda_k^* + \Omega_k} . \tag{3.20}$$

The Cauchy data, Ω_k^{ME} and Λ_k^{ME}, which minimize this quantity are

$$\Omega_k^{\text{ME}} = \pm w_k , \quad \Lambda_k^{\text{ME}} = \pm w_k . \tag{3.21}$$

It can be noticed that they agree with those obtained by Najmi and Ottewill[8] using another formalism.

Replacing solution (3.21) into (3.20), we obtain the energy VEV

$$\langle 0 | E | 0 \rangle = - \sum_{\mathbf{k}} 2 w_k , \tag{3.22}$$

where the minus sign has been chosen in (3.21) in order that $\psi_{\mathbf{k}}^{1,2}$ be positive-frequency solutions and $\psi_{\mathbf{k}}^{3,4}$ negative-frequency solutions.

IV. VACUUM STATE AND PARTICLE PRODUCTION

In a series of works, Castagnino *et al.*[10–13] introduce the idea of the vacuum state definition using the coincidence of a local property (singular structure of DeWitt-Schwinger kernel) with a global property (energy minimization). This idea is motivated in the Einstein semiclassical equation, since its source, the energy-momentum tensor, is generally composed by terms corresponding to particle creation depending on $|\alpha_{\mathbf{k}}|^2$, which are global by nature (they depend on Cauchy data given in a surface Σ) and by local terms resulting from its renormalization.[1,2] These local terms depend on the metric and its derivatives up to fourth adiabatic order. So, the local property is necessary in order that the vacuum expectation value of energy-momentum tensor be renormalizable. The usual renormalization technique[2] consists in subtracting the energy-momentum tensor $\langle T_{\mu\nu} \rangle_{\text{DS}}^{(4)}$ constructed with the DeWitt-Schwinger kernel $G_1^{\text{DS}}(x,x')$ up to fourth adiabatic order from the energy-momentum tensor computed using the kernel $G_1(x,x')$, corresponding to vacuum state $| 0 \rangle$, i.e.,

$$\langle 0 | T_{\mu\nu}(x) | 0 \rangle^{\text{ren}} = \langle 0 | T_{\mu\nu}(x) | 0 \rangle - \langle T_{\mu\nu} \rangle_{\text{DS}}^{(4)} . \tag{4.1}$$

Then, if in the coincidence limit $x \to x'$, $G_1(x,x')$ behaves like $G_1^{\text{DS}}(x,x')$ the vacuum expectation value of energy-momentum tensor turns out to be renormalizable. As this is a necessary condition, we shall demand that local property must be satisfied in these terms.[13] As the basis corresponding to the kernel $G_1^{\text{DS}}(x,x')$ up to fourth adiabatic order can be constructed using the adiabatic basis,[18] the Cauchy data for Ω_k and Λ_k functions defined in (2.23) must be those given by the adiabatic expansion (2.29).

As $G_1(x,x')$ can approximate $G_1^{\text{DS}}(x,x')$ in the limit $x \to x'$ in different ways, local property is not enough to define the vacuum state. The reason is that only analytic terms in the geodesic distance between x and x' in $G_1^{\text{DS}}(x,x')$ can be computed; and as it is known nonanalytic terms are also necessary to define the vacuum state.[19,2] Therefore, we need another property in order to complete the vacuum determination. So, we introduce a global property, the minimization of the energy, a global quantity. In agreement with Refs. 8 and 10, the quantity we have taken as energy in (3.17) is

$$\langle 0 | H | 0 \rangle^{\text{ren}} = \int_{\Sigma} \langle 0 | T |_{\mu\nu} | 0 \rangle^{\text{ren}} \eta^{\mu} d\Sigma^{\nu} , \tag{4.2}$$

where Σ is a Cauchy surface (i.e., the "time" associated to the defined vacuum). The vacuum state $|0\rangle_\Sigma$ related to surface Σ will be the quantum state that minimizes the energy (4.2).

In preceding work[10–13] it was shown that there is a good vacuum state in every case where it is possible to find a quantum state that satisfies simultaneously local and global properties. The way in which this coincidence is satisfied leads to the definition of different kinds of vacua: weak vacuum, when developing one or both properties for high energies (in powers of $w_k{}^{-1}$) we have coincidence to the lower order, and strong vacuum, when both properties are satisfied for all adiabatic orders.

Many interesting properties have been proved following these ideas: between two weak vacua the density of particles created and the energy density are finite for spin-0 particles in arbitrary metrics depending on time.[13] It has also been found that the only strong vacua are trivial Killing vacuum and the conformal vacuum, which are generally accepted in the literature. The spectrum of created particles between two strong vacua is an exponentially damped one,[11] as it approaches zero faster than any power of k^{-1} in the limit $k \to \infty$. For massive spin-1 fields in a spatially flat Robertson-Walker metric the results are similar to the spin-0 case for the transverse-polarization modes,[12] while for the longitudinal-polarization mode it was necessary to impose the condition $\dot{a}=0$ in the times where the density of created particles and the energy density are evaluated in order to obtain finite results.

In this work we obtain a different kind of result in the case of spin-$\frac{1}{2}$ particles. The Cauchy data corresponding to local property (2.29) are an adiabatic expansion, while those corresponding to global property (3.21) coincide in its exact expression with the first adiabatic order of (2.29). Therefore, in this case we can define a weak vacuum for any universe evolution, and it is defined by the energy minimization Cauchy data. It is interesting to note that in the scalar case,[13] this criterion for weak vacuum definition in arbitrary metrics led to the Robertson-Walker metrics and fixed the metric coupling constant at the value $\xi=\frac{1}{6}$. This value is known as "conformal coupling" because the Klein-Gordon equation turns out to be invariant under conformal transformations in the nonmassive case. Similarly, for spin-$\frac{1}{2}$ fields, the coincidence of local and global properties to the lowest order do not impose any condition; this is related to the fact that the Dirac equation is always invariant under conformal transformations in the $m=0$ case.

In order to estimate the density of particles created between two weak vacua, we need to calculate the $\alpha_{\mathbf{k}}$ matrix elements appearing in (3.15). They can be obtained using expression (3.11). The block of interest results:

$$\begin{bmatrix}\alpha_{31\mathbf{k}} & \alpha_{32\mathbf{k}}\\ \alpha_{41\mathbf{k}} & \alpha_{42\mathbf{k}}\end{bmatrix} = \begin{bmatrix}k_3 & k^-\\ k^+ & k_3\end{bmatrix}\left[\frac{(w_k^{\tau_0}+m)(\Omega_k^{\tau_0 *}+m)(w_k^{\tau}+m)(\Omega_k^{\tau}+m)}{2w_k^{\tau_0}(\Omega_k^{\tau_0 *}+\Lambda_k^{\tau_0})2w_k^{\tau}(w_k^{\tau *}+\Lambda_k^{\tau})}\right]^{1/2}$$
$$\times\left[\frac{1}{a^{\tau_0}}\left[\frac{1}{\Omega_k^{\tau_0 *}+m}-\frac{1}{w_k^{\tau_0}+m}\right]\left[1+\frac{k^2}{a_\tau{}^2(\Omega_k^{\tau}+m)(w_k^{\tau}+m)}\right]\exp\left[i\int_{\tau_0}^{\tau}\lambda_k dt\right]\exp\left[-i\int_{\tau_0}^{\tau}w_k dt\right]\right.$$
$$\left.-\frac{1}{a^{\tau}}\left[\frac{1}{\Omega_k^{\tau *}+m}-\frac{1}{w_k^{\tau}+m}\right]\left[1+\frac{k^2}{a_{\tau_0}{}^2(\Omega_k^{\tau_0}+m)(w_k^{\tau_0}+m)}\right]\exp\left[-i\int_{\tau_0}^{\tau}\lambda_k dt\right]\exp\left[i\int_{\tau_0}^{\tau}w_k dt\right]\right]. \tag{4.3}$$

As to zero adiabatic order, Ω_k and Λ_k coincide with w_k [see (2.29)] all the $\alpha_{\mathbf{k}}$ elements in (4.3) turn out to be null and therefore there is no particle creation to the lowest order. Considering the following adiabatic order, we obtain for $N_{1\mathbf{k}}$, $N_{2\mathbf{k}}$, $N_{3\mathbf{k}}$, and $N_{4\mathbf{k}}$ that

$$_{\tau_0}\langle 0|N_{\mathbf{k}}^{\tau}|0\rangle_{\tau_0}=\frac{k^2m^2}{16(w_k^{\tau_0})^2(w_k^{\tau})^2(w_k^{\tau}+m)^2(w_k^{\tau_0}+m)^2}\left|\frac{H(\tau)(w_k^{\tau}-m)(w_k^{\tau_0}-m)w_k^{\tau_0}}{a(\tau)(w_k^{\tau})^2}\exp\left[-2i\int_{\tau_0}^{\tau}w_k dt\right]-\frac{H(\tau_0)}{a(\tau_0)}\frac{(w_k^{\tau_0}-m)(w_k^{\tau}+m)w_k^{\tau}}{(w_k^{\tau_0})^2}\right|^2. \tag{4.4}$$

So, for high energies it behaves as

$$_{\tau_0}\langle 0|N_{\mathbf{k}}^{\tau}|0\rangle_{\tau_0}\sim 1/k^4\,. \tag{4.5}$$

Then, the density of particles created $N(\tau_0,\tau) = \int_{\tau_0}\langle 0|N_{\mathbf{k}}^{\tau}|0\rangle_{\tau_0}d^3k$ results finite.

In this calculus we have used only the first nontrivial order of the adiabatic expansion; nevertheless the results are valid in general, because the upper adiabatic orders have higher k powers in the denominator, and so they add finite terms.

With respect to energy density, we shall see that it is renormalizable in those evolutions with $H(\tau_0)=0$, but the energy-momentum tensor results are nonrenormalizable, as we shall see in the next section.

Finally, we shall study the condition for the existence of a strong vacuum. To have a strong vacuum, Cauchy data (2.29) must coincide with (3.21) to every adiabatic order. As the expression (3.21) coincides with the first order of (2.29), we deduce that we shall be able to define a strong vacuum only when all the upper orders in (2.29) are null.

It can be seen that the conditions that satisfy this restriction are as follows.

(a) $m=0$. In this case, Dirac's equation is invariant under conformal transformations. The resulting vacuum is known as conformal vacuum and is usually accepted in the literature as the adequate vacuum in the nonmassive case.

(b) $a=$const. In the static case we have the trivial Killing vacuum.

Only in these two cases we can define a strong vacuum continuously for all time. In both cases the vacuum obtained is the generally accepted one and is in complete agreement with the strong vacuum obtained for spin 0 and spin 1 using this criterion.[12,13]

We can also study the existence of a strong vacuum at a given time. This can be useful when we work with in-out models, so it is particularly interesting to investigate this point in the remote future ($t\rightarrow\infty$) and in the limit $t\rightarrow 0$. In both cases we shall restrict the study to evolutions $a(t)=t^{\alpha}$ (for $0<\alpha<1$, the most relevant cosmological evolutions are included) and we shall study the conditions resulting for the exponent.

(1) Remote future: It can be seen that if the exponent $\alpha>0$, then all the upper adiabatic terms in (2.29) approach zero in the limit $t\rightarrow\infty$.

(2) Singularity: In the limit $t\rightarrow 0$, all the upper adiabatic terms in (2.29) became null if $\alpha\geq 1$.

Consequently, we conclude that for $\alpha\geq 1$, we can calculate the density of particles created from $t=0$ to $t=\infty$ between two strong vacua, and it is finite. In addition, its spectrum is exponentially damped as can be seen in Ref. 10.

V. RENORMALIZATION

In this section we shall study if the weak vacuum, defined through energy minimization gives an energy-momentum tensor VEV renormalizable using the usual techniques.[2] As it is the Einstein equation source, its renormalizability is a very interesting point:

$$G_{\mu\nu}+\Lambda g_{\mu\nu}+\alpha H^{(1)}_{\mu\nu}+\beta H^{(2)}_{\mu\nu}=\langle T_{\mu\nu}\rangle^{\text{ren}} . \tag{5.1}$$

First, we shall find the expression of $\langle 0\,|\,T_{\mu\nu}(\mathbf{x},\tau)\,|\,0\rangle_{\tau_0}$. The vacuum state associated with the base which minimizes the energy in τ_0, $|\,0\rangle_{\tau_0}$, is in general a particle state at time τ, we shall call it $|\,n_{\mathbf{k},s}n'_{\mathbf{k}',s'}\rangle_\tau$. As we are studying fermions, for a given value of $\mathbf{k}$ and s, the particle and antiparticle numbers, n and n', can only take the values 0 or 1. Using (2.31) and (3.18), the T_{00} component, energy density results:

$$\langle n_{\mathbf{k},s}n'_{\mathbf{k}',s'}\,|\,T_{00}\,|\,n_{\mathbf{k},s}n'_{\mathbf{k}',s'}\rangle=\sum_{\mathbf{k},s}N_{\mathbf{k},s}(-ik_j\bar{u}_{\mathbf{k},s}\Gamma^j u_{\mathbf{k},s}-m\bar{u}_{\mathbf{k},s}u_{\mathbf{k},s})+\sum_{\mathbf{k},s}(-N'_{\mathbf{k},s}+1)(-ik_j\bar{v}_{\mathbf{k},s}\Gamma^j v_{\mathbf{k},s}-m\bar{v}_{\mathbf{k},s}v_{\mathbf{k},s}) . \tag{5.2}$$

Substituting the spinors (2.26) and the energy minimization conditions (3.21) we obtain

$$\langle n_{\mathbf{k},s}n'_{\mathbf{k}',s'}\,|\,T_{00}\,|\,n_{\mathbf{k},s}n'_{\mathbf{k}',s'}\rangle=\frac{1}{(2\pi a)^3}\left[\sum_{\mathbf{k}}(N_{1\mathbf{k}}+N_{2\mathbf{k}}+N_{3\mathbf{k}}+N_{4\mathbf{k}})w_k-\sum_{\mathbf{k}}2w_k\right] , \tag{5.3}$$

where $N_{\mathbf{k}}i$ are the particle and antiparticle number operator VEV in the state $|\,n_{\mathbf{k},s}n'_{\mathbf{k},s'}\rangle$, given by (4.4). The last term which appears in (5.3) is $\langle 0\,|\,T_{00}\,|\,0\rangle$. The first term shows that each particle or antiparticle with mass m and momentum k contributes to the energy with w_k. Repeating the former calculation the other energy-momentum components we obtain

$$_{\tau_0}\langle 0\,|\,T_{ii}(\mathbf{x},\tau)\,|\,0\rangle_{\tau_0}=\frac{1}{(2\pi a)^3}\left[\sum_{\mathbf{k}}(N_{1\mathbf{k}}+N_{2\mathbf{k}}+N_{3\mathbf{k}}+N_{4\mathbf{k}})\frac{k_i{}^2}{a^2}\frac{1}{w_k}-2\sum_{\mathbf{k}}\frac{k_i{}^2}{a^2}\frac{1}{w_k}\right] . \tag{5.4}$$

It is easy to see, substituting (4.4) in (5.3) and (5.4) that these quantities diverge. The divergent terms are

$$\text{div}_{\tau_0}\langle 0\,|\,T_{00}(\mathbf{x},\tau)\,|\,0\rangle_{\tau_0}=-\frac{2}{(2\pi a)^3}\int_0^\infty w_k d^3k+\frac{m^2H(\tau_0)H(\tau)}{(2\pi a)^3 4}\int_0^\infty\frac{d^3k}{w_k{}^3}+\frac{m^2H^2(\tau)}{(2\pi a)^3 4}\int_0^\infty\frac{d^3k}{w_k{}^3} , \tag{5.5a}$$

$$\text{div}_{\tau_0}\langle 0\,|\,T_{ii}(\mathbf{x},\tau)\,|\,0\rangle_{\tau_0}=-\frac{2}{3(2\pi a)^3}\int_0^\infty\frac{k^2}{a^2}\frac{d^3k}{w_k}-\frac{m^2H(\tau_0)H(\tau)}{(2\pi a)^3 12}\int_0^\infty\frac{d^3k}{w_k{}^3}-\frac{m^2H^2(\tau)}{(2\pi a)^3 12}\int_0^\infty\frac{d^3k}{w_k{}^3} . \tag{5.5b}$$

It can be seen from (5.5) that the energy-momentum tensor VEV has quartic and logarithmic divergencies. The last appear in two terms: one proportional to $H^2(\tau)$ and another one proportional to $H(\tau)H(\tau_0)$. The latter can be eliminated when we consider evolutions with the initial condition $H(\tau_0)=0$.

In order to see if expressions (5.5) are renormalizable, we shall use the adiabatic regularization scheme.[20,2] We shall subtract from (5.5) the energy-momentum tensor VEV constructed with the fourth adiabatic order expansion of the fields modes, so the renormalized quantities are

$$\langle 0 | T_{\mu\nu} | 0 \rangle^{\text{ren}} = \langle 0 | T_{\mu\nu}(x) | 0 \rangle - \langle 0^{(4)} | T_{\mu\nu}(x) | 0^{(4)} \rangle \, |^4 \, . \tag{5.6}$$

The symbol $|^4$ indicates that only the terms up to fourth adiabatic order in $\langle T_{\mu\nu} \rangle$ must be taken into account.

However, as it can be seen in (5.5), the divergences in $\langle T_{\mu\nu} \rangle$ correspond to terms up to second adiabatic order. So it is enough to calculate $\langle 0^{(2)} | T_{\mu\nu}(x) | 0^{(2)} \rangle \, |^2$, because we shall show that its divergences are not the same that which appear in (5.5). And the remaining divergences corresponding to second adiabatic order cannot be canceled with terms corresponding to $\langle 0^{(4)} | T_{\mu\nu}(x) | 0^{(4)} \rangle^{(4)}$.

The T_{00} component can be obtained by substituting (2.29) in (3.20):

$$\langle 0^{(2)} | T_{00} | 0^{(2)} \rangle \, |^2 = \frac{1}{(2\pi a)^3} \left[-2 \int d^3k \, w_k + \tfrac{1}{4} \int d^3k \frac{m^2}{w_k^5} (w_k^2 - m^2) H^2(\tau) \right] . \tag{5.7}$$

The divergent terms are

$$\text{div} \langle 0^{(2)} | T_{00} | 0^{(2)} \rangle \, |^2 = \frac{1}{(2\pi a)^3} \left[-2 \int d^3k \, w_k + \tfrac{1}{4} \int d^3k \frac{m^2}{w_k^3} H^2(\tau) \right] . \tag{5.8}$$

We see that the divergences in (5.5a) cancel those of (5.8) if we consider evolutions satisfying the condition $H(\tau_0)=0$. $\langle 0^{(A)} | T_{ii} | 0^{(A)} \rangle \, |^A$ can be obtained substituting the expansion (2.29) in the expression

$$\langle 0^{(A)} | T_{ii} | 0^{(A)} \rangle \, |^A = \tfrac{2}{3} \sum_k \frac{k^2}{a^2} \frac{(\Omega_k^A + \Omega_k^{A*} + 2m)}{k^2/a^2 + (\Omega_k^A + m)(\Omega_k^{A*} + m)} \, . \tag{5.9}$$

Up to second adiabatic order, the result is

$$\langle 0^{(2)} | T_{ii} | 0^{(2)} \rangle \, |^2 = \frac{1}{(2\pi a)^3} \left\{ -\frac{2}{3} \int \frac{d^3k}{w_k} \frac{k^2}{a^2} - \int d^3k \frac{w_k + m}{w_k^2} \left[\left[\frac{1}{4} \frac{m^2}{w_k^2} - \frac{1}{4} \frac{m^3}{w_k^3} + \frac{5}{4} \frac{m^4}{w_k^4} - \frac{5}{4} \frac{m^5}{w_k^5} \right] H^2 - \frac{1}{4} \frac{m^2}{w_k^3} (w_k - m) \frac{R}{6} \right] \right\} . \tag{5.10}$$

The divergent terms are

$$\text{div} \langle 0^{(2)} | T_{ii} | 0^{(2)} \rangle \, |^2 = \frac{1}{(2\pi a)^3} \left[-\frac{2}{3} \int \frac{d^3k}{w_k} \frac{k^2}{a^2} + \frac{1}{3} \int d^3k \left[\frac{1}{4} \frac{H^2 m^2}{w_k^3} - \frac{1}{4} \frac{m^2}{w_k^3} \frac{R}{6} \right] \right] . \tag{5.11}$$

Comparing the expression (5.5b) and (5.11), we see that even taking the condition $H(\tau_0)=0$, the divergences do not cancel exactly, but they differ in the quantity $-\frac{1}{2}(R/6) \int d^3k (m^2/w_k^3)$. So, the energy-momentum tensor VEV obtained with energy minimization criterion is not renormalizable using the standard techniques.

This result is in agreement with that obtained by Najmi and Ottewill,[8] who worked with the singularity structure of the energy minimization propagator $G_M(x,x') = \langle [\psi_M(x), \psi_M(x')] \rangle$ and showed that the energy-momentum tensor trace has divergences which are not of the same form as the Hadamard structure. They found that the discrepant term is a logarithmic divergency proportional to $R/6$, like the one obtained in this work.

The alternative method used in this section for studying the renormalizability of the energy-momentum tensor leads to simpler calculation and gives the possibility of a direct generalization to the case of more complex metrics.

VI. CONCLUSIONS

From the results obtained by the implementation of local and global properties, we see that for any universe evolution $a(t)$ a weak vacuum can be defined and it coincides with the one defined using energy minimization. On the other hand, it is only possible to define a strong vacuum in everytime in the nonmassive case (conformal vacuum) and when $a(t)=\text{const}$ (Killing vacuum). In these cases, the vacuum states obtained are the ones usually accepted in the literature. We have also found some evolutions in which it is possible to define a strong vacuum asymptotically in the future and in the singularity.

On the other hand, we have computed the Bogoliubov transformation coefficients between two energy minimization bases, and using them we have proved that the density of particles created is finite.

Finally, we have calculated the energy-momentum tensor VEV with energy minimization Cauchy data and we have proved that it is nonrenormalizable comparing its divergences with those of $\langle 0^{(4)} | T_{\mu\nu} | 0^{(4)} \rangle |^4$ (adiabatic regularization). This result shows that if we accept standard renormalization techniques, vacuum state definition through the energy minimization requirement leads to physical quantity expectation values in those states which are not finite.

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[1]B. S. DeWitt, Phys. Rep. **19C**, 297 (1975).

[2]N. D. Birrell and P. C. W. Davies, *Quantum Fields in Curved Space* (Cambridge University Press, Cambridge, 1982).

[3]L. Parker, Phys. Rev. **183**, 1057 (1969); Phys. Rev. D **3**, 346 (1971).

[4]A. A. Grib and S. G. Mamaev, Sov. J. Nucl. Phys. **10**, 722 (1970).

[5]M. R. Brown and C. R. Dutton, Phys. Rev. D **18**, 4422 (1978).

[6]D. Deutsch and A. M. Najmi, Phys. Rev. D **28**, 1907 (1983).

[7]M. R. Brown, J. Math. Phys. **25**, 136 (1984).

[8]A. M. Najmi and A. C. Ottewill, Phys. Rev. D **30**, 2573 (1984).

[9]A. M. Najmi and A. C. Ottewill, Phys. Rev. D **30**, 1733 (1984); Gen. Relativ. Gravit. **17**, 573 (1985).

[10]M. Castagnino, Gen. Relativ. Gravit. **15**, 1149 (1983).

[11]M. Castagnino and F. D. Mazzitelli, Phys. Rev. D **31**, 742 (1985).

[12]M. Castagnino, L. P. Chimento, and C. Laciana, Prog. Theor. Phys. **17**, 5 (1986).

[13]M. Castagnino and L. P. Chimento, preceding paper, Phys. Rev. D **34**, 3676 (1986).

[14]S. Weinberg, *Gravitation and Cosmology* (Wiley, New York, 1972).

[15]M. Castagnino, Ann. Inst. Henri Poincaré **25**, 55 (1981).

[16]M. Castagnino, L. P. Chimento, D. Harari, and C. J. Nuñez, J. Math. Phys. **25**, 360 (1984).

[17]J. D. Bjorken and S. D. Drell, *Relativistic Quantum Fields* (McGraw-Hill, New York, 1965).

[18]M. Castagnino, D. Harari, and C. Núñez, in *Unification of the Fundamental Particle Interactions—II,* edited by J. Ellis and S. Ferrara (Ettore Majorana Science Series-Physical Sciences, Vol. 15) (Plenum, New York, 1983).

[19]C. Bernard and S. Duncan, Ann. Phys. (N.Y.) **107**, 201 (1977).

[20]L. Parker and S. Fulling, Phys. Rev. D **9**, 341 (1974).