Multipole moments for stationary, non-asymptotically-flat systems in general relativity

Wai-Mo Suen*

Theoretical Astrophysics 130-33, California Institute of Technology, Pasadena, California 91125

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A formulation of multipole moments generalizing that of Thorne is proposed for the stationary, vacuum region of spacetime surrounding a source of gravity, without assuming asymptotic flatness. In this formalism, such a region of spacetime is characterized by four sets of moments, the internal mass and current moments (those of the internal source) and the external mass and current moments (those of the external universe), which are read out from a de Donder coordinate expansion of the metric density. These moments uniquely determine the vacuum region of spacetime. The interactions between a gravitating body and an external gravitational field can be described in terms of these moments, in close analogy with Newtonian theory. A derivation, using the vacuum Einstein equation alone, is given of the laws of force and torque for an isolated body acted on by an external field. These laws generalize the results of Thorne and Hartle and of Zhang.

I. INTRODUCTION

The concept of multipole moments for curved spacetime is significant in many ways: Through analogy with Newtonian systems, multipole moments can provide important physical insights into solutions of the Einstein equations. Also, they provide a way to extract the information carried in a metric. Indeed, in the stationary spacetimes we are studying, we will show that the multipole moments contain all the information about the vacuum region of spacetime; the entire metric can be constructed from the multipole moments; in the words of Beig,[1] the multipole moments act as a "complete set of variables for the state space." In view of the success of solution-generating methods for the stationary axisymmetric vacuum Einstein equations,[2] a scheme using the multipole moments to classify and understand these solutions is clearly desirable. Besides their use in analyzing given metrics, multipole moments are also useful in constructing model spacetimes; we will give explicit examples of this in this paper. Multipole moments are also valuable in studying the structure of spatial infinity; indeed, the Geroch-Hansen definition of multipole moments for stationary asymptotically flat spacetimes is intimately tied to the structure of spatial infinity.

Many efforts have been made to define the multipole moments of stationary asymptotically flat spacetimes.[3–9] The recent works essentially concentrate on two approaches. The first approach works in the conformal completion of the 3-manifold of timelike Killing trajectories and defines the multipole moments as symmetric trace-free tensors at the point corresponding to spatial infinity. This approach was initiated by Geroch and Hansen[4] and continued by many authors.[5,6] The beauty of the resulting definition is that it is completely geometric. The only possible arbitrariness in determining the moments comes from the choice of the conformal factor. But Geroch[4] (see also Beig[1]) has shown that by introducing into the definition terms involving the Ricci tensor of the conformal space, an arbitrary change of the conformal factor affects the multipole moments in exactly the same way as translation affects the Newtonian moments. More importantly, it has been shown that the moments so defined have many of the properties which we would like multipole moments to have.[6]

By contrast, the second approach defines multipole moments as the coefficients of certain coordinate expansions of certain metric functions in physical spacetime using specially chosen coordinates;[7,8] this generalizes the usual procedure of reading the mass and angular momentum from the metric. Thorne's formalism[7] expands the metric in asymptotically Cartesian and mass centered (ACMC) coordinates, whereas Beig and Simon[8] expand the Hansen potentials[4] in similar coordinates. At first sight it appears that these formalisms have the unpleasant feature of depending crucially on the choice of coordinates. Both the Thorne formalism and the Beig-Simon formalism have solved this problem by showing that the moments so defined are independent of the coordinate system so long as one stays within the chosen class of coordinates, that they have a number of desirable properties, and that, in fact, they coincide with the geometrically defined Geroch-Hansen moments.[8,9] Compared with the Geroch-Hansen approach, these formalisms are closely tied to physical spacetime, in the sense that (1) one can read the moments directly from the metric of physical spacetime as the coefficients of a coordinate expansion, and (2) the formalisms are fortified with algorithms which in a straightforward manner reconstruct the metric from the multipole moments in terms of a series expansion. Hence they are rather convenient for application to physical problems. Thorne's formalism is also tied to gravitational wave generation, and has been used in a number of astrophysical studies.[10] On the other hand, the development of the metric into series expansions creates problems: (1) Given a metric that is a solution to the Einstein equation, is the expansion in those specially chosen coordinates always convergent? (2) Given a set of multipoles, under what conditions will the expansions of the constructed metric converge? These questions have not been thoroughly investigated in either the Thorne formalism or the Beig-

Simon formalism.

All of the formalisms discussed above deal only with bodies in *asymptotically flat* spacetime. Can one also analyze a system consisting of an isolated body in an externally imposed gravitational field in terms of multipole moments? This is the question that we want to answer in this paper. Surely in Newtonian gravitation such a system is well described in terms of multipole moments: from the expansion of the potential $\Phi(\nabla^2\Phi=0)$ in positive and negative powers of the radial coordinate r one can read off a set of internal multipole moments characterizing the structure of the central body (and its gravitational field) and a set of external multipole moments characterizing the imposed external field (and its sources). Then the gravitational interaction can be described as follows. (i) The external l-pole field will distort the central body, and hence induce a change in the internal l moment. (ii) The external l-pole field will couple to the internal l moment (both intrinsic and induced) to produce a torque on the body, if their principal axes are not aligned. (iii) The external $(l+1)$-pole field coupled to the internal l moment will produce an acceleration of the body.

What we wish to show in this paper is that the external and internal multipole moments of a stationary vacuum spacetime can be defined by a natural extension of Thorne's formalism, and that the gravitational interaction of an isolated body with an external universe can be set in exactly the same language in general relativity as in Newtonian theory. In the case of an asymptotically flat, empty external universe (vanishing external moments), the internal moments of the analysis reduce to those of Thorne;[7] and in the case of no internal body (vanishing internal moments), the external moments are closely related to those of Zhang.[11]

The spirit of our analysis is rather different from that of the recent work by Thorne and Hartle[12] and Zhang[13] on the gravitational fields of isolated bodies interacting with an external universe. Briefly, they permit the gravitational field to be slowly varying with time, whereas we insist that it be stationary (except in Sec. III and Appendix B below where we generalize to slow time variations); they restrict attention to the lowest few multipole moments, whereas we consider all moments; and they regard the moments as defined only up to an uncertainty determined by the effects of coupling of the body to the external universe, whereas our moments are defined precisely. We will discuss these issues at greater length in the body of this paper.

In Sec. II we make precise the kind of system that we want to study and propose a definition of the multipole moments for such systems; and we describe an algorithm which enables us to construct the metric from a given set of multipole moments (basically a repetition of Thorne's algorithm for the asymptotically flat case). In Sec. III we first investigate the constraints on the moments in generating a stationary vacuum spacetime, then we relax the exact stationary condition to allow for slow time variation, and obtain the force and torque laws in terms of the multipole moments. In Sec. IV we summarize and discuss the results.

II. MULTIPOLE MOMENTS FOR STATIONARY SYSTEMS

We begin with a brief discussion of the systems to which our formalism applies and the situation where this formalism is most useful. We consider a stationary system with a gravitating body located in an external universe. Surrounding the world tube of the body (the region of spacetime which either has $T_{\mu\nu}\neq 0$ or is inside a horizon) there is a region of spacetime satisfying the vacuum Einstein equations. Call this region D. We shall define our multipole moments in terms of coordinate expansions of the metric density, which is a solution of the vacuum equations in D. It does not matter whether D extends to spatial infinity or not; in particular, *no asymptotic flatness* is assumed. Indeed, if we assume the spacetime to be asymptotically flat, then our external multipole moments will vanish, and our internal moments will trivially reduce to those of Thorne.[7] Where there are gravitational fields generated by external sources, we will have an additional set of moments, the external moments, to characterize the structure of the vacuum spacetime. Also, we need not make explicitly the assumption that the gravitational field is weak in D. However, in general the concept of a multipole expansion of a field is useful only when the field is smooth enough that it can be characterized by the first few terms of the expansion and the higher multipole moments can be neglected. In the same sense, the multipole expansion that we shall construct will be useful mainly for an "isolated" body in an external universe, for which the multipole expansion converges rapidly. We use the word "isolated" in the sense of Thorne and Hartle:[12] the external material is distant enough that it generates a Riemann curvature tensor near the central body having length scales $\mathscr{R},\mathscr{L} \gg L,M$ where $\mathscr{R}$ is the radius of curvature of external Riemann tensor, $\mathscr{L}$ is the inhomogeneity scale of external Riemann tensor, M is the mass of the central body, and L is the length scale (size) of the central body. (The separation into external and internal quantities will be made precise in Sec. III B. For the discussion here precise separation is not necessary.) For such a body there exists a "buffer" zone in the vacuum region D, at a typical radius r with $(M,L)\ll r\ll(\mathscr{L},\mathscr{R})$. In this buffer region, the multipole expansion typically is dominated by the first few moments, and the multipole formalism is most useful here.

A. The construction of the stationary vacuum metric in terms of multipole moments

Following Thorne,[7] our formalism is built on a de Donder coordinate system. We assume that there is a single coordinate system which satisfies the de Donder gauge condition in the vacuum region D (though the origin of the coordinate may lie outside D). The structure of the spacetime in this vacuum region is given by a tensor field $\bar{h}^{\mu\nu}$, which is related to the metric density by

$$\mathfrak{g}^{\alpha\beta}=\sqrt{-g}\,g^{\alpha\beta}=\eta^{\alpha\beta}-\bar{h}^{\alpha\beta}\ ,$$
$$\eta^{\alpha\beta}=\mathrm{diag}(-1,1,1,1),\quad g=\det(g_{\mu\nu})\ . \qquad (2.1)$$

From $g_{\mu\nu}$ it is straightforward to determine $\bar{h}^{\mu\nu}$, and vice versa, provided the metric is nondegenerate. We assume that the metric satisfies this requirement throughout the paper. In the following discussion we sometimes make no differentiation between $g_{\mu\nu}$ and $\bar{h}^{\mu\nu}$ and refer to both of them loosely as the metric. The Einstein equation in de Donder coordinates reads

$$\Box \bar{h}_{\alpha\beta} = -\bar{h}_{\alpha\beta,00} + \bar{h}_{\alpha\beta,jj} = -16\pi W_{\alpha\beta} , \tag{2.2}$$

and the de Donder coordinate condition is

$$\bar{h}_{00,0} = \bar{h}_{0j,j}, \ \bar{h}_{j0,0} = \bar{h}_{jk,k} . \tag{2.3}$$

Here $\alpha,\beta = 0,1,2,3; i,j = 1,2,3$; commas denote partial derivatives; and indices on all quantities except $g^{\mu\nu}$ and $g_{\mu\nu}$ are raised and lowered with the flat metric $\eta_{\alpha\beta}$ (which permits us to interchange upper and lower spatial indices according to convenience). The summation convention is used not only when one index is up and the other down, but also for latin (spatial) indices when both are down. The $W_{\alpha\beta}$ in (2.2) is given by

$$W_{\alpha\beta} = (-g) t^{(\mathrm{LL})}_{\alpha\beta} + \frac{1}{16\pi} (\bar{h}_{\alpha\mu,\nu} \bar{h}_{\beta}^{\ \nu,\mu} - \bar{h}_{\alpha\beta,\mu\nu} \bar{h}^{\mu\nu}) . \tag{2.4}$$

Here $t^{(\mathrm{LL})}_{\alpha\beta}$ is the Landau-Lifshitz pseudotensor:

$$\begin{aligned} 16\pi(-g) t^{\alpha\beta(\mathrm{LL})} = {} & \mathfrak{g}^{\alpha\beta}{}_{,\lambda} \mathfrak{g}^{\lambda\mu}{}_{,\mu} - \mathfrak{g}^{\alpha\lambda}{}_{,\lambda} \mathfrak{g}^{\beta\mu}{}_{,\mu} + \tfrac{1}{2} g^{\alpha\beta} g_{\lambda\mu} \mathfrak{g}^{\lambda\nu}{}_{,\rho} \mathfrak{g}^{\rho\mu}{}_{,\nu} \\ & - (g^{\alpha\lambda} g_{\mu\nu} \mathfrak{g}^{\beta\nu}{}_{,\rho} \mathfrak{g}^{\mu\rho}{}_{,\lambda} + g^{\beta\lambda} g_{\mu\nu} \mathfrak{g}^{\alpha\nu}{}_{,\rho} \mathfrak{g}^{\mu\rho}{}_{,\lambda}) + g_{\lambda\mu} g^{\nu\rho} \mathfrak{g}^{\alpha\lambda}{}_{,\nu} \mathfrak{g}^{\beta\mu}{}_{,\rho} \\ & + \tfrac{1}{8} (2 g^{\alpha\lambda} g^{\beta\mu} - g^{\alpha\beta} g^{\lambda\mu})(2 g_{\nu\rho} g_{\sigma\tau} - g_{\rho\sigma} g_{\nu\tau}) \mathfrak{g}^{\nu\tau}{}_{,\lambda} \mathfrak{g}^{\rho\sigma}{}_{,\mu} . \end{aligned} \tag{2.5}$$

The integrability condition for Eqs. (2.2) and (2.3) will be particularly important in later discussion; it is the Bianchi identity, which in de Donder coordinates read

$$W^{\alpha\beta}{}_{,\alpha} = 0 . \tag{2.6}$$

We shall now describe a systematic way of constructing the solution $\bar{h}_{\mu\nu}$ of the Einstein and gauge equations (2.2) and (2.3), in terms of expansions. This is essentially a repetition of the analysis in Ref. 7, except for the contributions coming from the "external field." We present this construction in detail here since the formulas will be referred to frequently later.

We start by writing

$$\bar{h}_{\alpha\beta} = \sum_{p=1}^{\infty} G^p \gamma^p_{\alpha\beta} , \tag{2.7}$$

where G is a "nonlinearity" bookkeeping parameter, whose numerical value can be set to one. Then for each order in p, we have, from Eqs. (2.2) and (2.3),

$$\gamma^p_{00,0} = \gamma^p_{0j,j}, \ \ \gamma^p_{j0,0} = \gamma^p_{jk,k} , \tag{2.8}$$

$$\Box \gamma^p_{\alpha\beta} = -16\pi W^p_{\alpha\beta}(\gamma^q_{\mu\nu}; q < p) , \tag{2.9}$$

where $W^p_{\alpha\beta}$ is a polynomial in $\gamma^q_{\mu\nu}(q<p)$ and its first two derivatives. These can be regarded as the defining equations for $\gamma^p_{\mu\nu}$. Throughout this paper we consider only the solutions of the Einstein equation which admit such "post-Minkowskian" expansions.[7,14] This assumption amounts to requiring the metric to be obtainable to arbitrary accuracy by iterating the linearized solution. It is physically reasonable to expect that in the weak-field buffer region, all solutions admit such an expansion.

Next we specialize to the stationary situation. It is clear that we can always choose de Donder coordinates such that

$$\frac{\partial}{\partial x^0} = \frac{\partial}{\partial t} = \text{timelike Killing vector} . \tag{2.10}$$

Hence all $\partial/\partial t$ give zero. For $p=1$ we have, from (2.8) and (2.9),

$$\gamma^1_{\alpha\beta,kk} = 0 , \tag{2.11}$$

$$\gamma^1_{\mu j,j} = 0 . \tag{2.12}$$

The general solution to Eq. (2.11) is, in symmetric trace-free (STF) tensor form

$$U_{00} = \sum_{l=0}^{\infty} \mathscr{A}_{A_l} \partial_{A_l} \left[\frac{1}{r} \right] + \sum_{l=0}^{\infty} \mathscr{A}'_{A_l} X_{A_l} , \tag{2.13}$$

$$\begin{aligned} U_{0j} = {} & \sum_{l=1}^{\infty} \left[\epsilon_{jpq} \mathscr{B}_{pA_{l-1}} \partial_{qA_{l-1}} \left[\frac{1}{r} \right] + \epsilon_{jpq} \mathscr{B}'_{pA_{l-1}} \hat{X}_{qA_{l-1}} \right] \\ & + \sum_{l=1}^{\infty} \left[\mathscr{C}_{jA_{l-1}} \partial_{A_{l-1}} \left[\frac{1}{r} \right] + \mathscr{C}'_{jA_{l-1}} X_{A_{l-1}} \right] + \sum_{l=0}^{\infty} \left[\mathscr{D}_{A_l} \partial_{jA_l} \left[\frac{1}{r} \right] + \mathscr{D}'_{A_l} X_{\langle jA_l \rangle} \right] , \end{aligned} \tag{2.14}$$

$$\begin{aligned} U_{ij} = {} & \sum_{l=0}^{\infty} \left[\delta_{ij} \mathscr{E}_{A_l} \partial_{A_l} \left[\frac{1}{r} \right] + \delta_{ij} \mathscr{E}'_{A_l} X_{A_l} \right] + \sum_{l=2}^{\infty} \left[\mathscr{F}_{ijA_{l-2}} \partial_{A_{l-2}} \left[\frac{1}{r} \right] + \mathscr{F}'_{ijA_{l-2}} X_{A_{l-2}} \right] \\ & + \sum_{l=2}^{\infty} \left[\epsilon_{pq\langle j} \mathscr{G}_{i\rangle pA_{l-2}} \partial_{qA_{l-2}} \left[\frac{1}{r} \right] + \epsilon_{pq\langle j} \mathscr{G}'_{i\rangle pA_{l-2}} \hat{X}_{qA_{l-2}} \right] + \sum_{l=1}^{\infty} \left[\mathscr{H}_{A_{l-1}\langle i} \partial_{j\rangle A_{l-1}} \left[\frac{1}{r} \right] + \mathscr{H}'_{A_{l-1}\langle i} \hat{X}_{j\rangle A_{l-1}} \right] \\ & + \sum_{l=1}^{\infty} \left[\mathscr{I}_{pA_{l-1}} \epsilon_{pq\langle i} \partial_{j\rangle qA_{l-1}} \left[\frac{1}{r} \right] + \mathscr{I}'_{pA_{l-1}} \epsilon_{pq\langle i} \hat{X}_{j\rangle qA_{l-1}} \right] + \sum_{l=0}^{\infty} \left[\mathscr{K}_{A_l} \partial_{ijA_l} \left[\frac{1}{r} \right] + \mathscr{K}'_{A_l} \hat{X}_{ijA_l} \right] . \end{aligned} \tag{2.15}$$

Here we adopt the conventions of Refs. 14 and 15 that (i) all indices between $\langle\,\rangle$ are to be symmetrized and made trace-free, and (ii) a caret over a tensor indicates that all its indices are to be symmetrized and made trace-free. All other conventions follow Thorne,[7] namely, (iii) a sequence of l indices is denoted by $S_{A_l} \equiv S_{a_1 a_2 a_3 \cdots a_l}$, (iv) $r \equiv (x_i x_i)^{1/2}, n_i \equiv x_i / r, x_{A_l} \equiv r^l N_{A_l} \equiv x_{a_1} x_{a_2} \cdots x_{a_l}$, (v) capital script letters denote symmetric trace-free tensors, $\mathscr{B}_{A_l} \equiv \hat{\mathscr{B}}_{A_l} \equiv \mathscr{B}_{\langle a_1 a_2 \cdots a_l \rangle}$, and (vi) ϵ_{ijk} is the alternating (flat-space Levi-Civita) symbol. One can easily see that the U's given in the form of Eqs. (2.13)–(2.15) satisfy the Laplace equation. (A thorough discussion of the properties of the $1/r^l$ part of these solutions and their relationship to various kinds of spherical harmonics is given in Ref. 7; the r^l part follows trivially. See also Pirani[16] for STF tensors.) Note that $\partial_{A_l}(1/r) = \hat{\partial}_{A_l}(1/r)$, and that there is no need to set a caret on those X_{A_l} that are contracted into an STF tensor. The structure of the terms in Eqs. (2.13)–(2.15) should be clear.

To obtain $\gamma^1_{\mu\nu}$, we substitute the U' into the stationary de Donder gauge condition (2.12) and arrive at

$$\gamma^1_{00} = \sum_{l=0}^{\infty} \left[\mathscr{A}_{A_l} \left[\frac{1}{r} \right]_{,A_l} + \mathscr{A}'_{A_l} X_{A_l} \right] , \tag{2.16}$$

$$\gamma^1_{0j} = \sum_{l=1}^{\infty} \left[\epsilon_{jpq} \mathscr{B}_{pA_{l-1}} \left[\frac{1}{r} \right]_{,qA_{l-1}} + \epsilon_{jpq} \mathscr{B}'_{pA_{l-1}} \hat{X}_{qA_{l-1}} \right] + \sum_{l=1}^{\infty} (\mathscr{C}'_{jA_{l-1}} X_{A_{l-1}}) + \sum_{l=0}^{\infty} \left[\mathscr{D}_{A_l} \left[\frac{1}{r} \right]_{,ijA_l} \right] , \tag{2.17}$$

$$\begin{aligned} \gamma^1_{ij} = {} & \sum_{l=2}^{\infty} (\mathscr{F}'_{ijA_{l-2}} X_{A_{l-2}}) + \sum_{l=2} (\epsilon_{pq\langle j} \mathscr{G}'_{i\rangle qA_{l-2}} \hat{X}_{pA_{l-2}}) + \sum_{l=1}^{\infty} \left[\mathscr{I}_{qA_{l-1}} \epsilon_{pq\langle i} \partial_{j\rangle pA_{l-1}} \left[\frac{1}{r} \right] \right] \\ & + \sum_{l=0} \left[\mathscr{K}_{A_l} \partial_{ijA_l} \left[\frac{1}{r} \right] \right] + \sum_{l=1}^{\infty} \left[-\frac{1}{6} \mathscr{H}_{A_l} \delta_{ij} \partial_{A_l} \left[\frac{1}{r} \right] + \mathscr{H}_{A_{l-1}\langle i} \partial_{j\rangle A_{l-1}} \left[\frac{1}{r} \right] \right] \\ & + \sum_{l=1}^{\infty} \left[-\frac{1}{6} \frac{(2l^2+5l+3)}{l(2l-1)} \mathscr{H}'_{A_l} \delta_{ij} X_{A_l} + \mathscr{H}'_{A_{l-1}\langle i} \hat{X}_{j\rangle A_{l-1}} \right] . \end{aligned} \tag{2.18}$$

The forms of these terms will be important both in building the metric from multipole moments and in identifying multipole moments from a given metric, as we shall see.

Next we make use of the "residual" gauge freedom to make what is remaining in $\gamma^1_{\mu\nu}$ assume a form close to that of the Newtonian potential. Under a gauge transformation, $\gamma^1_{\mu\nu}$ transforms as

$$\gamma^{1\,\text{new}}_{\mu\nu} = \gamma^1_{\mu\nu} + \xi_{\mu,\nu} + \xi_{\nu,\mu} - \eta_{\mu\nu} \xi^{\alpha}{}_{,\alpha} . \tag{2.19}$$

With a ξ_μ satisfying

$$\Box \xi_\mu = 0 , \tag{2.20}$$

i.e., without leaving de Donder coordinates, we can gauge $\gamma^1_{\mu\nu}$ into the form (the superscript "new" has been dropped and we have renamed the coefficients)

$$\begin{aligned} \gamma^1_{00} = {} & \sum_{l=0}^{\infty} (-1)^l \frac{4}{l!} \mathscr{I}_{A_l} \left[\frac{1}{r} \right]_{,A_l} \\ & + \sum_{l=1}^{\infty} \frac{4(2l-1)!!}{l!} \mathscr{Q}_{A_l} X_{A_l} , \end{aligned} \tag{2.21}$$

$$\begin{aligned} \gamma^1_{0j} = {} & -\sum_{l=1}^{\infty} (-1)^l \frac{4l}{(l+1)!} \epsilon_{jpq} \mathscr{S}_{pA_{l-1}} \left[\frac{1}{r} \right]_{,qA_{l-1}} \\ & - \sum_{l=1}^{\infty} \frac{4l(2l-1)!!}{(l+1)!} \epsilon_{jpq} \mathscr{C}_{pA_{l-1}} \hat{X}_{qA_{l-1}} , \end{aligned} \tag{2.22}$$

$$\gamma^1_{ij} = 0 . \tag{2.23}$$

We call $\mathscr{I}_{A_l}$ and $\mathscr{S}_{A_l}$ the internal mass and current l-pole moments, and $\mathscr{Q}_{A_l}$ and $\mathscr{C}_{A_l}$ the external mass and current l-pole moments. The choice of normalization factors will be obvious later. Note that we have put in a rescaling of the spacetime coordinates to remove the constant parts from γ_{00}, i.e., to make the coordinate-independent part of g_{00} equal to -1; and, as a result, the summation for the external moments begins at $l=1$.

For this definition of multipole moments to make sense, we have to make sure that there is no more gauge freedom left. This is guaranteed by the following theorem.

Theorem 1. For any stationary second-rank tensorial solution of the flat-spacetime wave equation (i.e., $\Box\gamma_{\mu\nu}=0, \partial_t \gamma_{\mu\nu}=0$), if $\partial_j \gamma_{\mu j}=0$, then there exists a *unique* $\gamma^{\text{new}\,\mu\nu}$ with the form given by (2.21)–(2.23) which is related to $\gamma_{\mu\nu}$ by the gauge transformation (2.19) with $\partial_t \xi_\mu = 0$. The proof of the theorem is straightforward and we omit it.

Equations (2.21)–(2.23) define the multipole moments to G^1 order. What about the general nonlinear situation where we include in $\bar{h}_{\mu\nu}$ the terms of order G^n, with $n>1$? With the $\gamma^1_{\mu\nu}$ given by Eqs. (2.21)–(2.23), we can generate $\gamma^p_{\mu\nu}$ by the following algorithm.

Algorithm A. (i) From $\gamma^1_{\mu\nu}$ of Eqs. (2.21)–(2.23), calculate $W^2_{\mu\nu}$ as the $O(G^2)$ part of Eq. (2.4). (ii) Invert the $p=2$ case of Eq. (2.9) (with vanishing time derivatives) to obtain

$$\gamma^2_{\alpha\beta} = -16\pi \Delta^{-1} W^2_{\alpha\beta} + U^2_{\alpha\beta} , \tag{2.24}$$

where $\Delta^{-2} W^2_{\mu\nu}$ denotes a special solution and $U^2_{\mu\nu}$ satisfies $\nabla^2 U^2_{\mu\nu}=0$. (See Appendix A for the construction of a special solution; however, the algorithm does not depend on how the special solution is constructed.) (iii) Next make use of the freedom of $U^2_{\mu\nu}$ to require $\gamma^2_{\mu\nu}$ to satisfy the stationary gauge condition $\partial_j \gamma^2_{\mu j}=0$, i.e.,

$$\partial_j U^2_{\mu j} - 16\pi \partial_j \Delta^{-1} W^2_{\mu j} = 0 \; . \tag{2.25}$$

(Sometimes this equation will have no solution. We will discuss this point in detail in Sec. III.) This requirement determines $U_{\mu\nu}$ partially; the undetermined parts of $U_{\mu\nu}$ have the forms of (2.16)–(2.18). (iv) Now use the gauge freedom of Eq. (2.19) to guarantee that there be no such Laplace-free and divergence-free terms in γ^2_{ij}, and that the only such terms in γ^2_{0j} have the form given by Eq. (2.22). Then the freedom in $U_{\mu\nu}$ amounts to a free choice of the G^2-order multipole moments. That this can always be done is guaranteed by theorem 1. Therefore if we are given the G^2-order moments, the $\gamma^2_{\mu\nu}$ is uniquely determined. (v) In the same way, we can obtain $\bar{h}_{\mu\nu}$ to arbitrary order in G; and the structure at arbitrary order will be such that the mass and current moments to that order are given by the Laplace-free terms in $\bar{h}_{00}$ and $\bar{h}_{0j}$, i.e., by terms of the form of Eqs. (2.21) and (2.22). All terms having different structure, i.e., different combinations of N_{A_l} and r^m in $\bar{h}_{00}$, $\bar{h}_{0i}$, and $\bar{h}_{0j}$ come from the nonlinear coupling of these multipole moments. The mathematical formulas needed in algorithm A are given in Appendix A. Examples of the construction are given in Appendix B.

B. The general structure of the metric generated

What kind of structure will the metric generated by algorithm A have? We make the following observations.

(i) Logarithmic terms. It is well known that in de Donder coordinates the metric often contains logarithmic terms, cf. Refs. 7 and 14. In algorithm A, a logarithm will be produced in inverting the Laplacian operator for a source with the structure (Laplacian-free function)/r^2. Further iterations of such a logarithmic term give logarithms raised to integer powers. In general the power of $\ln(r)$ can be p [cf. Eq. (2.9)] after p iterations. However, all the logarithmic terms in de Donder coordinates in previous studies[7,14] are connected with dynamical effects, e.g., tail phenomena, phase shifts, propagation on wrong characteristics, etc. This makes us suspect that there may be no logarithmic terms generated in the present stationary case. Indeed, Blanchet and Damour[14] (see also Ref. 7) have shown that there will be no logarithmic terms generated in the case of a stationary vacuum spacetime which is asymptotic flat, i.e., without the external universe. On the other hand, when there is only the external universe and no internal body, it is also easy to see that there will also be no logarithmic terms. In the region of consideration (vacuum, stationary spacetime with nondegenerate metric), $\bar{h}_{\mu\nu}$ satisfies an elliptical equation. Rearranging Eq. (2.2) gives

$$g^{ij}\partial_i\partial_j \bar{h}_{\mu\nu} = -16\pi(-g)t^{\rm LL}_{\mu\nu} - \bar{h}_{\alpha\mu,\nu}\bar{h}_{\beta}{}^{\nu,\beta} \; . \tag{2.26}$$

On the left-hand side g^{ij} is positive definite. The right-hand side is an analytic function of $\bar{h}_{\alpha\beta}$ and its derivatives. [We see this by rewriting $g^{\mu\nu}g_{\rho\sigma}$ in $t^{(\rm LL)}_{\alpha\beta}$ as $g^{\mu\nu}(g^{\rho\sigma})^{-1}$, which is analytic in $\bar{h}_{\mu\nu}$ since $\det(g^{\mu\nu})=\det(g_{\mu\nu})\neq 0$.] Thus, by Morrey's theorem,[17] the solution of Eq. (2.26) is analytic in the coordinates. Hence $\bar{h}_{\mu\nu}$ is a *real analytic* function of the coordinates and contains no logarithmic terms. Next we ask, in the case where there are both an internal body and an external universe, will the coupling of the internal moments and the external moments produce logarithmic terms? We have checked explicitly that in $W^2_{\mu\nu}$ (cf. Sec. III B) all dangerous terms of the form (Laplacian-free function)/r^2 cancel exactly with each other. Moreover, in all the G^3 to G^6 cases we have spot checked, we also find miraculous cancellation. Therefore we conjecture that any metric generated as a post-Minkowskian expansion [cf. Eq. (2.7)] by algorithm A will contain no logarithmic terms.

The absence of logarithmic terms is not necessary for the algorithm to work, but it certainly makes the formula cleaner and the formalism nicer to work with.

(ii) General form of the metric. The $\bar{h}_{\mu\nu}$ generated by algorithm A has the following form:

$$\bar{h}_{00} = \left[\sum_{l=0}^{\infty} (-1)^l \frac{4}{l!} \mathscr{I}_{A_l} \left[\frac{1}{r}\right]_{,A_l} + \sum_{l=1}^{\infty} \frac{4(2l-1)!!}{l!} \mathscr{Q}_{A_l} X_{A_l} \right] + \left\{ \sum_m \sum_l (\mathscr{A}_{A_l} \hat{N}_{A_l}) r^m \right\} , \tag{2.27}$$

$$\bar{h}_{0j} = \left[-\sum_{l=1}^{\infty} (-1)^l \frac{4l}{(l+1)!} \epsilon_{jpq} \mathscr{S}_{pA_{l-1}} \left[\frac{1}{r}\right]_{,qA_{l-1}} - \sum_{l=1}^{\infty} \frac{4l(2l-1)!!}{(l+1)!} \epsilon_{jpq} \mathscr{C}_{pA_{l-1}} \hat{X}_{qA_{l-1}} \right] + \left\{ \sum_m \sum_l (\epsilon_{ijk} \mathscr{B}_{jA_l} \hat{N}_{kA_l}) r^m \right\} , \tag{2.28}$$

$$\bar{h}_{ij} = \left\{ \sum_m \sum_l (\mathscr{D}_{ijA_l} N_{A_l} + \mathscr{E}_{A_l (i} N_{j) A_l} + \mathscr{F}_{A_l} \hat{N}_{ijA_l} + \mathscr{G}_{A_l} N_{A_l} \delta_{ij}) r^m \right\} . \tag{2.29}$$

The terms in [ ] are the multipole-moment terms ("multipole terms") which we use to generate the metric, whereas the term in { } are those generated from the coupling of the moments ("coupling terms"). (In this paper we will always break any functions of the coordinates into sums of the form const$\times \hat{N}_{A_l} r^m$ or const$\times \hat{N}_{A_l} r^m$ $\times$[polynomial in $\ln(r)$] if there are any $\ln(r)$.) By "Laplacian-free term," we shall mean terms having the structure $\hat{N}_{A_l} r^l$ or $\hat{N}_{A_l}/r^{(l+1)}$. Note that the "multipole terms" are "Laplacian-free." In the coupling terms, $\mathscr{A}_{A_l}, \mathscr{B}_{A_l}, \mathscr{D}_{A_l}, \mathscr{E}_{A_l}, \mathscr{F}_{A_l}, \mathscr{G}_{A_l}$, are either STF constant tensors or STF constant tensors times a polynomial in

$\ln(r)$. In the coupling terms the summations on m and l run over all integers which do not produce a term that is both Laplacian-free and divergence-free; i.e., the coupling-term sums contain no terms with the forms (2.16) and (2.17). [We have gauged the Laplacian-free and divergence-free terms away in step (iv) (and its higher-order counterpart) of algorithm A, except for terms of the form (2.22), which are multipole terms rather than coupling terms.] Next we note that in (2.27)–(2.29) the occurrences of ϵ_{ijk} in $\bar{h}_{\mu\nu}$ are determined by time-reversal symmetry; i.e., $\partial/\partial t \to -(\partial/\partial t)$, $\bar{h}_{00}\to\bar{h}_{00}, \bar{h}_{0i}\to -\bar{h}_{0i}$, and $\bar{h}_{ij}\to\bar{h}_{ij}$, which implies

$$\mathscr{I}_{A_l}\to\mathscr{I}_{A_l}, \mathscr{D}_{A_l}\to\mathscr{D}_{A_l}, \mathscr{S}_{A_l}\to-\mathscr{S}_{A_l}, \mathscr{C}_{A_l}\to-\mathscr{C}_{A_l} ,$$

and

$$W_{00}\to W_{00}, W_{0j}\to -W_{0j}, W_{ij}\to W_{ij} .$$

Note that there are no time-symmetry changing operations in forming $W_{\mu\nu}$ from $\bar{h}_{\mu\nu}$ [cf. Eq. (2.4)]. We let $n=$(the number of $\bar{h}_{0j}$ or its derivatives in a term in $W_{\mu\nu}$)=(the number of S_{A_l}) + (the number of C_{B_m}). Then clearly n is even in W_{00} and W_{ij}, and odd in W_{0j}. Since there is an ϵ_{ijk} associated with each current moment and since the product of two ϵ_{ijk} can be reduced to a set of Kronecker δ's, we conclude that there is exactly one ϵ_{ijk} in W_{0j} and hence in $\bar{h}_{0j}$, and no ϵ_{ijk} in W_{00} and W_{ij} and hence in $\bar{h}_{00}$ and $\bar{h}_{ij}$.

(iii) The choice of G. Here we note that the choice of the expansion parameter G in Eq. (2.7) is of no significance for our definition of multipole moments. G can have any numerical value and the metric generated by algorithm A will still satisfy the Einstein equation. Besides the requirement that to G^0 order the metric should be Minkowskian, we are free to choose G to be any small parameter arising in the specific problem we are dealing with. In most cases a convenient choice for our multipole study is to choose all multipole terms to be of order G. This makes all higher-order terms in G come only from the nonlinear coupling of the multipoles (coupling terms). We will make this choice throughout the rest of this paper unless we specify otherwise.

(iv) The reading out of moments from a given metric. From the general form given by Eqs. (2.27)–(2.29) we can read out the multipole moments for a given metric without first going through the generation process. Assume that a suitable metric (stationary, vacuum, admitting "post-Minkowskian expansion") has been given in arbitrary coordinates. Pick a de Donder coordinate system, and transform the given metric to this system. (In general it is a very hard task to transform a metric into a de Donder coordinate system exactly. However, in most cases we need only the first few moments and do not need an exact transformation. We will give an example of this in the following paper.[18]) In general the $\bar{h}_{\mu\nu}$ thereby obtained will contain Laplacian-free and divergence-free terms in the "wrong" places. In this case, use the remaining gauge freedoms to get rid of the offending terms and bring the metric into the canonical de Donder form Eqs. (2.27)–(2.29); and from this metric read out the multipole moments. In the next subsection we will show that the multipole moments so obtained are unique (i.e., independent of the chosen de Donder coordinate system), up to Newtonian-type transformations among themselves induced by Euclidean-type translations and rotations of the coordinates.

C. The residual coordinate freedom

It is obvious that with the requirement that the $\bar{h}_{\mu\nu}$ takes the form (2.27)–(2.29), our coordinate system is much more restricted than simply being stationary and de Donder [Eq. (2.3)]. Indeed we can easily show that the coordinate freedom has been restricted to Euclidean motions, i.e., to the freedom of choosing the origin of the coordinates and the orientation of the axes.

Suppose we have two metric densities $\mathfrak{g}'^{\mu\nu}(x')=\eta^{\mu\nu}-\bar{h}'^{\mu\nu}(x')$ and $\mathfrak{g}^{\mu\nu}(x)=\eta^{\mu\nu}-\bar{h}^{\mu\nu}(x)$. Both $\bar{h}'^{\mu\nu}(x')$ and $\bar{h}^{\mu\nu}(x)$ are in the required form of expansions Eqs. (2.27)–(2.29). We choose, for convenience, the parameter G in such a way that all the multipole terms in $\tilde{g}^{\mu\nu}$ are linear in G and all nonlinear terms are coupling terms [see the discussion in point (iii) of Sec. II B].

Suppose the coordinates are related by

$$x'^{\mu}=x^{\mu}+\lambda f^{\mu}(x^i) . \tag{2.30}$$

[We only have to consider infinitesimal transformations, i.e., keep the calculation to λ^1 order and drop all terms with $\lambda^n (n\geq 2)$ since finite transformations can be built by e-folding infinitesimal ones. Since both t' and t are tied to the Killing vector, we have f^{μ} independent of t.] Next we expand $f^{\mu}(x^i)$ in G, and obtain

$$f^{\mu}=f_0^{\mu}+Gf_1^{\mu}+G^2f_2^{\mu}+\cdots . \tag{2.31}$$

The metric densities are related to each other by

$$[\eta^{\mu\nu}-\bar{h}'^{\mu\nu}(x'^i)]=\frac{1}{L}L^{\mu}{}_{\alpha}L^{\nu}{}_{\beta}[\eta^{\alpha\beta}-\bar{h}^{\alpha\beta}(x^i)] , \tag{2.32}$$

where $L^{\mu}{}_{\alpha}=(\partial x'^{\mu})/(\partial x^{\alpha})$ and $L=|\det(L^{\mu}{}_{\alpha})|$. From (2.30)–(2.32) we obtain, to G^0 order,

$$f_0{}^{\mu,\nu}+f_0{}^{\nu,\mu}-\eta^{\mu\nu}f_0{}^{k}{}_{,k}=0 ; \tag{2.33}$$

and, to G^1 order,

$$\begin{aligned}\bar{h}'^{\mu\nu}(x)=\bar{h}^{\mu\nu}(x)&+\lambda\bar{h}^{\alpha\beta}(f_0{}^{\mu}{}_{,\alpha}\delta^{\nu}{}_{\beta}+f_0{}^{\nu}{}_{,\beta}\delta^{\mu}{}_{\alpha}-\delta^{\mu}{}_{\alpha}\delta^{\nu}{}_{\beta}f_0{}^{k}{}_{,k})\\&+G\lambda(f_1{}^{\nu,\mu}+f_1{}^{\mu,\nu}-\eta^{\mu\nu}f_1{}^{k}{}_{,k}) ,\end{aligned} \tag{2.34}$$

and likewise to higher order in G. From Eq. (2.33) we immediately know that

$$\begin{aligned}&f_0{}^0=\text{const} ,\\&f_0{}^i=\text{Killing vector fields of Euclidean 3-space}\\&\quad=\epsilon^i{}_{jk}\omega^j x^k+d^i ,\end{aligned} \tag{2.35}$$

where ω^j and d^i are constant vectors. Next we look at the case of G^1 order. Having already studied and understanding the G^0-order freedom, we set $f_0{}^{\mu}=0$. Then (2.34) just represents a gauge transformation. However, theorem 1 tells us that our choice of the forms of $\gamma^{\mu\nu}$ and $\gamma'^{\mu\nu}$ leaves no gauge freedom and hence we have

$$f_1{}^{\mu,\nu}+f_1{}^{\nu,\mu}-\eta^{\mu\nu}f_1{}^{k}{}_{,k}=0\ , \tag{2.36}$$

which gives again the Euclidean motion as in (2.35). Using this argument repeatedly, we see that to arbitrary order in G, the freedom is no more than choosing the origin of the coordinates and the orientation of the axes. Hence we have the following important result.

If $\bar{h}_{\mu\nu}$ is in the form (2.27)–(2.29), the most general coordinate freedoms are Euclidean motions (2.35).

We well know from Newtonian theory that under a Euclidean motion the multipole moments of a body mix among themselves (e.g., a displacement ξ_j couples to the mass $\mathscr{I}$ to produce a change in the mass dipole moment $\delta\mathscr{I}_j=\mathscr{I}\xi_j$). In an analogous manner the Euclidean motions described above will cause a mixing of our multipole moments among themselves. Aside from this mixing, our moments are uniquely determined for any given vacuum stationary region of spacetime D.

D. Relationship of the multipole moments to their sources

In Newtonian theory the multipole moments read off from Φ are related intimately to the internal structures of their sources. In general relativity integration over the source may not always be meaningful (e.g., for a black hole). In the case of our present analysis, our de Donder coordinates might not always be extendible into the interior of the source (even if there is no spacetime singularity), unless the material source is gravitating weakly enough. Therefore, we do not in general have something which corresponds to a Newtonian integral over the source. However, if gravity is weak enough that we can use linearized theory (approximation of order G^1), we have [by including the material stress-energy tensor in Eq. (2.4)]

$$\mathscr{I}_{A_l}=\int T^{\rm in}_{00}r^l N_{\langle A_l\rangle}d^3x\ , \tag{2.37}$$

$$\mathscr{S}_{jA_l}=\int \epsilon_{pq\langle j}X_{A_l\rangle}x_p(-T^{\rm in}_{0q})d^3x\ , \tag{2.38}$$

$$Q_{A_l}=\int T^{\rm ext}_{00}\frac{N_{\langle A_l\rangle}}{r^{l+1}}d^3x\ , \tag{2.39}$$

$$\mathscr{C}_{jA_l}=\int \epsilon_{pq\langle j}N_{A_l\rangle}n_p(-T^{\rm ext}_{0q})\frac{1}{r^{l+1}}d^3x\ , \tag{2.40}$$

where $T^{\rm in}_{\mu\nu}$ and $T^{\rm ext}_{\mu\nu}$ are the material stress-energy tensor for the interior body and the external universe, respectively. Notice that we have chosen the normalization factor in Eqs. (2.21) and (2.22) or Eqs. (2.27) and (2.28), so that Eqs. (2.37)–(2.40) have the "expected" form. The physical meaning of the multipoles is clear in these formulas.

III. THE LAWS OF FORCE AND TORQUE

A. Constraints on the multipole moments for a stationary spacetime

We begin by asking the question, if we specify a set of moments, does it always generate a stationary spacetime? There are two problems that may arise. The first problem is that the expansion of $\bar{h}^{\mu\nu}$ generated by algorithm A may not converge. In general relativity, this problem is much more serious than in the corresponding Newtonian expansion due to the nonlinear coupling. We will not try to solve the question of what the requirement is on the multipole moments such that the algorithm gives a convergent series but will merely restrict attention to sets of multipole moments which do so.

The second problem is also well known. Given a set of moments, it is straightforward to generate a solution to Eqs. (2.2). However, this solution may or may not satisfy the time-independent gauge condition [Eq. (2.3) plus Eq. (2.10)], so that it may or may not be a solution of the stationary Einstein equations. We will now examine this problem.

We look at step (iii) of the algorithm for the generation of $\bar{h}^{\mu\nu}$. The question is what are the constraints, if any, on the multipole moments such that we can find a homogeneous solution $U_{\mu\nu}$ to Eq. (2.25), thereby making $\bar{h}_{\mu\nu}$ satisfy the gauge condition? Suppose we have generated $\bar{h}_{\mu\nu}$ to order $p-1$ and are now trying to carry the algorithm to order p. Since $\bar{h}_{\mu\nu}$ is to all orders explicitly time independent, we have to find U^p_{0j} and U^p_{ij} such that [Eq. (2.25)]

$$\partial_j U^p_{0j}+4\pi\partial_j\int_{\bar{D}}\frac{W^p_{0j}}{|x-x'|}d^3x'=0\ , \tag{3.1}$$

$$\partial_j U^p_{ij}+4\pi\partial_j\int_{\bar{D}}\frac{W^p_{ij}}{|x-x'|}d^3x'=0\ , \tag{3.2}$$

with

$$\nabla^2 U^p_{0j}\equiv U^p_{0j,kk}=0,\quad \nabla^2 U^p_{ij}\equiv U^p_{ij,kk}=0\ , \tag{3.3}$$

where $\bar{D}$ is a $t=$const hypersurface in D, the vacuum spacetime sandwiched between the internal and external sources. First we notice that the second terms of Eqs. (3.1) and (3.2) are ∇^2-free, i.e., they are scalar and vector harmonics, respectively, as guaranteed by the integrability condition [Eq. (2.6)]. Therefore they can always be expanded as in Eqs. (2.13) and (2.14). It is easy to show that all terms of the form (2.13) can be obtained from the divergence of the vector harmonic U_{0j} except for a term of the form (i) A/r. Likewise any term of the form (2.14) can be obtained from the divergence of the tensor harmonic U_{ij}, except for terms having the form (ii) C_i/r and (iii) $\epsilon_{ipq}\mathscr{B}_p(1/r)_{,q}$. Therefore, for possible failure of the construction of $\gamma^p_{\mu\nu}$ (the pth-order part of $\bar{h}_{\mu\nu}$), we have only to search for terms with these forms (i)–(iii) in the differentiated integrals of Eqs. (3.1) and (3.2).

Consider, first, dangerous A/r terms in the differential integral of (3.1), which can be written as

$$\partial_j \int_{\bar{D}} \frac{W^p_{0j}(x')}{|x-x'|} d^3x' = -\oint_{\partial\bar{D}} \frac{W^p_{0j}(x')}{|x-x'|} d^2_jx'$$
$$= -\oint_{\partial\bar{D}} W^p_{0j}(x') \left[\sum_{lm} \frac{4\pi}{2l+1} \frac{r^l_<}{r^{l+1}_>} \mathscr{Y}^{lm}_{J_l} \mathscr{Y}^{*lm}_{K_l} N_{J_l} N'_{K_l} \right] d^2_jx' \ . \tag{3.4}$$

Here we have expanded $1/|x-x'|$ in terms of symmetric trace-free tensors. The $\mathscr{Y}^{lm}_{J_l}$ are defined in Thorne;[7] see also Appendix A. Immediately we notice an interesting feature: If there is only an external universe and no internal body, then $r_>$ is always r' and no $1/r$ term can appear [i.e., there are no dangerous terms of types (i) above]. It is also easy to see that if there is no external universe (i.e., all external moments are zero), the integral is also zero. We hence look at the case where both a central body and an external universe exist. The coefficient of the $1/r$ term is given by the following integral over the "inner" surface $\partial_i\bar{D}$ of $\bar{D}$ (the intersection of a t=const surface and a 2-surface bounding the central body's world tube):

$$\int_{\partial_i\bar{D}} (-W^p_{0j}) d^2_jx' = \int_{\partial_i\bar{D}} (-W^p_{0j}) n'_j r'^2 d\Omega' \ . \tag{3.5}$$

Notice that despite the appearance of r'^2, the integral is independent of r', as guaranteed by $\partial_j W^p_{0j}=0$. Next we notice that the vector field W^p_{0j} can always be expanded as

$$W^p_{0j} = \sum_l [\mathscr{E}_{jA_{l-1}}(r) N_{A_{l-1}}]^T + n_j \mathscr{R}_{A_l}(r) N_{A_l} + [\epsilon_{jpq} n_p \mathscr{B}_{qA_{l-1}}(r) N_{A_{l-1}}] \ , \tag{3.6}$$

where $\mathscr{E}_{A_l}$, $\mathscr{B}_{A_l}$, and $\mathscr{R}_{A_l}$ are STF tensors depending only on the radius r. T indicates taking the transverse part [cf. Ref. 7, Eq. (2.25b)]. On the other hand, from the time-reversal symmetry considerations of Sec. II B, we know that W_{0j} has exactly one ϵ_{abc} in each of its terms. Therefore in Eq. (3.6) only the last terms inside square brackets are nonzero. Inserting this result into Eq. (3.5) gives zero. Hence we have no constraints arising from $\partial_j \gamma^p_{0j}=0$.

The search for dangerous C_i/r and $\epsilon_{ipq}\mathscr{B}_p(1/r)_{,q}$ terms in (3.2) proceeds similarly. Again, if there is either no internal body or no external universe, there is no constraint. Otherwise the constraint requires

$$\int_{\partial_i\bar{D}} (-W^p_{ij}) n'_j r'^2 d\Omega' = 0 \tag{3.7}$$

and

$$\int_{\partial_i\bar{D}} \epsilon_{iml} (-W^p_{ij}) n'_m n'_j r'^3 d\Omega' = 0 \ . \tag{3.8}$$

This time we can find no symmetry requirement to force the surface integrals to vanish. We shall evaluate these integrals in the next subsection in terms of the moments and show that in order for the spacetime to be stationary, the moments *are constrained.*

As Newtonian theory would suggest, additional constraints on the moments for a stationary spacetime are expected when both internal and external material are present. The question is, do these constraints have the correct physical origin? We give a positive answer to this question in the next subsection.

It is clear that once the stationary gauge conditions [(2.8) plus (2.10)] are satisfied there will be no further complication in the construction of the metric from the multipole moments, so we conclude this section with the following result.

Given any set of multipole moments, assuming that algorithm A generates a convergent series, the metric generated as a post-Minkowskian expansion (2.7) will satisfy the stationary vacuum Einstein equations (2.8), (2.9), and (2.10) to order p in the region D if and only if Eqs. (3.7) and (3.8) are satisfied up to that order.

B. Force and torque laws and related discussions

When the constraints (3.7) and (3.8) are not observed (i.e., the stationary gauge condition is violated) there will be a time-evolving momentum and angular momentum; and these give us the laws of motion and precession (force and torque laws) for the central body.

Let us consider the case where we are given a certain set of moments, $\mathscr{I}_{A_l}$, $\mathscr{Q}_{A_l}$, $\mathscr{S}_{A_l}$, and $\mathscr{C}_{A_l}$ each of order G. Let G be small, so that we will keep terms only up to G^2. To construct a metric satisfying the Einstein equations to order G^2, we proceed according to algorithm A. To order G^2, $W_{\mu\nu}$ is given by

$$-16\pi W^2_{00} = -\tfrac{3}{2}\gamma^1_{0j,k}\gamma^1_{0k,j} - \tfrac{1}{2}\gamma^1_{0j,k}\gamma^1_{0j,k} + \tfrac{7}{8}\gamma^1_{00,k}\gamma^1_{00,k} \ , \tag{3.9}$$

$$-16\pi W^2_{0i} = \gamma^1_{00,k}\gamma^1_{i0,k} - \gamma^1_{00,k}\gamma^1_{0k,i} \ , \tag{3.10}$$

$$-16\pi W^2_{ij} = -\tfrac{1}{4}(\gamma^1_{00,i}\gamma^1_{00,j} - \tfrac{1}{2}\delta_{ij}\gamma^1_{00,l}\gamma^1_{00,l}) + (\gamma^1_{0m,i}\gamma^1_{0m,j} - \tfrac{1}{2}\delta_{ij}\gamma^1_{0m,l}\gamma^1_{0m,l}) + (\gamma^1_{0i,k}\gamma^1_{0j,k} - \gamma^1_{0i,k}\gamma^1_{0k,j}) - (\gamma^1_{0j,k}\gamma^1_{0k,i} - \tfrac{1}{2}\delta_{ij}\gamma^1_{0l,k}\gamma^1_{0k,l}) \ . \tag{3.11}$$

After inserting $\gamma^1_{\mu\nu}$ of (2.21)–(2.23) into Eqs. (3.9)–(3.11), we can carry out the Poisson integration and determine the homogeneous term U_{ij} that makes $\gamma_{ij,j}$ zero, as described in the algorithm A. (In appendix B we carry out this process explicitly.) Then, as discussed before, when we come to terms of the form $\mathscr{C}_i/r$ and $\epsilon_{ipq}\mathscr{B}_p(1/r)_{,q}$, we are stuck. We have

$$\gamma^2_{ij,j} = -\frac{4}{r}\dot{\mathscr{P}}_i - 2\epsilon_{ijk}\dot{\mathscr{S}}_j\frac{n_k}{r^2} ,$$

with

$$\dot{\mathscr{P}}_i = \sum_{l=1}^{\infty}\left[\frac{(2l-1)!!}{(l-1)!}\mathscr{I}_{A_{l-1}}\mathscr{Q}_{iA_{l-1}} - \frac{4(2l-1)!!}{l(l-2)!}\mathscr{C}_{iB_{l-1}}\mathscr{S}_{B_{l-1}}\right] , \tag{3.12}$$

$$\dot{\mathscr{S}}_j = -\sum_{l=1}^{\infty}\left[\frac{(2l-1)!!}{(l-1)!}\epsilon_{iab}\mathscr{Q}_{aA_{l-1}}\mathscr{I}_{bA_{l-1}} - \frac{4l^2(2l-1)!!}{(l+1)!}\epsilon_{iab}\mathscr{C}_{aA_{l-1}}\mathscr{S}_{bA_{l-1}}\right] . \tag{3.13}$$

(The reason for this notation will be clear shortly.) No choice of time-independent homogeneous term can annul this. Therefore to satisfy the gauge conditions Eqs. (2.3) we are forced to include terms in γ_{00} and γ_{0j} which are explicitly dependent on time, and the resulting $\bar{h}_{\mu\nu}$ read, up to order G^2,

$$\bar{h}_{00} = 4\frac{\mathscr{I}}{r} + 4\left[\mathscr{I}_a + \dot{\mathscr{P}}_a\frac{t^2}{2}\right]\frac{n_q}{r^2} + \sum_{l=2}^{\infty}(-1)^l\frac{4}{l!}\mathscr{I}_{A_l}\left[\frac{1}{r}\right]_{,A_l} + \sum_{l=1}^{\infty}\frac{4(2l-1)!!}{l!}\mathscr{Q}_{A_l}X_{A_l} + \text{coupling terms} , \tag{3.14}$$

$$\bar{h}_{0j} = -2\epsilon_{jpq}(\mathscr{S}_p + \dot{\mathscr{S}}_p t)\frac{n_q}{r^2} - \frac{4}{r}\dot{\mathscr{P}}_i t - \sum_{l=2}^{\infty}(-1)^l\frac{4l}{(l+1)!}\epsilon_{jpq}\mathscr{S}_{pA_{l-1}}\left[\frac{1}{r}\right]_{,qA_{l-1}} - \sum_{l=1}^{\infty}\frac{4l(2l-1)!!}{(l+1)!}\epsilon_{jpq}\mathscr{C}_{pA_{l-1}}\hat{X}_{qA_{l-1}} + \text{coupling terms} , \tag{3.15}$$

$$\bar{h}_{ij} = \text{coupling terms} . \tag{3.16}$$

[The coupling terms are time-independent terms of order G^2, contributed by the coupling of moments through $W^2_{\mu\nu}$ as discussed in algorithm A. In addition, in $\bar{h}_{00}$ there is an extra coupling term $-2n_a\dot{P}_a$ so that $\bar{h}_{00}$ will still satisfy Eq. (2.2) after the inclusion of the t^2 term. All these coupling terms have a combination of N_{A_l} and r^m different from the explicitly given "multiple terms." Their general structure is shown in Sec. II B, and they are completely determined by the multipole terms. Thus, they carry no extra information and are not interesting in the present study.] The $\bar{h}_{\mu\nu}$ of Eqs. (3.14)–(3.16) gives us a metric satisfying the Einstein equations to order G^2. We note that this metric is accurate only for a finite duration of time; i.e., t can be at most so large that $\dot{\mathscr{P}}_a t^2$ or $\dot{\mathscr{S}}_p t$ become of order G^1; otherwise the higher-order iterations can no longer be considered small. From Eqs. (3.14) and (3.15), we clearly would identify the multipole moments of the internal body at time t to be

$$\text{mass dipole} = \mathscr{I}_a + \dot{\mathscr{P}}_a\frac{t^2}{2} ,$$

$$\text{current dipole} = \mathscr{S}_p + \dot{\mathscr{S}}_p t ,$$

where $\mathscr{I}_a$ and $\mathscr{S}_p$ are the "given" values of the moments at time $t=0$. In other words, since

momentum = (rate of change of mass dipole moment) ,

we have

(rate of change of momentum of the internal body)

$$= \frac{d^2}{dt^2}(\text{dipole moment})_i$$

$$= \dot{\mathscr{P}}_i[\text{given by Eq. (3.12)}] ,$$

and

(rate of change of current moment)$_i$

$$= \dot{\mathscr{S}}_i[\text{given by Eq. (3.13)}] .$$

This is why the symbols $\dot{\mathscr{P}}_i$ and $\dot{\mathscr{S}}_i$, with the dot denoting the time derivative, are used. (For some relevant discussions, see Sec. 8 of Ref. 7.)

Some comments on the laws of motion and precession as given by Eqs. (3.12) and (3.13) are in order now. Although the calculation of the G^2-order terms does not require the assumption of a weak field, Eqs. (3.12) and (3.13) are good approximations to the laws of force and torque only when the contributions of G^3- (and higher-) order terms are negligible. That is, we require that there exist a weak-field region (buffer zone, cf. Ref. 12) in the spacetime under consideration, with typical radius r so that the G^1-order quantities $\mathscr{I}_{A_l}/r^{l+1}$, $\mathscr{Q}_{A_l}r^l$, $\mathscr{S}_{A_l}/r^{l+1}$, and $\mathscr{C}_{A_l}r^l$ are small, and we imagine our calculation to be carried out there. Notice also that we have placed no constraints on the central body; i.e., it can have a strong field, or even be a black hole. As long as it is isolated enough, the force and torque laws are given accurately by Eqs.

(3.12) and (3.13). Notice that this is exactly the same situation as is treated by Thorne and Hartle[12] and Zhang.[13] Thorne and Hartle[12] have considered only the case $\mathscr{I}_\alpha=\mathscr{Q}_\alpha=\mathscr{C}_\alpha=0$ (i.e., mass-centered and inertial coordinates). They derive the leading term ($l=2$) in Eqs. (3.12) and (3.13). Zhang derives the next corrections ($l=3$), as well as terms that entail time derivatives of the multipole moments and thus vanish for our quasistationary situation. If we denote the time scale of variation of the moments by T, in our analysis we have thrown away contributions to the force and torque laws which are of order $(1/T)G$. [If the time rate of change of the multipole moments results solely from the gravitational interaction, $1/T$ is at most of order G^2 and the contribution to the Zhang's time-derivative laws[13] to $\dot{\mathscr{P}}_i$ and $\dot{\mathscr{S}}_i$ will be at most of order G^3 which is beyond the accuracy of (3.12) and (3.13).]

Equations (3.12) and (3.13) determine the force and torque to first order in the moment-moment coupling for an arbitrary central body in an arbitrary external gravitational field; arbitrary in the sense that both the central body and the external gravitational field can have arbitrary multipole moments. With our present formulation, it is straightforward, though tedious, to carry the calculation to higher order in G (but zero order in $1/T$).

It has been argued by Thorne and Hartle that the force and torque laws for strongly relativistic bodies, in terms of multipole moments, should be the same as for a nearly Newtonian body with negligible self-gravity (cf. Ref. 12, Sec. I C). Indeed, when $\mathscr{S}_{A_l}=0$ and $\mathscr{C}_{A_l}=0$, Eqs. (3.12) and (3.13) reduce exactly to the formulas we would obtain, from Newtonian theory,

$$\dot{\mathscr{P}}_i=-\int \rho\nabla_i\Phi d^3x\ , \tag{3.17}$$

$$\dot{\mathscr{S}}_i=-\int \epsilon_{ijk}x_j\rho\nabla_k\Phi d^3x\ , \tag{3.18}$$

where Φ is the external universe's Newtonian potential ($g_{00}=-1-2\Phi+\cdots$).

The results of Thorne and Hartle[12] and Zhang[13] are expressed not in terms of the external multipole moments $\mathscr{Q}_{A_l}$ and $\mathscr{C}_{A_l}$, but in terms of the curvature produced by the "external universe," which they define in terms of an asymptotic expansion (cf. Ref. 12). In their way of separating out an external universe, there are uncertainties in the definitions of the mass, momentum, and angular momentum for the central body, which become precise only in the limit of vanishing external universe. In the present analysis, all the moments, including the mass as the monopole moment, the momentum as the time derivative of the mass dipole and the angular momentum as the current dipole, are uniquely and unambiguously defined. Of course, one can always question whether this specific choice of definition is desirable. To this end the formulas (3.12) and (3.13) which agree exactly with the Newtonian expressions again support a positive answer.

With the present definition of internal and external moments, we can define exactly an "internal spacetime" and an "external spacetime" corresponding to a given physical spacetime with a given set of internal and external moments. Suppose that from the stationary vacuum metric of a given physical spacetime we have read out the moments (Sec. II B). We then pick out the external moments and use algorithm A to construct from them a stationary metric. This we call the "external spacetime" or "external universe." Likewise we define the "internal universe" corresponding to the physical spacetime; and we can then use our formalism to discuss in an exact fashion the gravitational interactions between the internal and external spacetimes.

With this definition of internal and external spacetime, we can write down the force and torque laws to order G^2 (i.e., to the leading order in moment-moment coupling) in a geometrical form in a way analogous to Eq. (1.11) of Ref. 12. We refer to the coordinate system where $\bar{h}_{\mu\nu}$ takes the form (3.14)–(3.16) as the "instantaneous rest frame" of the central body at $t=0$. We separate out from the exact spacetime metric at $t=0$ a metric of the central body (built with $\mathscr{I}_{A_l}$ and $\mathscr{S}_{A_l}$) and a metric of the external universe (built with $\mathscr{Q}_{A_l}$ and $\mathscr{C}_{A_l}$). The force and torque laws will be written down in terms of a set of 4-vectors and 4-tensors, living at the origin of the external universe and defined as follows. (i) The 4-velocity of the central body is defined as the unit vector $\mathbf{U}$ in the direction of $\partial/\partial t$. (ii) $\mathscr{P}^\mu=\mathscr{I}U^\mu$ is the 4-momentum of the central body, $\mathscr{I}$ being the body's mass, i.e., its internal mass monopole moment. (iii) $\mathscr{I}_{\Omega_l}$, $\mathscr{S}_{\Omega_l}$, $\mathscr{Q}_{\Omega_l}$, and $\mathscr{C}_{\Omega_l}$ are 4-tensors orthogonal to $\partial/\partial t$ and with nonzero components in the body's instantaneous rest frame given by $\mathscr{I}_{A_l},\mathscr{S}_{A_l},\mathscr{Q}_{A_l},\mathscr{C}_{A_l}$ where $a_i=1,2,3;\omega_l=0,1,2,3$. Then to order G^2, we have

$$\mathscr{P}_{\alpha;\beta}U^\beta=\sum_{l=1}^{\infty}\left[\frac{(2l-1)!!}{(l-1)!}\mathscr{Q}_{\alpha\Omega_{l-1}}\mathscr{I}^{\Omega_{l-1}}-\frac{4(2l-1)!!}{l(l-2)!}\mathscr{C}_{\alpha\Omega_{l-1}}\mathscr{S}^{\Omega_{l-1}}\right], \tag{3.19}$$

$$\mathscr{S}_{\alpha;\beta}U^\beta=-\sum_{l=1}^{\infty}\left[\frac{(2l-1)!!}{(l-1)!}\epsilon_{\mu\alpha\beta\gamma}\mathscr{Q}^{\beta\Omega_{l-1}}\mathscr{I}_{\Omega_{l-1}}-\frac{4l^2(2l-1)!!}{(l+1)!}\epsilon_{\mu\alpha\beta\gamma}\mathscr{C}^{\beta\Omega_{l-1}}\mathscr{S}^{\gamma}{}_{\Omega_{l-1}}\right], \tag{3.20}$$

where $\epsilon_{\mu\alpha\beta\gamma}$ is the Levi-Civita tensor and semicolons denote covariant derivatives, in the external universe.

To be able to integrate Eqs. (3.19) and (3.20), we have to provide information on how the moments change (except the monopole and dipole moments). This requires the specification of the equation of state of the material making up the body and the external universe, as in general they are distorting each other and changing each other's multipole moments through gravitational interaction. (For more discussion of this point see Ref. 12.)

The present formulation suggests a way to define a rigid body in general relativity. If the body evolving forward in time in a quasistationary external universe changes only its mass dipole moment and current dipole moment, and all the other moments have values that can be related to those at $t=0$ by a rotation and translation, clearly we would like to say that the body is rigid. That is, a rigid body does not develop induced multipole moments. Note that the force and torque laws for such a rigid body can be obtained to arbitrary accuracy by the quasistationary calculation carried to higher order in G. It would be interesting to study how this notion of rigid body relates to the usual definition of constant proper distance between adjacent matter elements.

Although the present derivations of the force and torque laws are presented in terms of the secular changes in γ_{00} and γ_{0j} which are forced into existence by the gauge condition, this actually amounts to a calculation of the integrals

$$\dot{\mathscr{P}}_i=-\oint_{\partial_i\bar{D}} W_{ij} d_j^2x \ , \tag{3.21}$$

$$\dot{\mathscr{S}}_i=-\oint_{\partial_i\bar{D}} \epsilon_{ijk}x_j W_{kl} d_l{}^2x \ , \tag{3.22}$$

as discussed in Sec. III A. Since we have identified $\dot{\mathscr{P}}_i$ and $\dot{\mathscr{S}}_i$ as the change in momentum and angular momentum of the central body, W_{ij} has the physical meaning of a stress 3-tensor. By $\partial_\mu W^{\mu\nu}=0, W^{0i}$ is the energy flux and W^{00} is the energy density of the gravitational field. Indeed, repeating the same line of argument as that which leads to (3.21) and (3.22), we arrive at

$$\dot{M}=-\oint_{\partial_i\bar{D}} W^{0j} d_j{}^2x \ , \tag{3.23}$$

where $\dot{M}$ is the time rate of change of mass $M=\mathscr{I}$ of the central body. In our quasistationary approximation, (3.6) gives $\dot{M}=0$. Therefore, our identification of multipole moments has led us also to the identification of $W^{\mu\nu}$ as given by Eq. (2.4) as the gravitational stress-energy tensor in our special de Donder coordinate system (de Donder coordinate condition plus certain choice for fixing the residual gauge freedom). Note that this $W^{\mu\nu}$ differs from the Landau-Lifshitz pseudotensor by two additional terms.

To summarize, our present treatment of a stationary or quasistationary spacetime produces a very Newtonian-type picture: the gravitational field is characterized by a scalar potential $\bar{h}_{00}$ and a vector potential $\bar{h}_{0j}$, determined by the mass moments and current moments, respectively, evolving in a flat background with a nonlinear interaction between them; the gravitational interaction between gravitating bodies can be described in terms of the coupling of the multipole moments of $\bar{h}_{00}$ and $\bar{h}_{0j}$; and associated with this interaction there is a stress-energy tensor constructed from the gravitational field at quadratic order and higher.

Of course, we would not expect this picture to be useful in a highly dynamical situation, where no timelike Killing vector or nearly timelike Killing vector exists.

IV. DISCUSSION AND CONCLUSION

In this section we will first summarize the results of the preceding sections and then discuss some of the remaining issues. We have studied the structures of stationary vacuum spacetimes, without assuming asymptotic flatness, in terms of de Donder coordinate expansions in a way that can be regarded as a natural extension of Thorne's formalism.[7] We have succeeded in identifying some parts of the metric that carry all the information about the vacuum spacetime, namely, the multipole terms. Out of these we can read the multipole moments characterizing the spacetime. There are four sets of moments: internal mass multipoles $\mathscr{I}_{A_l}$, internal current multipoles $\mathscr{S}_{A_l}$, external mass multipoles $\mathscr{Q}_{A_l}$, and external current multipoles $\mathscr{C}_{A_l}$ characterizing, respectively, the central body and the external universe. In particular, the mass, the momentum, and the angular momentum of a body in an external universe are defined precisely in terms of the internal monopole and dipole moments. We have constructed an algorithm so that all other parts of the metric can be determined in terms of the multipole moments (alogorithm A). We have given explicit examples of this construction for the first few lowest moments (Appendix B). We have discussed the general structure of the metric obtained from the algorithm and we have given a prescription to read out the multipole moments for given stationary vacuum spacetimes. We have derived, using the vacuum Einstein equation alone, the force and torque laws in terms of the multipole moments in quasistationary situations. These laws generalize the results of Thorne and Hartle[12] and of Zhang[13] to arbitrary l poles. These laws are completely analogous to the Newtonian case even though the central body can be strongly gravitating. Related to these laws of motion and precession is an expression for the gravitational stress-energy tensor in our de Donder coordinate system.

We now turn to the remaining issues of the development. An important problem is to establish a criterion for the convergence of the series in our algorithm for building the metric from the coupling of the multipole moments. This is a generic problem common to all studies using series expansions.[7,8] However, it is also intuitively clear that in the buffer region (if it exists, cf. Sec. II) and for physically reasonable choices of the moments, the algorithm will generate convergent series. It is only of academic interest to prove it rigorously for the weak-field case.

Throughout the paper we have considered only metrics which are expandable in post-Minkowskian expansions in de Donder coordinate.[7,14] It is almost certain that there are stationary vacuum metrics which lie outside this class. It would be illuminating to find out explicitly what kind of solution is not expandable. However, it is again intuitively clear that in a weak-field buffer region, for which our formalism is intended, the linearized theory will produce the leading-order result and the metric can be obtained to arbitrary accuracy by iterating the linearized solution.

We have chosen some very specific coordinate conditions to study the geometric structure of the spacetime. How much of our study just reflects the choice of the coordinate conditions? How geometric are the multipole moments we have defined? This question will best be answered if we can find a coordinate-independent approach leading to the same set of moments. Indeed, when the spacetime is asymptotic flat, the external moments $(\mathcal{Q}_{A_l},\mathcal{C}_{A_l})$ vanish and the internal moments $(\mathcal{I}_{A_l},\mathcal{S}_{A_l})$ reduce to those of Thorne,[7] i.e., they are the same moments as defined by the Geroch-Hansen geometric approach. It would be desirable to have a study along the lines of Geroch and Hansen for spacetimes which are not asymptotic flat, i.e., spacetimes with both internal and external moments. Is it possible to invent some treatment that "folds up" the buffer zone to one point Λ analogous to the "point-at-infinity" Λ in Geroch's approach?

We have mentioned that in the Newtonian theory the three types of gravitational interactions, namely, the force and torque on and the distortion of a body in a gravitational field, can all be described in terms of multipole moments in an elegant way. We have shown in Sec. III that multipole moments in general relativity are useful in determining force and torque. The effects of distortion in terms of multipole moments will be studied in the following paper[18] using a model problem of a Schwarzschild black hole in an external gravitational field.

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APPENDIX A

In this appendix, we write down some useful formulas for symmetric trace-free tensors which are required to carry out algorithm A. As in other parts of this paper, we adopt the notation of Ref. 7. We will not repeat any formula which has already appeared in that article. Blanchet[15] also gives a collection of useful formulas for STF tensors.

1. Expansion formulas

One useful expansion formula is

$$\frac{1}{|x-x'|}=\sum_{lm}\frac{r_<^l}{r_>^{l+1}}\mathcal{Y}_{J_l}^{lm}\mathcal{Y}_{K_l}^{*\,lm}N_{J_l}N'_{K_l}\ . \tag{A1}$$

Here $J_l=j_1j_2\cdots j_l$ and $N_{J_l}=n_{j_1}n_{j_2}\cdots n_{j_l}$. Repeated indices are to be summed unless otherwise stated. The $\mathcal{Y}_{J_l}^{lm}$ with $-l\le m\le l$ form a basis for the $(2l+1)$-dimensional vector space of STF tensors; for their definition see Ref. 7, Sec. II C.

A useful formula for the contraction of STF tensors is

$$\sum_m \mathcal{Q}_{A_l}\mathcal{Y}_{A_l}^{lm}\mathcal{I}_{B_l}\mathcal{Y}_{B_l}^{*\,lm}=\frac{1}{4\pi}\frac{(2l+1)!!}{l!}\mathcal{Q}_{\langle A_l\rangle}\mathcal{I}_{\langle A_l\rangle}\ . \tag{A2}$$

The proof is trivial.

For breaking up the STF combination of N_{A_l}, we use

$$N_{\langle iA_l\rangle}=n_iN_{\langle A_l\rangle}-\frac{1}{2l+1}\sum_{s=1}^{l}\delta_{ia_s}N_{\langle a_1\cdots a_{s-1}a_{s+1}\cdots a_l\rangle}+\frac{2}{(2l-1)(2l+1)}\sum_{s<s'}\delta_{a_sa_{s'}}N_{\langle ia_1\cdots a_{s-1}a_{s+1}\cdots a_{s'-1}a_{s'+1}\cdots a_l\rangle}\ .$$

2. Differentiation formulas

The following formula is often used:

$$\partial_{A_l}\left[\frac{1}{r}\right]=(-1)^l(2l-1)!!\frac{\hat{N}_{A_l}}{r^{l+1}}\ .$$

3. Angular integration formulas

It can be shown that

$$\int N_{\langle K_l\rangle}N_{\langle A_m\rangle}N_{\langle B_n\rangle}d\Omega=4\pi I_0(K_l,A_m,B_n)\ , \tag{A3a}$$

$$I_0(K_l,A_m,B_n)\equiv\sum_s^{\min(m,n)}\frac{1}{(l+m+n+1)!!}C(m,s)C(n,s)s!l!\delta(A_m,A_{m-s}C_s)\delta(B_n,B_{m-s}C_s)\delta(K_l,A_{m-s}B_{m-s})\ , \tag{A3b}$$

where

$$C(m,s)=\frac{m!}{(m-s)!s!},\quad \delta(A_m,B_m)\equiv\delta_{a_1b_1}\delta_{a_2b_2}\cdots\delta_{a_mb_m}\ . \tag{A4}$$

Similarly,

$$\int N_{\langle K_l\rangle}N_{\langle A_m\rangle}N_{\langle B_n\rangle}n_j d\Omega=4\pi I_1(K_l,A_m,B_n,j)\ , \tag{A5a}$$

$$\begin{aligned}I_1(K_l,A_m,B_n,j)\equiv \sum_s^{\min(m,n)} \frac{1}{(l+m+n+2)!!}\, l!C(m,s)C(n,s)s! \\ \times[\delta(A_m,A_{m-s}C_s)\delta(B_n,B_{n-s},C_s)\delta(K_l,A_{m-s}B_{n-s}j) \\ +(m-s)\delta(A_m,jA_{m-s-1}C_s)\delta(B_n,B_{n-s}C_s)\delta(K_l,A_{m-s-1}B_{n-s}) \\ +(n-s)\delta(A_m,A_{n-s}C_s)\delta(B_n,jB_{n-s-1}C_s)\delta(K_l,A_{m-s}B_{n-s-1})]\ .\end{aligned} \tag{A5b}$$

Also,

$$\int N_{\langle K_l\rangle}N_{\langle A_m\rangle}N_{\langle B_n\rangle}n_i n_j d\Omega=4\pi I_2(K_l,A_m,B_n,i,j)\ , \tag{A6a}$$

$$\begin{aligned}I_2(K_l,A_m,B_n,i,j)= \sum_s^{\min(m,n)} \frac{1}{(l+m+n+3)!!}\, l!C(m,s)C(n,s)s! \\ \times\{\delta(A_m,A_{m-s}C_s)\delta(B_n,B_{n-s}C_s)[\delta(K_l,A_{m-s}B_{n-s}ij)+\delta(K_l,A_{m-s}B_{n-s})\delta_{ij}] \\ +(m-s)\delta(A_m,jA_{m-s-1}C_s)\delta(B_n,B_{n-s}C_s)\delta(K_l,iA_{m-1-s}B_{n-s})+(\text{exchange } i,j) \\ +(n-s)\delta(A_m,A_{m-s}C_s)\delta(B_n,jB_{n-s-1}C_s)\delta(K_l,iA_{m-s}B_{n-s-1})+(\text{exchange } i,j) \\ +(m-s)(m-s-1)\delta(A_m,ijA_{m-s-2}C_s)\delta(B_n,B_{n-s}C_s)\delta(K_l,A_{m-s-2}B_{n-s}) \\ +(n-s)(n-s-1)\delta(A_m,A_{m-s}C_s)\delta(B_n,ijB_{n-s-2}C_s)\delta(K_l,A_mB_{n-s-2}) \\ +(m-s)(n-s)\delta(K_l,A_{m-s-1}B_{n-s-1})[\delta(A_m,iA_{m-s-1}C_s)\delta(B_n,jB_{n-s-1}C_s) \\ +\delta(A_m,jA_{m-s-1}C_s)\delta(B_n,iB_{n-s-1}C_s)]\}\ .\end{aligned} \tag{A6b}$$

These complicated formulas can be easily understood: In $\int d\Omega$, the integral is zero unless all n_a are contracted. In Eq. (A3), this means that the largest l is given by $m+n$, so that $N_{\langle K_l\rangle}$ can be contracted with $N_{\langle A_m\rangle}N_{\langle B_n\rangle}$. Smaller l are possible: some a_i can be contracted with b_j. Hence we have the sum over s. In Eqs. (A4) and (A5) more terms arise since the indices i and j can be contracted freely with K_l,A_m,B_n and among themselves.

4. Solution of Poisson's equation

The three formulas (A3)–(A5) take care of all the angular integrations that may ever be needed to carry out the algorithm for calculations up to G^2 order. Indeed the calculation of $\Delta^{-1}W^2_{\mu\nu}$ is straightforward with these formulas. Here by $\Delta^{-1}W_{\mu\nu}$ we mean a special solution to the Poisson equation

$$\nabla^2\bar{h}_{\mu\nu}=W_{\mu\nu}\ . \tag{A7}$$

Solving this equation is the most involved part of the algorithm.

The Poisson equation appears in the algorithm in the following forms, and only in these forms to G^2 order. (For higher-order calculations, the reductions to these forms are sometimes tedious.) The specific solutions given are precisely the Poisson integral, except in the cases of $\ln r$ terms where we have chosen simpler expressions:

$$\Delta^{-1}(Q_{\langle A_m\rangle}I_{\langle B_n\rangle}N_{A_m}N_{B_n}r^p)=-\sum_{l=0}^{\infty} I_r(l,p)I_0(K_l,A_m,B_n)N_{\langle K_l\rangle}Q_{\langle A_m\rangle}I_{\langle B_n\rangle}\ , \tag{A8}$$

$$\Delta^{-1}(Q_{\langle A_m\rangle}I_{\langle B_n\rangle}N_{A_m}N_{B_n}n_jr^p)=-\sum_{l=0}^{\infty}I_r(l,p)I_1(K_l,A_m,B_n,j)N_{\langle K_l\rangle}Q_{\langle A_m\rangle}I_{\langle B_m\rangle}\ , \quad \text{(A9)}$$

$$\Delta^{-1}(Q_{\langle A_m\rangle}I_{\langle B_n\rangle}N_{A_m}N_{B_n}i,jr^p)=-\sum_{l=0}^{\infty}I_r(l,p)I_2(K_l,A_m,B_n,i,j)N_{\langle K_l\rangle}Q_{\langle A_m\rangle}I_{\langle B_n\rangle}\ . \quad \text{(A10)}$$

Here Q_{A_m} and I_{B_n} are arbitrary constant tensors, and

$$I_r(l,p)\equiv\frac{(2l-1)!!}{l}r^{2+p}\times\begin{cases}\ln r & \text{if } l+p+3=0\ ,\\ -\ln r & \text{if } 2+p-l=0\ ,\\ \left[\dfrac{1}{l+3+p}-\dfrac{1}{2+p-l}\right] & \text{otherwise}\ .\end{cases} \quad \text{(A11)}$$

With the foregoing formulas, each step of the algorithm is straightforward, though sometimes tedious.

APPENDIX B

In this appendix we give $\bar{h}_{\mu\nu}$ to first order in the coupling of multipoles, for the lowest few multipoles. With the formulas in Appendix A and $W_{\mu\nu}$ as given in Sec. III B [Eqs. (3.9)–(3.11)], the calculation is straightforward. It is also clear that to first order in the coupling we can separate the discussion into two moments at one time. Since only the coupling of external moments with internal moments gives rise to interesting results, we will not list the terms that arise from internal-internal or external-external coupling. Some expressions for internal-internal coupling can be found in Refs. 9 and 15.

The requirement that $\bar{h}_{\mu\nu}$ take up the forms (2.27)–(2.29) has greatly restricted the coordinate freedom. After these restrictions, we have left only the freedom of choosing the origin of the coordinates and the orientation of the axis (i.e., a Euclidean motion, see Sec. II C). We could have used this freedom to make our coordinates be mass centered, i.e., $\mathcal{I}_i=0$. However this results in no substantial simplification in our treatment. In fact $\mathcal{I}_i$ behave just like other multipoles but with a simpler structure. Hence it serves as a good example for studying the general behavior of multipole moments. This point will be clarified by the following examples.

(a) For $\mathcal{Q}_a$ with $\mathcal{I}_i$,

$$\gamma^1_{00}=4\mathcal{I}\frac{1}{r}+4\mathcal{Q}_an_ar,\ \gamma^1_{0j}=0,\ \gamma^1_{ij}=0\ , \quad \text{(B1a)}$$

$$\gamma^2_{00}=14\mathcal{Q}_a\mathcal{I}n_a+2\mathcal{I}\mathcal{Q}_a\frac{n_a}{r^2}t^2+\{-2\mathcal{I}\mathcal{Q}_an_a\}\ , \quad \text{(B1b)}$$

$$\gamma^2_{0i}=-4\mathcal{I}\mathcal{Q}_i\frac{t}{r},\ \gamma^2_{ij}=2\delta_{ij}Q_a\mathcal{I}n_a-4\mathcal{I}\mathcal{Q}_{(i}n_{j)}\ . \quad \text{(B1c)}$$

The terms with t are forced into $\bar{h}_{\mu\nu}$ by the gauge condition (see the discussion in Sec. III). From these terms we can read out the force on the central body, which arises from the failure of our coordinates to be locally inertial with respect to the external universe ($\mathcal{Q}_i\neq0$). The term in { } in γ^2_{00} is forced into existence by Eq. (2.9) and the presence of the t^2 term. In later expressions, terms in { } have the same origin.

(b) For $\mathcal{Q}_a$ with $\mathcal{I}_b$,

$$\gamma^1_{00}=4\mathcal{I}_b\frac{n_b}{r^2}+4\mathcal{Q}_an_ar,\ \gamma^1_{0j}=0,\ \gamma^1_{ij}=0\ , \quad \text{(B2a)}$$

$$\gamma^2_{00}=14\mathcal{Q}_a\mathcal{I}_b\frac{n_an_b}{r}\ , \quad \text{(B2b)}$$

$$\gamma^2_{0i}=-2\frac{t}{r^2}(\mathcal{Q}_i\mathcal{I}_b-\mathcal{Q}_b\mathcal{I}_i)n_b$$
$$=-2\epsilon_{ipq}n_p\frac{t}{r^2}(\epsilon_{qab}\mathcal{Q}_a\mathcal{I}_b)\ , \quad \text{(B2c)}$$

$$\gamma^2_{ij}=2\delta_{ij}\mathcal{I}_a\mathcal{Q}_b\frac{n_an_b}{r}-4\mathcal{I}_a\frac{\mathcal{Q}_{(i}n_{j)}n_a}{r}+\left[-2\delta_{ij}\frac{\mathcal{I}_a\mathcal{Q}_a}{r}\right]\ . \quad \text{(B2d)}$$

From the γ_{0i} term we can read out the torque on the central body, i.e., the increase of the body's orbital angular momentum with respect to the coordinate system, which results from the acceleration of our coordinates with respect to the external universe ($\mathcal{Q}_i\neq0$) together with the failure of our coordinates to be mass centered in the body ($\mathcal{I}_i\neq0$). The term in [ ] is a homogeneous term, i.e., ∇^2 free, which is forced into existence by the gauge requirement [step (iii) of algorithm A]. In later expressions, terms in [ ] have the same origin.

(c) For $\mathcal{Q}_a$ with $\mathcal{I}_{ab}$,

$$\gamma^1_{00}=6\mathcal{I}_{ab}\frac{n_an_b}{r^3}+4\mathcal{Q}_an_ar,\ \gamma^1_{0j}=0,\ \gamma^1_{ij}=0\ , \quad \text{(B3a)}$$

$$\gamma^2_{00}=21\mathcal{Q}_a\mathcal{I}_{bc}\frac{n_an_bn_c}{r^2},\ \gamma^2_{0i}=0\ , \quad \text{(B3b)}$$

$$\gamma^2_{ij}=3\delta_{ij}\mathcal{Q}_a\mathcal{I}_{bc}\frac{n_an_bn_c}{r^2}-6\mathcal{I}_{bc}\mathcal{Q}_{(i}n_{j)}\frac{n_bn_c}{r^2}+\left[-2\delta_{ij}\mathcal{Q}_a\mathcal{I}_{ab}\frac{n_b}{r^2}+2\mathcal{Q}_b\mathcal{I}_{ij}\frac{n_b}{r^2}\right]\ . \quad \text{(B3c)}$$

In this case we have no time-dependent term. Indeed, only when the external moment has the same number or one more number of indices than the internal moment, do we obtain secular evolution.

(d) For $\mathcal{Q}_{ab}$ with $\mathcal{I}$,

$$\gamma^1_{00}=4\mathcal{I}\frac{1}{r}+6\mathcal{Q}_{ab}n_an_br^2,\ \gamma^1_{0j}=0,\ \gamma^1_{ij}=0\ , \quad \text{(B4a)}$$

$$\gamma^2_{00}=21\mathcal{Q}_{ab}\mathcal{I}n_an_br,\ \gamma^2_{0i}=0\ , \tag{B4b}$$
$$\gamma^2_{ij}=3\delta_{ij}\mathcal{I}\mathcal{Q}_{ab}n_an_br-6\mathcal{I}\mathcal{Q}_{a(i}n_{j)}n_ar+6\mathcal{I}\mathcal{Q}_{ij}r\ .$$

(e) For $\mathcal{Q}_{ab}$ with $\mathcal{I}_c$,

$$\gamma^1_{00}=4\mathcal{I}_a\frac{n_a}{r^2}+6\mathcal{Q}_{ab}n_an_br^2,\ \gamma^1_{0j}=0,\ \gamma^1_{ij}=0\ , \tag{B5a}$$

$$\gamma^2_{00}=21\mathcal{Q}_{ab}\mathcal{I}_cn_an_bn_c+6\mathcal{Q}_{ba}\mathcal{I}_a\frac{n_bt^2}{r^2}+\{-6\mathcal{Q}_{ba}\mathcal{I}_an_b\}\ , \tag{B5b}$$

$$\gamma^2_{0i}=-12\mathcal{Q}_{ia}\mathcal{I}_a\frac{t}{r}\ , \tag{B5c}$$

$$\gamma^2_{ij}=3\delta_{ij}\mathcal{Q}_{ab}\mathcal{I}_cn_an_bn_c-6\mathcal{I}_b\mathcal{Q}_{a(i}n_{j)}n_an_b-6\mathcal{Q}_{ij}\mathcal{I}_an_a-6I_a\mathcal{Q}_{a(i}n_{j)}+6\mathcal{Q}_{a(j}\mathcal{I}_{i)}n_a\ . \tag{B5d}$$

(f) For Q_{ab} with $\mathcal{I}_{cd}$,

$$\gamma^1_{00}=6\mathcal{I}_{ab}\frac{n_an_b}{r^3}+6\mathcal{Q}_{ab}n_an_br^2,\ \gamma^1_{0i}=0,\ \gamma^1_{0j}=0\ , \tag{B6a}$$

$$\gamma^2_{00}=\frac{63}{2}\mathcal{Q}_{ab}\mathcal{I}_{cd}\frac{n_an_bn_cn_d}{r}\ , \tag{B6b}$$

$$\gamma^2_{0i}=-6\epsilon_{ipq}\frac{n_pt}{r^2}(\epsilon_{qab}\mathcal{Q}_{ac}\mathcal{I}_{cb})\ , \tag{B6c}$$

$$\gamma^2_{ij}=-9\mathcal{I}_{bc}\mathcal{Q}_{a(i}n_{j)}\frac{n_an_bn_c}{r}+\frac{9}{2}\delta_{ij}\mathcal{Q}_{ab}\mathcal{I}_{cd}n_an_bn_cn_d\frac{1}{r}-6\mathcal{I}_{ab}\mathcal{Q}_{a(i}n_{j)}\frac{n_b}{r}-3\mathcal{Q}_{ij}\mathcal{I}_{ab}\frac{n_an_b}{r}+6\mathcal{Q}_{a(i}\mathcal{I}_{j)b}\times\frac{n_an_b}{r}+\left[-3\delta_{ij}\mathcal{Q}_{ab}\mathcal{I}_{ab}\frac{1}{r}-6\mathcal{Q}_{a(i}\mathcal{I}_{j)a}\frac{1}{r}\right]\ . \tag{B6d}$$

The time-dependent term in γ^2_{0i} is due to the torque produced by coupling the body's quadrupole moment, $\mathcal{I}_{ab}$, to the quadrupole mass moment (electric part of Riemann curvature) of the external universe, $\mathcal{Q}_{ac}$.

(g) For $(\mathcal{I},\mathcal{S}_i)$ with $\mathcal{C}_i$,

$$\gamma^1_{00}=4\mathcal{I}\frac{1}{r}\ ,$$
$$\gamma^1_{0i}=-2\epsilon_{ipq}\mathcal{S}_p\frac{n_q}{r^2}-2\epsilon_{ipq}\mathcal{C}_pn_qr\ , \tag{B7a}$$
$$\gamma^1_{ij}=0\ .$$

We again will write down only those terms which arise from the coupling of internal and external moments ($\mathcal{I}$ with $\mathcal{C}_i$, $\mathcal{S}_i$ with $\mathcal{C}_j$):

$$\gamma^2_{00}=-4\mathcal{C}_a\mathcal{S}_b\frac{n_{\langle a}n_{b\rangle}}{r}\ , \tag{B7b}$$

$$\gamma^2_{0i}=8\epsilon_{ikq}\mathcal{C}_q\mathcal{I}n_k+4\epsilon_{iap}\frac{n_at}{r^2}(\epsilon_{pmn}\mathcal{C}_m\mathcal{S}_n)\ , \tag{B7c}$$

$$\gamma^2_{ij}=8\mathcal{S}_n\mathcal{C}_{(i}n_{j)}\frac{n_a}{r}-4\delta_{ij}\mathcal{S}_a\mathcal{C}_b\frac{n_an_b}{r}+4\delta_{ij}\mathcal{C}_a\mathcal{S}_a\frac{1}{r}\ . \tag{B7d}$$

The time-dependent term arises because our coordinates are rotating relative to the local inertial frames of the external universe with angular velocity $\mathcal{C}_m$, and the body's angular momentum $\mathcal{S}_n$ refuses to rotate with them (it insists on remaining inertial).

(h) For $(\mathcal{I},\mathcal{S}_i)$ with $\mathcal{C}_{ab}$,

$$\gamma^1_{00}=4\frac{\mathcal{I}}{r}\ ,$$
$$\gamma^1_{0i}=-2\epsilon_{ipq}\mathcal{S}_p\frac{n_q}{r^2}-4\epsilon_{ipq}\mathcal{C}_{pa}n_{\langle a}n_{q\rangle}r^2\ , \tag{B8a}$$
$$\gamma^1_{ij}=0\ ,$$

$$\gamma^2_{00}=-16\mathcal{C}_{ab}\mathcal{S}_an_b-2\mathcal{C}_{ab}\mathcal{S}_cn_an_bn_c-12n_b\frac{t^2}{r^2}\mathcal{C}_{ab}\mathcal{S}_a+\{12n_b\mathcal{C}_{ab}\mathcal{S}_a\}\ , \tag{B8b}$$

$$\gamma^2_{0i}=24\epsilon_{ipq}\mathcal{C}_{aq}\mathcal{I}n_an_pr+24\mathcal{C}_{aj}\mathcal{S}_a\frac{t}{r}\ , \tag{B8c}$$

$$\gamma^2_{ij}=12(\mathcal{S}_a\mathcal{C}_{a(j}n_{i)}-\mathcal{C}_{a(i}\mathcal{S}_{j)}n_a+\mathcal{C}_{ij}\mathcal{S}_an_a)+6(2\mathcal{S}_b\mathcal{C}_{a(i}n_{j)}n_an_b-\delta_{ij}\mathcal{C}_{ab}\mathcal{S}_cn_an_bn_c)\ . \tag{B8d}$$

The time-dependent terms are due to the force on the body caused by coupling of its spin angular momentum $\mathcal{S}_a$ to the external universe's curvature.

With these examples, it is clear that the algorithm A can be used easily to construct model spacetimes. The metric in the weak-field region can be written down in a straightforward manner once the multipoles of the chosen spacetime have been specified. For example, for a Kerr black hole in an external universe (say, a quadrupole external gravitational field), the metric can easily be written down showing explicitly the precession of the angular momentum of the hole.[12] This is the subject of an accompanying paper.

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*Present address: Physics Department, University of Florida, Gainesville, FL 32611.

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