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Gauge-invariant perturbations in anisotropic homogeneous cosmological models

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Perturbations in spatially flat anisotropic homogeneous cosmological models with arbitrary dimension N are classified into three types I, II, and III and gauge-invariant quantities are defined in each type. Equations for them are derived for arbitrary anisotropic flat models. It is found that density perturbations are described by two second-order differential equations, as in the treatment of Perko, Matzner, and Shepley for the pressureless fluid. The solutions are obtained for approximate Kasner-type anisotropic models and their characteristic behaviors are shown for the fluids with nonzero pressure as well as the pressureless fluid. They are consistent with the counterparts of Perko, Matzner, and Shepley for the pressureless fluid. The instability problem in a Kaluza-Klein multidimensional universe also is discussed.

I. INTRODUCTION

The present Universe is isotropic and homogeneous, but at the early stage the Universe may have had no such symmetries. Direct observational constraints to the deviation from isotropy and homogeneity are brought by the severe isotropy of the cosmic background radiation, but the Universe can be anisotropic at times earlier than the recombination time.

On the basis of the Kaluza-Klein theory, moreover, the following scenario of the evolution of the Universe has recently been considered by many workers:[1–3] the Universe starts with a multidimensional spacetime whose dimension is N (>4), and the external three-dimensional space always expands, while the $(N-4)$-dimensional internal space contracts and is compactified due to quantum effect. At the stage before compactification, the Universe is considered to be very anisotropic. If there really are these highly anisotropic stages in our Universe, it is necessary to investigate the behavior of the perturbations, because they are closely connected with the problems of the formation of galaxies and cosmic background radiation. If their growth and damping rates are derived, we shall be able to constrain the freedom of the scenario and the initial conditions.

The perturbations in anisotropic models have so far been studied by several authors.[4–7] They showed that, as the volume of the anisotropic model increases, the density perturbations grow faster than those in isotropic models, and that density perturbations couple with tensor perturbations, contrary to the case of isotropic models. However, most of them treated the perturbations using the synchronous gauge condition. As was already well known in isotropic models, unphysical perturbations appear together with physical ones and it is difficult to discriminate them from physical ones except for the pressureless fluid. In isotropic models we could avoid the unphysical perturbations by adopting a special gauge such as the comoving gauge[8] or using the gauge-invariant formalism introduced by Bardeen.[9] Recently Abbott, Bednarz, and Ellis[7] have derived the gauge-invariant theory of perturbations in a multidimensional anisotropic universe, whose background space is a product of two homogeneous spaces. In their treatment geometrical objects such as scalars, vectors, and tensors are constructed in each space and the idea of the "gauge invariance" corresponds to the transformations within each space. In this paper we shall define gauge-invariant quantities in the total space and treat systematically physical perturbations in anisotropic models. Because the dimension N of spacetime is arbitrary, our theory can be applied to multidimensional Kaluza-Klein universe models. It will be shown that the obtained results are consistent with those in the work of Perko, Matzner, and Shepley for the pressureless fluid and the behaviors of the solutions for the fluid with nonzero pressure are rather different from those in the pressureless case.

In Sec. II the background models are shown and in Sec. III the perturbations are classified into three types according to transformation properties and gauge-invariant quantities are defined. For simplicity the direction of the wave vector k_a is specified, so as to be one of the $N-1$ axial directions. In Sec. IV the perturbation equations are derived in arbitrary flat models, in Sec. V they are solved in approximate ordinary Kasner models, and in Sec. VI the behaviors of perturbations in a Kaluza-Klein model are examined. In Sec. VII concluding remarks are given. In Appendix A perturbed components of the Ricci tensor are shown and in Appendix B the derivation of the density perturbations in the pressureless case is shown.

II. BACKGROUND MODELS

The metric of spatially flat anisotropic homogeneous models is expressed as

$$ds^2=g_{\alpha\beta}dx^\alpha dx^\beta$$
$$=e^{2\alpha(\tau)}[-d\tau^2+\gamma_{ab}(\tau)dx^a dx^b] \qquad (2.1)$$

with

$$\gamma_{aa}=\exp[2\beta_a(\tau)], \quad \sum_{a=1}^{N-1}\beta_a(\tau)=0\ , \qquad (2.2)$$

and $\gamma_{ab}=0$ for $a\neq b$. Here α,β run from 0 to $N-1$, a,b run from 1 to $N-1$, τ is the conformal time, and the cosmic time t is related to τ by $dt=e^{\alpha(\tau)}d\tau$. In this paper the Universe is assumed to be filled with a perfect fluid. Then the Einstein equations

$$G^{\alpha}_{\beta}\equiv R^{\alpha}_{\beta}-\tfrac{1}{2}\delta^{\alpha}_{\beta}R=-\kappa T^{\alpha}_{\beta}$$

are expressed as

$$(N-1)\ddot{\alpha}+\sum_{a=1}^{N-1}(\dot{\beta}_a)^2=-\frac{\kappa}{N-2}[(N-3)E+(N-1)P]e^{2\alpha}\,, \tag{2.3}$$

$$\ddot{\alpha}+(N-2)\dot{\alpha}^2=\frac{\kappa}{N-2}(E-P)e^{2\alpha}\,, \tag{2.4}$$

$$(N-2)\dot{\alpha}\dot{\beta}_a+\ddot{\beta}_a=0\,, \tag{2.5}$$

$$\dot{E}+(E+P)(N-1)\dot{\alpha}=0\,, \tag{2.6}$$

where the energy-momentum tensor is given by $T^{\mu\nu}=(E+P)u^{\mu}u^{\nu}+Pg^{\mu\nu}$ and E and P are the energy density and the pressure, respectively. Elimination of $\ddot{\alpha}$ from Eqs. (2.3) and (2.4) gives

$$\kappa Ee^{2\alpha}=-\frac{1}{2}\left[\sum_c\dot{\beta}_c{}^2-(N-1)(N-2)\dot{\alpha}^2\right] \tag{2.7}$$

and, if $P/E=\text{const}\ (=c_s{}^2)$, we get from Eq. (2.6)

$$E\propto\exp[-(1+c_s{}^2)(N-1)\alpha]\,.$$

An overdot denotes differentiation with respect to τ and $\overline{G}=\kappa/(8\pi)$ is the N-dimensional gravitational constant.

The exact solutions for Eqs. (2.3)—(2.6) were derived by Lorentz-Petzold[10] for various equations of state in the case when β_a have only two different values, that is,

$$\beta_1=\beta_2=\cdots=\beta_d\,,\qquad \beta_{d+1}=\beta_{d+2}=\cdots=\beta_{d+D}\,, \tag{2.8}$$

where $d+D=N-1$. Later we shall consider the case given by Eq. (2.8) at the stage near the singularity. Near the singularity e^{α}, e^{β_1}, and $e^{\beta_{d+1}}$ are expressed by the multidimensional Kasner solutions in the lowest-order approximation. The next-order terms depend on the equation of state. Their approximate expressions are shown in the case $P/E=\text{const}$:

$$e^{\alpha}=\tau^{1/(N-2)}\left[1+\frac{\gamma}{N-2}\tau^{\gamma}+O(\tau^{2\gamma})\right],$$
$$e^{\beta_1}=\tau^{-\mu}[1+\mu\tau^{\gamma}+O(\tau^{2\gamma})]\,, \tag{2.9}$$
$$e^{\beta_{d+1}}=\tau^{\nu}[1-\nu\tau^{\gamma}+O(\tau^{2\gamma})]\,,$$

where

$$\gamma\equiv\frac{N-1}{N-2}(1-c_s{}^2)\,. \tag{2.10}$$

Here μ and ν are defined by

$$\mu\equiv\pm\left[\frac{D}{d}\Big/(N-2)\right]^{1/2}$$

and (2.11)

$$\nu\equiv\pm\left[\frac{d}{D}\Big/(N-2)\right]^{1/2},$$

respectively, and satisfy the relations

$$d\mu-D\nu=0$$

and (2.12)

$$d\mu^2+D\nu^2=\frac{N-1}{N-2}\,.$$

From Eq. (2.9) we get the relations

$$\tau\dot{\alpha}=\frac{1}{N-2}(1+\gamma^2\tau^{\gamma})\,,$$
$$\tau\dot{\beta}_1=\mu(-1+\gamma\tau^{\gamma})\,, \tag{2.13}$$
$$\tau^2\sum_c\dot{\beta}_c{}^2=\frac{N-1}{N-2}(1-2\gamma\tau^{\gamma})\,,$$

and

$$e^{2\alpha}\kappa E=\frac{N-1}{N-2}\gamma(\gamma+1)\tau^{\gamma-2}[1+\gamma(\gamma-2)\tau^{\gamma}]\,. \tag{2.14}$$

III. CLASSIFICATION OF PERTURBATIONS

In isotropic homogeneous models the perturbations of metric components and fluid variables are uniquely classified into three types—scalar, vector, and tensor perturbations—and the three types of perturbations are decoupled geometrically and dynamically. In anisotropic models they are coupled in a complicated manner because of the shear motion of the background. Accordingly the geometric classification due to tensor analysis which is conserved for time evolution is not possible in the anisotropic case. In this paper we consider the classification of the perturbations due to their transformation properties. The method we adopt is as follows. First, we divide all gauge transformations into those of two types whose generators are scalarlike and vectorlike. Second, we classify the perturbations into three types—I, II, and III, so that the perturbations of type I are closed for scalarlike transformations, the perturbations of type II are closed for vectorlike transformations, and the perturbations of type III do not change in these transformations. Third, we define in each type the invariant quantities corresponding to each transformation.

The gauge transformations are expressed as

$$\overline{\tau}=\tau+\xi^0=\tau+T(\tau)Q(x^c)\,, \tag{3.1}$$

$$\overline{x}^a=x^a+\xi^a=x^a+L(\tau)Q^a(x^c)+L^a(\tau)Q(x^c)\,, \tag{3.2}$$

where $T(\tau)$ and $L(\tau)$ are arbitrary functions of τ generating the perturbations of type I, and $L^a(\tau)$ are functions generating those of type II. Because the space is flat, we use the plane-wave harmonics, that is,

$$Q(x^c)=\exp\left[i\sum_{c=1}^{N-1}k_c x^c\right],$$

and $Q^a(x^c)$ is defined by

$$Q^a(x^c)\equiv -k^{-1}Q(x^c)^{|a}=-ik^a k^{-1}Q(x^c)\ , \quad (3.3)$$

where a vertical bar denotes the covariant derivative with respect to $\gamma_{ab}(\tau)$ in Eq. (2.1),

$$k^2=\sum_{c=1}^{N-1}k_c k^c\ ,$$

and $k^c=\gamma^{cb}k_b$. A vector $L^a(\tau)$ is orthogonal to a wave vector k_a, i.e., $k_a L^a=0$. It should be noted that k^a is not constant in time, while k_a is constant, so that the harmonics such as $Q^a(x^c)$ is not constant in time.

For simplicity we consider in the following the case in which the vector k_a is in an axial direction of x^a ($a=1$ to d) or x^2 ($a=d+1$ to $d+D$). Then without loss of generality we can express it as $k_a=\delta_a{}^1$, so that $k^a=e^{-2\beta_1}\delta_1^a$ and $k=e^{-\beta_1}$.

Let metric perturbations be $h_{\alpha\beta}(x^\gamma)$. The differences of their perturbations due to the transformations (3.1) and (3.2) are

$$\bar{h}_{\alpha\beta}=h_{\alpha\beta}-e^{2\alpha(\tau)}\Delta_{\alpha\beta}\ ,$$

$$\Delta_{\alpha\beta}=e^{-2\alpha}(\xi_{\alpha;\beta}+\xi_{\beta;\alpha})\ ,$$

where a semicolon means the covariant derivative with respect to the background metric $g_{\alpha\beta}$. The explicit forms of $\Delta_{\alpha\beta}$ are

$$\Delta_{00}/Q=-2(\dot{T}+\dot{\alpha}T)\ , \quad (3.4)$$

$$\Delta_{0a}/Q=-ik_a k^{-1}\left[kT+\dot{L}-\frac{\dot{\gamma}_{bc}k^b k^c}{2k^2}L\right]+i(\dot{L}_a-\dot{\gamma}_{ab}\gamma^{bc}L_c)\ , \quad (3.5)$$

$$\Delta_{ab}/Q=2(\dot{\alpha}\gamma_{ab}+\tfrac{1}{2}\dot{\gamma}_{ab})T+2\frac{k_a k_b}{k}L-(k_a L_b+k_b L_a)\ . \quad (3.6)$$

A. Type-I perturbations

The metric perturbations which are closed for the transformations with T and L are

$$h_{00}=-2e^{2\alpha}AQ\ , \quad (3.7)$$

$$h_{0a}=-e^{2\alpha}BQ_a\ , \quad (3.8)$$

$$h_{ab}=2e^{2\alpha}\left[-\frac{k_a k_b}{k}H_T+(\dot{\alpha}\gamma_{ab}+\tfrac{1}{2}\dot{\gamma}_{ab})H_M+G_{ab}^{(1)}\right]Q\ . \quad (3.9)$$

These metric perturbations include not only the components [$A(\tau)$, $B(\tau)$, $H_T(\tau)$, and $H_M(\tau)$] corresponding to density perturbations, but also a component [$G_{ab}^{(1)}(\tau)$] for the gravitational-wave perturbations associated with them. Here $G_{ab}^{(1)}$ is transverse and traceless, i.e., $G_{ab}^{(1)}\gamma^{ab}=0$, $G_{ab}^{(1)}k^a=0$, and $G_{ab}^{(1)}k^b=0$. Under the transformations (3.1) and (3.2) the gravitational-wave perturbations are invariant. They are, however, indispensable here, in order that the perturbations of this type may be dynamically closed and decoupled from the perturbations of the other types. This situation will be explained in the next section. The variation of the other components is

$$A-\bar{A}=\dot{T}+\dot{\alpha}T\ ,$$

$$B-\bar{B}=-\left[kT+\dot{L}-\frac{\dot{\gamma}_{ab}k^a k^b}{2k^2}L\right],$$

$$H_T-\bar{H}_T=-kL\ ,$$

$$H_M-\bar{H}_M=T\ ,$$

so that the following two other invariant quantities are obtained:

$$\Phi_1\equiv A-\dot{H}_M-\dot{\alpha}H_M\ , \quad (3.10)$$

$$\Phi_2\equiv B+kH_M-\dot{H}_T/k\ . \quad (3.11)$$

Here it is interesting to notice that two traceless tensors appear in Eq. (3.9): $k_a k_b/k^2-(N-1)^{-1}\gamma_{ab}$ and $\dot{\gamma}_{ab}$. The latter is proportional to the background shear tensor σ_{ab}. From the two tensors we can construct a traceless transverse quantity. This suggests that the type-I perturbations are closely connected with the gravitational-wave perturbations. In the isotropic case in which $\gamma_{ab}=\delta_{ab}$, $\dot{\gamma}_{ab}=0$, and $G_{ab}^{(1)}=0$, Eq. (3.9) is reduced to Bardeen's corresponding expression by the use of H_L defined by

$$\dot{\alpha}H_M\equiv H_L+(N-1)^{-1}H_T$$

and Φ_H and Φ_A are related to Φ_1 and Φ_2 as

$$\Phi_H=\dot{\alpha}k^{-1}\Phi_2$$

and

$$\Phi_A=\Phi_1+(\Phi_2/k)^{\cdot}+\dot{\alpha}\Phi_2/k\ .$$

For fluid variables also we take up the energy density contrast $\delta\epsilon(\tau)Q$ and the velocity perturbations $\delta u^a=u^0 v(\tau)Q^a$ which are closed for the transformations with T and L. Here $u^0(=e^{-\alpha})$ is the zeroth component of the background N velocity. Because $\delta\epsilon-\overline{\delta\epsilon}=\dot{E}T$ and

$$v-\bar{v}=-\dot{L}+\dot{\gamma}_{ab}k^a k^b(2k^2)^{-1}L\ ,$$

the following gauge-invariant quantities corresponding to them can be defined in the same way as in the isotropic case:

$$\epsilon_m=\frac{\delta\epsilon}{E}-\frac{\dot{E}}{E}k^{-1}(v-B)\ , \quad (3.12)$$

$$v_s=v-k^{-1}\dot{H}_T\ . \quad (3.13)$$

Here ϵ_m is equal to the density contrast in the comoving gauge.

B. Type-II perturbations

The metric perturbations which are closed for the transformations with L^a are

$$h_{0a} = -e^{2\alpha} i B_a(\tau) Q \ ,$$
$$h_{ab} = e^{2\alpha} k^{-1}[k_a H_b(\tau) + k_b H_a(\tau)] Q \ , \tag{3.14}$$

where

$$B^a k_a = 0$$

and

$$H^a k_a = 0 \ . \tag{3.15}$$

Because the variations of B_a and H_a given by Eqs. (3.5) and (3.6) are

$$B_a - \overline{B_a} = -\dot{L}_a + 2\dot{\beta}_a L_a \ ,$$
$$H_a - \overline{H_a} = -L_a k \ ,$$

the gauge-invariant quantities are

$$\Psi_a \equiv B_a - \gamma_{ab}(H^b/k)^{\cdot}$$

or

$$\Psi^a \equiv \gamma^{ab}\Psi_b = B^a - (H^a/k)^{\cdot} \ . \tag{3.16}$$

From Eq. (3.15) we get $\Psi^a k_a = 0$.

For the corresponding velocity perturbation $\delta u^a = u^0 v^a(\tau) Q$ with $v^a k_a = 0$, we obtain the relation

$$v^a - \overline{v^a} = -\dot{L}^a \ ,$$

so that the invariant vector is

$$v_s^a = v^a - B^a \ . \tag{3.17}$$

The perturbations of this type give the free rotational perturbations which do not couple with the perturbations of the other types.

C. Type-III perturbations

The perturbations of this type are automatically gauge invariant, and give free gravitational-wave perturbations which are traceless and transverse. This type consists of only free waves, while the waves associated with density perturbations are included in type I. The metric perturbations are expressed as

$$h_{ab} = 2e^{2\alpha} G_{ab}^{(3)}(\tau) Q \ , \tag{3.18}$$

where $G_{ab}^{(3)}$ satisfy the relations

$$G_a^{(3)a} \equiv \gamma^{ab} G_{ab}^{(3)} = 0 \tag{3.19}$$

and

$$k_b G_a^{(3)b} = k^b G_b^{(3)a} = 0 \ . \tag{3.20}$$

IV. PERTURBATION EQUATIONS

Equations for metric perturbations are derived from the perturbed Einstein equations

$$\delta G^\alpha_\beta \equiv \delta R^\alpha_\beta - \tfrac{1}{2}\delta^\alpha_\beta \delta R = -\kappa \delta T^\alpha_\beta \tag{4.1}$$

and the fluid-dynamical equations are from the equations of energy-momentum conservation

$$\delta(T^\alpha_{\beta;\alpha}) = 0 \ . \tag{4.2}$$

In the following the perturbation equations for gauge-independent variables are derived by the use of Eqs. (4.1) and (4.2). For fluid motions only the adiabatic perturbations are assumed. The components of the Ricci tensor are given in Appendix A.

First we take up the equation

$$\delta G^0_0 - \frac{N-1}{k^2}\dot{\alpha}(\delta G^0_a)^{|a} = -\left[\delta T^0_0 - \frac{N-1}{k^2}\dot{\alpha}(\delta T^0_a)^{|a}\right] , \tag{4.3}$$

to which only type-I perturbations contribute. By the use of Eqs. (A1)–(A3) in Appendix A this equation is reduced to

$$[(N-2)\dot{\alpha} - \dot{\beta}_{(k)}]k\Phi_2 - \left[(N-1)\dot{\alpha}\dot{\beta}_{(k)} - \sum \dot{\beta}_b{}^2\right]\Phi_1 - [\dot{\xi} + (2N-3)\dot{\alpha}\xi] = e^{2\alpha}\kappa E \epsilon_m \ , \tag{4.4}$$

where

$$\dot{\beta}_{(k)} \equiv \sum_{a=1}^{N-1} \dot{\beta}_a k_a k^a / k^2$$

and ξ is defined by

$$\xi \equiv \sum_a \dot{\beta}_a G_a^{(1)a} \ .$$

Next we consider the relations obtainable from Eq. (4.1):

$$k^a k_b \delta R^b_a - \frac{k^2}{N-1}\delta R^c_c = 0 \ , \tag{4.5}$$

$$l^a_{(m)} l_{(n)b} \delta R^b_a - \frac{l^c_{(m)} l_{(n)c}}{N-1}\delta R^b_b = 0 \ , \tag{4.6}$$

$$l^a_{(m)} k_b \delta R^b_a = k^a l_{(m)b} \delta R^b_a = 0 \ , \tag{4.7}$$

where

$$l_{(m)a} = \delta^m_a - k_a k^m / k^2 \ .$$

To Eq. (4.5) only type-I perturbations contribute, to Eq. (4.5) type-I and -III perturbations contribute, and to Eq. (4.7) only type-II perturbations contribute. Accordingly we obtain from Eq. (4.5)

$$\dot{\Phi}_2 + k\Phi_1 + [(N-2)\dot{\alpha} - \dot{\beta}_{(k)}]\Phi_2 - \frac{N-1}{N-2}\dot{\beta}_{(k)}(\dot{\Phi}_1/k - \Phi_2) = 0 \ . \tag{4.8}$$

Now let us express the components of transverse traceless metric perturbations $G_b^{(1)a}$ and $G_b^{(3)a}$ as

$$G^{(i)}_{(mn)} \equiv G_a^{(i)b} l^a_{(m)} l_{(n)b} / (l_{(m)} l_{(n)})$$

with $i = 1$ and 3, where $G^{(i)}_{(mn)} = G^{(i)}_{(nm)}$ and

$$l_{(m)} \equiv (l_{(m)a} l^a_{(m)})^{1/2} \ .$$

Equation (4.6) is an inhomogeneous differential equation for traceless transverse metric perturbations with type-I perturbations as a source term. Of those metric perturbations $G^{(1)}_{(mn)}$ should be connected with the source term and $G^{(3)}_{(mn)}$ should be independent of it. Because in Eq. (4.6) the source term vanishes for $m \neq n$, $G^{(1)}_{(mn)}$ with $m \neq n$ vanishes and the following equation for $G^{(1)}_{(mm)}$ is obtained from Eq. (4.6) with $m = n$:

$$\ddot{G}^{(1)}_{(mm)} + (N-2)\dot{\alpha}\dot{G}^{(1)}_{(mm)} + k^2 G^{(1)}_{(mm)} = (\dot{\Phi}_1 - k\Phi_2)\left[\dot{\beta}_{(m)} + \frac{1}{N-2}\dot{\beta}_{(k)}\right] , \qquad (4.9)$$

where

$$\beta_{(m)} \equiv \sum_{a=1}^{N-1} \beta_a l_{(m)a} l^a_{(m)} / l_{(m)}{}^2 .$$

Equation (4.9) shows that $G^{(1)}_{(mn)}$ is indispensable in type-I perturbations, because they are not closed dynamically without it. For $k_a = \delta_a{}^1$ we have the relations

$$G^{(1)}_{(11)} = 0, \quad (d-1)G^{(1)}_{(22)} + DG^{(1)}_{(d+1,d+1)} = 0$$

and

$$\xi = \sum_a \dot{\beta}_a G^{(1)}_{(aa)} = (d-1)(\dot{\beta}_{(1)} - \dot{\beta}_{(d+1)}) G^{(1)}_{(22)} . \qquad (4.10)$$

The unique solution of Eq. (4.9) can be obtained as a Fourier component of the retarded solution $G_{\rm ret}$ of

$$\ddot{G} + (N-2)\dot{\alpha}\dot{G} - \nabla^2 G = F ,$$

where

$$\nabla^2 \equiv \gamma^{ab}\partial_a\partial_b = \sum_{a=1}^{N-1} e^{-2\beta_a}\partial_a\partial_a ,$$

$$F(\tau,\mathbf{x}) \equiv \prod_{c=1}^{N-1} \int dk_c \left[\dot{\Phi}_1 - k\Phi_2 \times \left[\dot{\beta}_{(m)} + \frac{1}{N-2}\dot{\beta}_{(k)}\right]\right] e^{ik_a x^a} .$$

The solution $G_{\rm ret}$ is formally expressed in terms of the retarded Green's function as

$$G_{\rm ret} = \int d\mathbf{x}' D_{\rm ret}(\mathbf{x},\mathbf{x}') F(\mathbf{x}') .$$

From Eq. (4.6), moreover, we obtain the following homogeneous equation for $G^{(3)}_{(mn)}$ which is independent of Φ_1 and Φ_2:

$$\ddot{G}^{(3)}_{(mn)} + (N-2)\dot{\alpha}\dot{G}^{(3)}_{(mn)} + k^2 G^{(3)}_{(mn)} + 2(\dot{\beta}_{(m)} - \dot{\beta}_{(n)})\dot{G}^{(3)}_{(mn)} = 0 . \qquad (4.11)$$

This equation describes free gravitational-wave perturbations. The last term in Eq. (4.11) vanishes for $1 \le m < n \le d$ or $d+1 \le m < n \le d+D = N-1$.

From Eq. (4.7) we get equations for type-II perturbations

$$k_a[\dot{\Psi}^b + (N-2)\dot{\alpha}\Psi^b] + k^b\gamma_{ac}[\dot{\Psi}^c + (N-2)\dot{\alpha}\Psi^c] - 2(\beta_b - \beta_a)(k_a\Psi^b - k^b\Psi_a) = 0$$

from which the following equation for Ψ^a is obtained

$$\dot{\Psi}^a + [(N-2)\dot{\alpha} + 2(\dot{\beta}_a - \dot{\beta}_{(k)})]\Psi^a = 0 . \qquad (4.12)$$

Remaining components of Eq. (4.1) are

$$k^a \delta R^0_a = -\kappa k^a \delta T^0_a \qquad (4.13)$$

and

$$l^a_{(m)} \delta R^0_a = -\kappa l^a_{(m)} \delta T^0_a . \qquad (4.14)$$

From Eq. (4.13) we get

$$k[(N-2)\dot{\alpha} - \dot{\beta}_{(k)}]\Phi_1 - \left[(N-2)(\ddot{\alpha} - \dot{\alpha}^2) + \sum \dot{\beta}_c{}^2\right]\Phi_2 - k\xi = \kappa(E+P)e^{2\alpha}v_s \qquad (4.15)$$

and from Eq. (4.14)

$$k^2\Psi^a = 2(E+P)e^{2\alpha}v^a_s . \qquad (4.16)$$

Moreover the equations of continuity and motion are derived as follows. First from the zero and k^a components of Eq. (4.2) we obtain two equations for type-I perturbations,

$$\dot{\epsilon}_m - (N-1)c_s{}^2\dot{\alpha}\epsilon_m + (1+c_s{}^2)[k\Phi_2 - (N-1)\dot{\alpha}\Phi_1] + \left[k^2 + (N-1)\left[2\dot{\alpha}\dot{\beta}_{(k)} + \frac{1-c_s{}^2}{2(N-2)}\sum\dot{\beta}_c{}^2 + \tfrac{1}{2}(N-1)(1+c_s{}^2)\dot{\alpha}^2\right]\right] \times \{\xi - [(N-2)\dot{\alpha} - \dot{\beta}_{(k)}]\Phi_1\} \Big/ \left[\sum\dot{\beta}_c{}^2 - (N-1)(N-2)\dot{\alpha}^2\right] = 0 , \qquad (4.17a)$$

$$\dot{v}_s + (\dot{\alpha} + \dot{\beta}_{(k)})v_s = k\{\Phi_1 + k^{-1}[\dot{\Phi}_2 + (\dot{\alpha} + \dot{\beta}_{(k)})\Phi_2]\} + \frac{kE}{E+P}c_s{}^2\epsilon_m , \qquad (4.17b)$$

and from the $l^a_{(m)}$ components the equation of motion for type-II perturbations,

$$\dot{v}_s^a+\left[\dot{\alpha}+2\dot{\beta}_a+\frac{\dot{P}}{E+P}\right]v_s^a=0 . \tag{4.18}$$

Equation (4.18) can be derived also by inserting Eq. (4.16) into Eq. (4.12).

For type-I perturbations we have four equations (4.8), (4.15), (4.17a), and (4.17b). If we eliminate v_s from Eqs. (4.15) and (4.17b), we get

$$e^{-\alpha-\beta_{(k)}}\left\{e^{-\alpha+\beta_{(k)}}(E+P)^{-1}\left[k[(N-2)\dot{\alpha}-\dot{\beta}_{(k)}]\Phi_1-\left[(N-2)(\ddot{\alpha}-\dot{\alpha}^2)+\sum\dot{\beta}_c{}^2\right]\Phi_2-k\xi\right]\right\}^{\cdot}$$
$$=k[\Phi_1+\dot{\alpha}\Phi_2/k+(\Phi_2/k)^{\cdot}]+k(E+P)^{-1}e^{-2\alpha}c_s{}^2\left[k[(N-2)\dot{\alpha}-\dot{\beta}_{(k)}]\Phi_2\right.$$
$$\left.-\left[(N-1)\dot{\alpha}\dot{\beta}_{(k)}-\sum\dot{\beta}_c{}^2\right]\Phi_1-[\dot{\xi}+(2N-3)\dot{\alpha}\xi]\right] . \tag{4.19}$$

By use of Eqs. (4.8) and (2.3)–(2.7), Eq. (4.19) is reduced to

$$\dot{\Phi}_1=kc_s{}^2\Phi_2+(1-c_s{}^2)[\dot{\xi}+(N-2)\dot{\alpha}\xi]/[(N-2)\dot{\alpha}-\dot{\beta}_{(k)}] . \tag{4.20}$$

Moreover, using Eqs. (4.9) and (4.20) we obtain an equation for ξ:

$$\ddot{\xi}+\{3(N-2)\dot{\alpha}-\lambda(1-c_s{}^2)/[(N-2)\alpha-\beta_{(k)}]\}\dot{\xi}$$
$$+\{k^2+(N-2)[\ddot{\alpha}+2(N-2)\dot{\alpha}^2]-\lambda\dot{\alpha}[(N-2)(1-c_s{}^2)]/[(N-2)\dot{\alpha}-\beta_{(k)}]\}\xi=k(c_s{}^2-1)\lambda\Phi_2 , \tag{4.21a}$$

where λ is defined by

$$\lambda\equiv(d-1)(\dot{\beta}_1)^2+D(\dot{\beta}_{d+1})^2-\frac{1}{N-2}(\dot{\beta}_{(k)})^2 . \tag{4.21b}$$

In the present case we have

$$\lambda=\left[\frac{N-1}{N-2}\right]\frac{1-\mu^2}{\tau^2}(1-2\gamma\tau^\gamma) .$$

Next eliminating Φ_1 from Eqs. (4.8) and (4.20), we obtain the second-order differential equation for Φ_2:

$$\ddot{\Phi}_2+[(N-2)\dot{\alpha}+\gamma\dot{\beta}_{(k)}]\dot{\Phi}_2+\left[k^2c_s{}^2+(N-2)^2(\gamma/2-1)\dot{\alpha}^2+(N-2)(2-\gamma)\dot{\alpha}\dot{\beta}_{(k)}+(\gamma-1)\dot{\beta}_{(k)}{}^2-\tfrac{1}{2}(1-c_s{}^2)\sum_c\dot{\beta}_c{}^2\right.$$
$$\left.+(1-c_s{}^2)\gamma\dot{\beta}_{(k)}\lambda/[(N-2)\alpha-\beta_{(k)}]\right]\Phi_2=J , \tag{4.22}$$

where

$$k[(N-2)\dot{\alpha}-\dot{\beta}_{(k)}]J\equiv(1-c_s{}^2)[A\dot{\xi}+(N-2)B\xi] ,$$

$$A\equiv-k^2+\frac{N-1}{N-2}\dot{\beta}_{(k)}\left[-3(N-2)\dot{\alpha}+2\dot{\beta}_{(k)}+\frac{\lambda(1+c_s{}^2)-(N-2)(\ddot{\alpha}+\dot{\alpha}\dot{\beta}_{(k)})}{(N-2)\dot{\alpha}-\dot{\beta}_{(k)}}\right] ,$$

$$B\equiv-\left[\dot{\alpha}+\frac{N-1}{(N-2)^2}\dot{\beta}_{(k)}\right]k^2+\dot{\beta}_{(k)}\left[-3(N-1)\dot{\alpha}^2+2\frac{N-1}{N-2}\dot{\alpha}\dot{\beta}_{(k)}+\frac{\gamma\lambda\dot{\alpha}-(N-1)(\ddot{\alpha}+\dot{\alpha}\dot{\beta}_{(k)})\dot{\alpha}}{(N-2)\dot{\alpha}-\dot{\beta}_{(k)}}\right] .$$

Equation (4.22) describes density perturbations together with Eqs. (4.4), (4.10), (4.17a), and (4.21). A more convenient one of Eqs. (4.4) and (4.17a) can be used for the derivation of ϵ_m.

The invariant variables and equations to be solved in each type are shown in Table I.

V. SOLUTIONS OF PERTURBATION EQUATIONS

It is difficult to derive even for a simple equation of state analytic solutions of perturbation equations in the previous section. Here we derive the approximate solutions for two extreme cases $(k\tau)^2\gg1$ and $(k\tau)^2\ll1$. As

TABLE I. Classification of types and equations.

Type	Physical properties	Invariant variables	Equations
I	Density Perturbation	Φ_1 Φ_2 ϵ_m v_s	(4.4), (4.8), (4.15), (4.17a), (4.17b), (4.20), (4.21), (4.22)
	Gravitational waves induced by density perturbations	$G^{(1)}_{(mm)}$ ξ	(4.9), (4.10)
II	Free rotational perturbations	Ψ^a, v_s^a	(4.12), (4.16), (4.18)
III	Free gravitational-wave perturbations	$G^{(3)}_{(mn)}$	(4.11)

the background model we use a model with $P/E=\text{const}$, which is shown in Sec. II, and take only the lowest-order terms, unless the higher-order terms are necessary.

A. Density perturbations

For the type-I perturbations there are four independent solutions, since we have two second-order equations or fourth-order equations.

1. $(k\tau)^2 \ll 1$

One of the solutions is obtained when ξ and Φ_2 are higher order compared with $\dot{\alpha}\Phi_1$ with respect to $(k\tau)^2$:

$$\Phi_1=(\Phi_1)_0+O((k\tau)^2),\quad \xi=(\Phi_1)_0\tau^{-1}O((k\tau)^2)\,,$$
$$\Phi_2=\left[2+\frac{N-3}{N-2}\mu\right]^{-1}(\Phi_1)_0 k\tau[1+O((k\tau)^2)]\,, \tag{5.1}$$

and

$$\epsilon_m=\frac{\mu+1}{\gamma(\gamma+1)}\tau^{-\gamma}(\Phi_1)_0[1+O((k\tau)^2)]\,, \tag{5.2}$$

where $(\Phi_1)_0$ is a constant and Eqs. (2.10) and (2.13)–(2.15) were used.

The other solutions, in which Φ_1, Φ_2, and ξ are comparable in Eq. (4.4), are derived by assuming the forms

$$\xi=\tau^a[1+O((k\tau)^2)]$$
and
$$\Phi_2=\zeta\tau^{a-\mu}\,, \tag{5.3}$$

where a and ζ are constants. First for $a+1\neq 0$ we get from Eq. (4.20)

$$\Phi_1=(1+\mu)^{-1}(a+1)^{-1}\times[(1-c_s{}^2)(a+1)+\zeta(1+\mu)c_s{}^2]\tau^{a+1}+\text{const}\,. \tag{5.4}$$

Using Eqs. (4.8) and (4.21a) we obtain

$$a^2+\left[2-\left[\frac{N-1}{N-2}\right](1-\mu)(1-c_s{}^2)\right]a+1+\left[\frac{N-1}{N-2}\right](1-\mu^2)(1-c_s{}^2)[\zeta-1/(1+\mu)]=0 \tag{5.5}$$

and

$$\zeta=-\left[\frac{N-1}{N-2}\right]\frac{\mu}{1+\mu}\frac{(1-c_s{}^2)x}{x+\mu(c_s{}^2-1)}\,, \tag{5.6}$$

where

$$a+1\equiv\left[\frac{N-1}{N-2}\right]x\,.$$

Eliminating ζ from Eqs. (5.5) and (5.6) we obtain

$$x^2[x-(1-c_s{}^2)]=0\,, \tag{5.7}$$

whose nonzero solutions are $x=1-c_s{}^2$, so that

$$a+1=\gamma\,. \tag{5.8}$$

Then we get from Eq. (5.6)

$$\zeta=-\gamma\mu/(1-\mu^2)\,, \tag{5.9}$$

and substituting Eqs. (5.3) and (5.4) into Eq. (4.4) we obtain

$$e^{2\alpha}E\epsilon_m=-\frac{N-1}{N-2}(1-\mu)^{-1}\tau^{\gamma-2}+\text{const}\times\tau^{-2}\,. \tag{5.10}$$

This constant is the same as the one appearing in the expression for Φ_1 in Eq. (5.4) and can be included in the former solution in Eq. (5.1). Accordingly we get

$$\epsilon_m\propto\tau^0\,. \tag{5.11}$$

In the case $a+1=0$ (or $x=0$), it is necessary to derive the higher-order terms with respect to $(k\tau)^2$ and τ^γ, in order to get the nonzero values of ϵ_m. Here let us put the solutions in the form

$$\xi=\xi_0+\xi_1,\quad \Phi_1=\Phi_{10}+\Phi_{11}$$
$$\text{and} \tag{5.12}$$
$$\Phi_2=\Phi_{20}+\Phi_{21}\ ,$$

where ξ_1/ξ_0, Φ_{11}/Φ_{10}, and Φ_{21}/Φ_{20} are $\sim(k\tau)^2$. The solutions for Eqs. (4.8), (4.20), and (4.41) are easily derived in the lowest order with respect to $(k\tau)^2$ and ξ, Φ_1, and Φ_2 can be expressed as

$$\begin{aligned}
&\xi_0=\tau^{-1}+G_0\tau^{\gamma-1},\quad \xi_1=b\tau^{2\mu+1}+G_1\tau^{2\mu+\gamma+1}\ ,\\
&\Phi_{10}=(\mu+1)^{-1}+G_2\tau^\gamma\ ,\\
&\Phi_{11}=b(\mu+1)^{-1}\tau^{2\mu+2}+G_3\tau^{2\mu+\gamma+2}\ ,\\
&\Phi_{20}=G_4\tau^{-\mu+\gamma-1}\ ,
\end{aligned} \tag{5.13}$$

and

$$\Phi_{21}=2b\tau^{\mu+1}+G_5\tau^{\mu+\gamma+1}\ ,$$

where $b\equiv-\frac{1}{4}(\mu+1)^{-2}$. The constants G_i $(i=0-5)$ are determined so as to satisfy Eqs. (4.8), (4.20), and (4.21). For G_0, G_2, and G_4 we get

$$\begin{aligned}
&G_0=\gamma\left[\frac{\mu-1}{\mu+1}(\gamma+1)-1\right],\\
&G_2=\frac{\gamma(\gamma+1)}{(\mu+1)^2}\left[\mu-\frac{N-2}{N-1}\gamma\right],\\
&G_4=\mu\gamma^2(\gamma+1)/(\mu+1)^2\ ,
\end{aligned} \tag{5.14}$$

and the derivation of G_1, G_3, and G_5 is shown in Appendix B. For ϵ_m it is found from Eqs. (4.4) and (5.13) that

$$\epsilon_m\propto\tau^{2\mu+2}\ . \tag{5.15}$$

The fourth solution can be derived in a similar way to the third one. When we put

$$\xi=\xi_0+\xi_1,\quad \Phi_1=\Phi_{10}+\Phi_{11}$$
$$\text{and} \tag{5.16}$$
$$\Phi_2=\Phi_{20}+\Phi_{21}$$

as in Eq. (5.12), their components are expressed as

$$\begin{aligned}
&\xi_0=\tau^{-1}\ln\tau+\tau^{\gamma-1}(G_0\ln\tau+H_0)\ ,\\
&\xi_1=\tau^{2\mu+1}b\left[\ln\tau-\frac{1}{\mu+1}\right]+\tau^{2\mu+1+\gamma}(G_1\ln\tau+H_1)\ ,\\
&\Phi_{10}=\frac{1}{\mu+1}\ln\tau+\tau^\gamma(G_2\ln\tau+H_2)\ ,\\
&\Phi_{11}=\frac{b}{\mu+1}\tau^{2\mu+2}\left[\ln\tau-\frac{1}{\mu+1}\right]\\
&\qquad+\tau^{2\mu+2+\gamma}(G_3\ln\tau+H_3)\ ,\\
&\Phi_{20}=\frac{1}{\mu+1}\tau^{-\mu-1}+\tau^{-\mu-1+\gamma}(G_4\ln\tau+H_4)\ ,\\
&\Phi_{21}=\frac{b}{\mu+1}\tau^{\mu+1}\left[2\ln\tau-\frac{1}{\mu+1}\right]\\
&\qquad+\tau^{\mu+1+\gamma}(G_5\ln\tau+H_5)\ .
\end{aligned} \tag{5.17}$$

The constants G_i and H_i $(i=0-5)$ are determined similarly so as to satisfy Eqs. (4.8), (4.20), and (4.21). As a result we get the values of G_0, G_2, and G_4 of quite the same form as those in Eq. (5.14):

$$\begin{aligned}
&G_0=\gamma\left[\frac{\mu-1}{\mu+1}(\gamma+1)-1\right],\\
&G_2=\frac{\gamma(\gamma+1)}{(\mu+1)^2}\left[\mu-\frac{N-2}{N-1}\gamma\right],\\
&G_4=\frac{\mu\gamma^2(\gamma+1)}{(\mu+1)^2}\ .
\end{aligned} \tag{5.18}$$

Their derivation is shown in Appendix B. For ϵ_m it is found for the above values of G_i and H_i that

$$\epsilon_m\propto\tau^{2\mu+2}(\ln\tau+\text{const})\ . \tag{5.19}$$

Accordingly the solutions of the last two types are expressed as

$$\epsilon_m=\tau^{2(\mu+1)}(d_1+d_2\ln\tau)\ , \tag{5.20}$$

where d_1 and d_2 are constant. Thus the solutions for ϵ_m consist of four components: (5.2), (5.18), and (5.20). They are consistent with the result of Perko, Matzner, and Shepley[5] in which the cosmic time t is used. In terms of t Eq. (5.20) is expressed in the pressureless case as

$$\epsilon_m\propto t^{2(\mu+1)/\gamma}(d_1'+d_2'\ln t)\ , \tag{5.21}$$

where d_1' and d_2' are constant.

2. $(k\tau)^2\gg1$

First we assume $1>c_s>0$. Then there are the following two different types of oscillatory solutions:

$$\begin{bmatrix}\xi\\ \Phi_1\\ \Phi_2\end{bmatrix}=\begin{bmatrix}\bar\xi\\ \bar\Phi_1\\ \bar\Phi_2\end{bmatrix}\exp\left[i\int k\,d\tau\right], \tag{5.22}$$

$$\begin{bmatrix}\xi\\ \Phi_1\\ \Phi_2\end{bmatrix}=\begin{bmatrix}\bar\xi\\ \bar\Phi_1\\ \bar\Phi_2\end{bmatrix}\exp\left[ic_s\int k\,d\tau\right], \tag{5.23}$$

where $(\ln\bar\xi)^\cdot$, $(\ln\bar\Phi_1)^\cdot$, and $(\ln\bar\Phi_2)^\cdot$ are $\sim1/\tau$.

Oscillatory solution 1. To get the nonzero values of ϵ_m it is necessary to derive ξ, Φ_1, and Φ_2 to next-order terms with respect to $(k\tau)^{-1}$ and τ^γ. Let us express them as

$$\begin{aligned}
&\bar\xi=\bar\xi_0+\bar\xi_1,\quad \bar\Phi_1=\bar\Phi_{10}+\bar\Phi_{11}\ ,\\
&\bar\Phi_2=\bar\Phi_{20}+\bar\Phi_{21}\ ,
\end{aligned} \tag{5.24}$$

where $\bar{\xi}_1/\bar{\xi}_0$, $\bar{\Phi}_{11}/\bar{\Phi}_{10}$, and $\bar{\Phi}_{21}/\bar{\Phi}_{20}$ are $\sim 1/(k\tau)$. Then the lowest-order solutions with respect to $(k\tau)^{-1}$ for Eqs. (4.20)–(4.22) are

$$\bar{\xi}_0=\exp[\tfrac{1}{2}\beta_{(k)}-\tfrac{3}{2}(N-2)\alpha] =\tau^{-\mu/2-3/2}[1+\tfrac{1}{2}(\mu-3\gamma)\tau^{\gamma}] ,$$

$$\bar{\Phi}_{10}=\bar{\xi}_0/[(N-2)\dot{\alpha}-\dot{\beta}_{(k)}] =\frac{1}{\mu+1}\tau^{-\mu/2-1/2} \times\{1+[\tfrac{1}{2}(\mu-3\gamma)-\gamma(\gamma-\mu)/(\mu+1)]\tau^{\gamma}\} , \quad (5.25)$$

$$\bar{\Phi}_{20}=i\bar{\Phi}_{10} .$$

The second-order solutions whose derivations are shown in Appendix B are

$$\bar{\xi}_1=-\frac{i}{8}(\mu+1)\tau^{-3\mu/2-5/2}[1+O(\tau^{\gamma})] ,$$
$$\bar{\Phi}_{11}=-\frac{i}{8}\tau^{-3\mu/2-3/2}[1+O(\tau^{\gamma})] , \quad (5.26)$$
$$\bar{\Phi}_{21}=-\tfrac{3}{8}\tau^{-3\mu/2-3/2}[1+O(\tau^{\gamma})] .$$

For ϵ_m we obtain from Eq. (4.4)

$$\epsilon_m \propto \bar{\epsilon}_m \exp\left[i\int k\,d\tau\right] ,$$
$$\bar{\epsilon}_m=\text{const}\times\tau^{-3\mu/2-3/2} . \quad (5.27)$$

This expression is independent of $c_s{}^2$, and in the case $c_s{}^2=0$, it is consistent with the counterpart which is derived for $N=4$ in the formalism of Perko, Matzner, and Shepley.[5]

Oscillatory solution 2. Substituting Eqs. (5.23) into Eqs. (4.21) and (4.22) as in solution 1, we obtain

$$\bar{\xi}=-\frac{\lambda}{k}\bar{\Phi}_2 \quad (5.28)$$

and

$$2\dot{\bar{\Phi}}_2+\bar{\Phi}_2\left[-\dot{\beta}_{(k)}+(N-2)\dot{\alpha}+\frac{N-1}{N-2}(1-c_s{}^2)\dot{\beta}_{(k)}\right]=-k\frac{1-c_s{}^2}{(N-2)\dot{\alpha}-\dot{\beta}_{(k)}}\bar{\xi} . \quad (5.29)$$

Solving these equations, $\bar{\Phi}_2$ is determined:

$$\bar{\Phi}_2=\tau^{-(\gamma-\mu-1)/2} . \quad (5.30)$$

For Φ_1 we get from Eq. (4.20)

$$\bar{\Phi}_1=-ic_s\bar{\Phi}_2 \quad (5.31)$$

and from Eq. (4.4)

$$\bar{\epsilon}_m \propto \tau^{-(2\gamma-1-\mu)2} . \quad (5.32)$$

B. Free rotational perturbations

Type-II perturbations represent free rotational perturbations. The time dependence of v_s^a is determined by Eq. (4.18):

$$v_s^a \propto \begin{cases} \tau^{2\mu-[1-(N-1)c_s{}^2]/(N-2)} & \text{for } a=1,\ldots,d, \\ \tau^{-2\nu-[1-(N-1)c_s{}^2]/(N-2)} & \text{for } a=d+1,\ldots,N-1 . \end{cases} \quad (5.33)$$

It is found that v_s^1 and v_s^{d+1} increase with time, if $\mu>0$ and $\nu<0$, respectively. This is because the scale factor $\exp(\alpha+\beta_a)$ decreases with time in these directions.

Ψ^a are obtained from Eq. (4.16):

$$\Psi^a \propto \tau^{-1}e^{2(\beta_{(k)}-\beta_a)} , \quad (5.34)$$

so that for $k_b=\delta_b{}^1$

$$\Psi^a \propto \begin{cases} \tau^{-1} , \\ \tau^{-1-2(\mu+\nu)} , \end{cases} \quad (5.35)$$

where a is the same as in Eq. (5.33).

C. Gravitational-wave perturbations

Free gravitational-wave perturbations $G^{(3)}_{(mn)}$ are obtained by solving Eq. (4.11).

Case 1. $1\le m<n\le d$ or $d+1\le m<n\le N-1$. For $k\tau \ll 1$

$$G^{(3)}_{(mn)}\simeq\int d\tau\, e^{-(N-2)\alpha}=\ln\tau+\text{const} , \quad (5.36)$$

and for $k\tau\gg 1$

$$G^{(3)}_{(mn)}\simeq(k\tau)^{-1/2}\exp\left[i\int k\,d\tau\right] . \quad (5.37)$$

The amplitude in Eq. (5.37) is $\tau^{-(1+\mu)/2}$ for $k_a \propto \delta_a{}^1$, which decrease with τ, except for $\nu=1$.

Case 2. $1\le m\le d$ and $d+1\le n\le N-1$. Equation (4.11) is

$$\ddot{G}^{(3)}_{(mn)}+[1-2(\mu+\nu)]\tau^{-1}\dot{G}^{(3)}_{(mn)}+k^2G^{(3)}_{(mn)}=0 . \quad (5.38)$$

The solutions are, for $k\tau\ll 1$,

$$G^{(3)}_{(mn)}\simeq\tau^{2(\mu+\nu)} \quad (5.39)$$

and, for $k\tau\gg 1$,

$$G^{(3)}_{(mn)}\simeq\tau^{\mu+\nu}(k\tau)^{-1/2}\exp\left[i\int k\,d\tau\right] . \quad (5.40)$$

The amplitude changes as $\tau^{(\mu-1)/2+\nu}$ for $k_a \propto \delta_a{}^1$, which increase generally with τ.

Gravitational-wave perturbations $G^{(1)}_{(m,m)}$ associated with the density perturbations are expressed by use of ξ, as in Eq. (4.9c).

VI. INSTABILITY OF A KALUZA-KLEIN UNIVERSE MODEL

Kaluza-Klein multidimensional universe models were proposed by Chodos and Detweiler[1] and later by Sahdev[2] and others.[3] In this section we discuss the instability of the multidimensional models at the final stage expressed approximately by the time-reversal Kasner spacetime, in which the total volume decreases with time τ. In these models the space with x^a of $a=1$ to d and the space with x^a of $a=d+1$ to $N-1$ are called "external" and "internal" spaces. Since the external space becomes our present space after the compactification of the internal space, the dimension of the external space is $d=3$ (or $D=3$), and the internal dimension is $D=N-4$ (or $d=N-4$), where $N>4$ is assumed. If the internal space is closed, the periodic condition must be imposed and the wave number k_a takes discrete numbers. If the two points with the coordinate distance r are identified, we have the relation $k_a r=2n\pi$ (n is an integer), so that k_a take discrete values $2n\pi/r$.

As for the background models, the time-reversal Kasner solution is derived from the ordinary solution by replacing τ with $\tau_0-\tau$ without changing parameters μ and ν. In the same way the perturbation equations and their solutions in the approximate time-reversal background can be obtained from those in the ordinary background by replacing τ by $\tau_0-\tau$, $\tau\dot{\alpha}$ by $-(\tau_0-\tau)\dot{\alpha}$, and $\tau\dot{\beta}_{ab}$ by $-(\tau_0-\tau)\dot{\beta}_{ab}$.

Now let us exemplify the behaviors of the perturbations in the most interesting case when the Universe is radiation dominated [$c_s{}^2=1/(N-1)$ and $\gamma=1$] and the dimension d or D of the external space is 3.

A. Density perturbations

For $(k\tau)^2 \ll 1$,

$$\epsilon_m = a_1\tau^{-1}+a_2\tau^0+\tau^{2(\mu+1)}(a_3+a_4\ln\tau) ,$$

and for $(k\tau)^2 \gg 1$,

$$\epsilon_m = b_1\tau^{-3(\mu+1)/2}e^{ik\tau/(\mu+1)}+b_2\tau^{-(1-\mu)/2}e^{ic_s k\tau/(\mu+1)} ,$$

where a_i and b_i are constant.

If $d=3$ or $D=3$,

$$\mu=\pm\left[\frac{N-4}{3(N-2)}\right]^{1/2}$$

or

$$\pm\left[\frac{3}{(N-2)(N-4)}\right]^{1/2} ,$$

respectively. In Table II these powers are shown for a moderate total dimension $N=10$. It is found that, as τ increase, the perturbations with the powers $2(\mu+1)$ grow, and as τ decreases, those with the power -1 grow. Oscillatory perturbations grow only as τ decreases.

B. Rotational perturbations

$$(v_s{}^1, v_s^{d+1}) \propto (\tau^{2\mu}, \tau^{-2\nu}) ,$$

$$(\Psi^1, \Psi^{d+1}) \propto (\tau^{-1}, \tau^{-1-2(\mu+\nu)}) .$$

TABLE II. Dimensions and powers in the Universe with $N=10$.

d	D	μ	ν
3	6	0.50	0.25
		-0.50	-0.25
6	3	0.25	0.50
		-0.25	-0.50

As τ increases or decreases, one of $v_s{}^1$ and v_s^{d+1} always grow, while, as τ decreases, Ψ^1 and Ψ^{d+1} ($\mu>0$) increase and Ψ^{d+1} ($\mu<0$) decreases.

C. Gravitational-wave perturbations

For $(k\tau)^2 \ll 1$,

$$G^{(3)}_{(11)}, G^{(3)}_{(d+1,d+1)} \propto \ln\tau ,$$

$$G^{(3)}_{(1,d+1)} \propto \tau^{2(\mu+\nu)} .$$

For $(k\tau)^2 \gg 1$,

$$G^{(3)}_{(11)}, G^{(3)}_{(d+1,d+1)} \propto \tau^{-(1+\mu)/2}e^{ik\tau/(\mu+1)} ,$$

$$G^{(3)}_{(1,d+1)} \propto \tau^{(\mu-1)/2+\nu}e^{ik\tau/(\mu+1)} .$$

In the present examples, we have $-(1+\mu)/2<0$ in all cases and for $\mu>0$ (<0), we have $\mu+\nu>0$ (<0), and $(\mu-1)/2+\nu\geq 0$ (<0), respectively.

VII. CONCLUDING REMARKS

On the basis of transformation properties the perturbations in anisotropic models were clarified into three types and the equations for gauge-invariant quantities in each type were derived in the case of $c_s=0$ and $1>c_s>0$. The equations for the type-I perturbations which include density perturbations consist of two second-order equations and there are four independent solutions, which are consistent with the results of Perko, Matzner, and Shepley for $c_s=0$.

In Sec. VI it was shown that in a Kaluza-Klein model many perturbations increase as τ approaches τ_0 (or τ approaches 0). How large their final values are depends on the initial condition and how nearly τ is close to τ_0 before the compactification. Unless the perturbations are smoothed out at the stage when the internal space is compactified and the external space becomes Friedmann type, the remarkable isotropy of the cosmic background radiation may impose severe conditions upon the initial condition and the evolution of the multidimensional universe at the precompactification stage.

In this paper we were confined to a simple case in which the wave vector k_a points to the axial directions. If it points to general directions, our formulation will be modified and the behaviors of perturbations will be complicated, because there appear more mode couplings. In a future work we will take up the formulation of perturbation theory in this general case.

The behaviors of derived perturbations may be due to the kinematical effect as well as the gravitational curvature effect. These effects must be discriminated by the use of quantities such as intrinsic curvature, but we have no invariant physical quantities good at the discrimination. Moreover, we have not considered in this paper any viscous effect which may be brought by gravitational-wave transport,[11] and any quantum effects such as particle creation[12] and the Casimir effect.[13] It is beyond the scope of this paper to study the behaviors of the perturbations in the presence of these effects.

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APPENDIX A: THE PERTURBED RICCI TENSOR

We write down the general expressions for the perturbed Ricci tensor components corresponding to the metric perturbations in Eqs. (3.7)—(3.9), (3.14), and (3.18):

$$\begin{aligned} e^{2\alpha}\delta R^0_0/Q = {} & (N-1)\dot\alpha\dot A + \left[2\left[(N-1)\ddot\alpha + \sum_{b=1}^{N-1}\dot\beta_b{}^2\right] - k^2\right]A - k\{\dot B + \dot\alpha B + [\dot\gamma_{ab}k^a k^b/(2k^2)]B\} \\ & - \Big[-\ddot H_T + (N-1)(\dot\alpha H_M)^{\cdot\cdot} + \dot\alpha[-\dot H_T + (N-1)(\alpha H_M)^{\cdot}] - [\dot\gamma_{ab}k^a k^b/(k^2)]\dot H_T \\ & - [\dot\gamma_{ac}\dot\gamma^{ab}k^c k_b/k^2 + (\dot\gamma_{ab}k^a k^b/k^2)^2]H_T + 2\sum_{b=1}^{N-1}(\dot\beta_b{}^2\dot H_M + \dot\beta_b\ddot\beta_b H_M)\Big] \\ & + 2\dot\beta_b\sum_b\{k_b[B^b - (H^b/k)^{\cdot}] - G^b_b\} , \end{aligned} \tag{A1}$$

$$\begin{aligned} e^{2\alpha}\delta R^0_a/Q = {} & ik_a[(N-2)\dot\alpha - \dot\beta_a]A + i\{\tfrac{1}{2}k^c(\dot\gamma_{ca} - \dot\gamma_{cb}k_a k^b/k^2)H_T + k_a[-(N-2)(\dot\alpha H_M) + \dot\beta_a(\dot H_M + \dot\alpha H_M) - \dot\beta_b{}^2 H_M]\} \\ & - \tfrac{1}{2}ik^2[B_a - \gamma_{ab}(H^b/k)^{\cdot}] - ik_a\dot\beta_a G^a_a , \end{aligned} \tag{A2}$$

$$\begin{aligned} e^{2\alpha}\delta R^a_b/Q = {} & \{(\dot\alpha + \dot\beta_a)\dot A + 2[\ddot\alpha + (N-2)\dot\alpha^2]A\}\delta^a_b - k^a k_b A \\ & - \{k\dot B + [(N-2)\dot\alpha - (\dot\beta_a + \dot\beta_b) + \dot\gamma_{cd}k^c k^d/(2k^2)]kB\}k^a k_b/k^2 \\ & - (\dot\alpha + \dot\beta_a)\delta^a_b kB + (k^a k_b/k^2)\ddot H_T + (\dot\alpha + \dot\beta_a)\delta^a_b \dot H_T + [(N-2)\dot\alpha - 2(\dot\beta_a + \dot\beta_b) + 2\dot\gamma_{cd}k^c k^d/k^2](k^a k_b/k^2)\dot H_T \\ & + \{4\dot\beta_a\dot\beta_b + [(N-2)\dot\alpha - 2(\dot\beta_a + \dot\beta_b)]\dot\gamma_{cd}k^c k^d/k^2 - \ddot\gamma^{cd}k_c k_d/k^2 + 2(\dot\gamma_{cd}k^c k^d/k^2)^2\}(k^a k_b/k^2)H_T \\ & + [-(N-3)\dot\alpha + \dot\beta_a + \dot\beta_b]k^a k_b H_M - \{[(\dot\alpha + \dot\beta_a)H_M]^{\cdot\cdot} + k^2(\dot\alpha + \dot\beta_a)H_M \\ & \qquad + [(2N-3)\dot\alpha + (N-1)\dot\beta_a](\dot\alpha H_M)^{\cdot} + (N-2)\dot\alpha(\dot\beta_a H_M)^{\cdot}\}\delta^a_b \\ & + \tfrac{1}{2}k_b\{\dot B^a - (H^a/k)^{\cdot\cdot} + (N-2)\dot\alpha[B^a - (H^a/k)^{\cdot}]\} + \tfrac{1}{2}k^a\gamma_{bc}\{[\dot B^c - (H^c/k)^{\cdot\cdot}] + (N-2)\dot\alpha[\dot B^c - (H^c/k)^{\cdot}]\} \\ & + (\dot\beta_a - \dot\beta_b)\{k_b[B^a - (H^a/k)^{\cdot}] - k^a[B_b - \gamma_{bc}(H^c/k)^{\cdot}]\} - \ddot G^a_b + [-(N-2)\dot\alpha - 2(\dot\beta_a - \dot\beta_b)]\dot G^a_b - k^2 G^a_b , \end{aligned} \tag{A3}$$

where $G^a_b = G^{(1)a}_b + G^{(3)a}_b$ and no summation is taken over an index a in Eq. (A2) and indices a and b in Eq. (A3), even if they appear twice. In the above derivation, Eq. (2.5) was used.

While these expressions seem complicated, the first-order equations in terms of the invariant variables can be reduced to simple and compact forms such as in Sec. IV.

APPENDIX B: DERIVATIONS OF PERTURBED QUANTITIES

1. $(k\tau)^2 \ll 1$

In the case $a+1=0$, the substitution of Eq. (5.20) into Eqs. (4.8), (4.20), and (4.21) leads to

$$\left[\gamma - \frac{N-1}{N-2}\mu\right]G_4 + \frac{N-1}{N-2}\mu\gamma G_2 = 0 , \tag{B1}$$

$$G_2 + \left[2\mu + \gamma + 2 - \frac{N-1}{N-2}\mu\right]G_5 + \frac{N-1}{N-2}\mu(2\mu + \gamma + 2)G_3 - \mu/(\mu+1) + 2b\left[\gamma(\gamma - \mu) + \frac{N-1}{N-2}\mu^2\right] = 0 , \tag{B2}$$

$$c_s{}^2 G_4 + (1 - c_s{}^2)\gamma(G_0 + \gamma)/(\mu+1) - \gamma G_2 = 0 , \tag{B3}$$

$$(2\mu+\gamma+2)G_3=-2\mu b c_s{}^2+c_s{}^2G_5+[b\gamma(2\mu-\gamma)+(2\mu+\gamma+2)G_1](1-c_s{}^2)/(1+\mu)\,, \tag{B4}$$

$$G_4=(G_0+\gamma)\gamma\mu/(\mu^2-1)\,, \tag{B5}$$

$$G_0+(2\mu+\gamma+2)(2\mu+2+\gamma\mu)G_1+\gamma(1-\mu^2)G_5-2\mu+b\gamma^2\{6(\mu+1)+\gamma+2(1-\mu)[\gamma/2-\mu-\mu(\mu+1)/\gamma]\}=0\,, \tag{B6}$$

where $b\equiv-1/(\mu+1)^2$. Equations (B1), (B3), and (B5) for G_0, G_2, and G_4 are not independent and one of them is arbitrary. If we specify them as in Eq. (5.14), the other solutions are given by the summation of the solution with Eq. (5.14) and the second type solutions (5.3)—(5.11). Accordingly we consider only the set of G_0, G_2, and G_4 in Eq. (5.14). Corresponding to this set, G_1, G_3, and G_5 are uniquely determined from Eqs. (B2), (B4), and (B6). The value of ϵ_m is derived by substituting Eq. (5.13) with this G_i into Eq. (4.4) and the result is given in Eq. (5.15).

For the fourth solution in the text, the substitution of Eq. (5.17) into Eqs. (4.8), (4.20), and (4.21) leads to

$$\left[\gamma-\frac{N-1}{N-2}\mu\right]G_4+\frac{N-1}{N-2}\mu\gamma G_2=0\,, \tag{B7}$$

$$G_4+\gamma H_4+\frac{N-1}{N-2}\mu\left[\frac{\mu}{\mu+1}+G_2+\gamma H_2-G_4\right]+\frac{\gamma(\gamma-\mu)}{\mu+1}=0\,, \tag{B8}$$

$$G_5+\left[2+\gamma+\left[2-\frac{N-1}{N-2}\right]\mu\right]H_5+H_2+\frac{N-1}{N-2}\mu[G_3+(2\mu+2+\gamma)H_3]+\frac{2b}{\mu+1}\left[\gamma(\gamma-\mu)+\frac{N-1}{N-2}\mu^2\right]=0\,, \tag{B9}$$

$$\left[2+\gamma+\left[2-\frac{N-1}{N-2}\right]\mu\right]G_5+G_2-\frac{\mu}{\mu+1}+\frac{N-1}{N-2}\mu(2\mu+2+\gamma)G_3+2b\left[\gamma(\gamma-\mu)+\frac{N-1}{N-2}\mu^2\right]=0\,, \tag{B10}$$

$$c_s{}^2G_4+(1-c_s{}^2)\gamma(G_0+\gamma)/(\mu+1)-\gamma G_2=0\,, \tag{B11}$$

$$-(G_2+H_2)+c_s{}^2\left[-\frac{\mu}{\mu+1}+G_4\right]+\frac{1-c_s{}^2}{\mu+1}\left[G_0+\gamma H_0-\frac{\gamma(\gamma-\mu)}{\mu+1}\right]=0\,, \tag{B12}$$

$$G_3(2\mu+2+\gamma)=c_s{}^2(-2\mu b+G_5)+\frac{1-c_s{}^2}{\mu+1}[(2\gamma+2+\gamma)G_1+b\gamma(2\mu-\gamma)]\,, \tag{B13}$$

$$G_3+(2\mu+2+\gamma)H_3=c_s{}^2\left[\frac{\mu b}{\mu+1}+H_5\right]+\frac{1-c_s{}^2}{\mu+1}[G_1+(2\mu+2+\gamma)H_1-b\mu\gamma/(\mu+1)], \tag{B14}$$

$$G_4=\mu\gamma(G_0+\gamma)/(\mu^2-1)\,, \tag{B15}$$

$$(\mu+1)G_0+\mu\gamma H_0+(1-\mu^2)H_4+3\gamma+\mu(\mu-1)+\gamma(\gamma-\mu)\frac{1-\mu}{1+\mu}=0\,, \tag{B16}$$

$$G_0+(2\mu+\gamma+2)(2\mu+2+\gamma\mu)G_1+\gamma(1-\mu^2)G_5-2\mu+b\{\gamma^2[6(\mu+1)+\mu\gamma+2(1-\mu)(\gamma-\mu)]-2\mu\gamma(1-\mu^2)\}=0\,, \tag{B17}$$

$$(\mu+1)(\gamma+4)G_1+[(2\mu+1+\gamma)(2\mu+3+\gamma\mu)+1-\gamma+\mu]H_1+H_0$$
$$+\gamma(1-\mu^2)H_5+b\gamma\left[\gamma\left[\mu-5-\frac{\gamma-\mu}{\mu+1}\right]+(1-\mu)(2\gamma+\mu)\right]=0\,. \tag{B18}$$

Equations (B7), (B11), and (B15) for G_0, G_2, and G_4 are the same as Eqs. (B1), (B3), and (B5) in the third solution, but in this case G_0, G_2, and G_4 cannot be arbitrary. The reason is that they are included in Eqs. (B8), (B12), and (B16) for H_0, H_2, and H_4, which are not independent, and the consistency condition determines uniquely the values of G_0, G_2, and G_4. They are given by Eq. (5.18) which is the same as Eq. (5.14). Since Eqs. (B8), (B12), and (B16) are not independent in the present case, one of H_0, H_2, and H_4 can be arbitrarily given. If we specify H_0 as

$$H_0=\left[-\mu(\mu-1)+2\gamma+\frac{N-1}{N-2}\left[\gamma(\gamma+2)(\mu-1)+\mu-\gamma-3+2\frac{\gamma+1}{\mu+1}\right]\right]\Big/\left[\gamma(\mu+1)+\frac{N-1}{N-2}\right]\,, \tag{B19}$$

then H_2 and H_4 are determined from Eqs. (B8) and (B12), and furthermore G_i and H_i ($i=1$, 3, and 5) are determined from Eqs. (B9), (B10), (B13)—(B15), (B17), and (B18).

For ϵ_m we obtain Eq. (5.19) substituting Eq. (5.17) with the specified values of G_i and H_i into Eq. (4.4).

2. $(k\tau)^2 \gg 1$

For the oscillatory solution 1 with Eq. (5.22) we get from Eq. (4.21a)

$$\ddot{\bar{\xi}}+2ik\dot{\bar{\xi}}+i\dot{k}\bar{\xi}+\{3(N-2)\dot{\alpha}-\lambda(1-c_s{}^2)/[(N-2)\dot{\alpha}-\dot{\beta}_{(k)}]\}(\dot{\bar{\xi}}+ik\bar{\xi})$$
$$+\{(N-2)[\ddot{\alpha}-2(N-2)\dot{\alpha}^2]-\lambda\dot{\alpha}(N-2)(1-c_s{}^2)/[(N-2)\dot{\alpha}-\dot{\beta}_{(k)}]\}\bar{\xi}+k(1-c_s{}^2)\lambda\bar{\Phi}_2=0\ . \qquad \text{(B20)}$$

If we substitute Eqs. (5.42) and (5.25) into Eq. (B20), we obtain

$$ik\left[2\dot{\bar{\xi}}_1+\left[-\dot{\beta}_{(k)}+3(N-2)\dot{\alpha}-\frac{(1-c_s{}^2)\lambda}{(N-2)\dot{\alpha}-\dot{\beta}_{(k)}}\right]\bar{\xi}_1\right]+k(1-c_s{}^2)\lambda\bar{\Phi}_{21}$$
$$+\tfrac{1}{2}[-(N-2)\ddot{\alpha}+\tfrac{1}{2}\dot{\beta}_{(k)}{}^2-(N-2)\dot{\alpha}\dot{\beta}_{(k)}-\tfrac{1}{2}(N-2)^2\dot{\alpha}^2+(1-c_s{}^2)\lambda]\bar{\xi}_0=0\ , \qquad \text{(B21)}$$

where $\dot{k}=-\dot{\beta}_{(k)}k$ and $\gamma=[(N-1)/(N-2)](1-c_s{}^2)$.

From Eq. (4.22) we obtain, on the other hand,

$$-k^2(1-c_s{}^2)\bar{\Phi}_2+2ik\dot{\bar{\Phi}}_2+i\dot{k}\Phi_2+[(N-2)\dot{\alpha}+\gamma\dot{\beta}_{(k)}]ik\bar{\Phi}_2=Q-R\ , \qquad \text{(B22)}$$

where

$$Q\equiv-k(1-c_s{}^2)\left[ik\bar{\xi}+\dot{\bar{\xi}}+(N-2)\left[\dot{\alpha}+\frac{N-1}{(N-2)^2}\dot{\beta}_{(k)}\right]\bar{\xi}\right]\Big/[(N-2)\dot{\alpha}-\dot{\beta}_{(k)}]\ , \qquad \text{(B23)}$$

$$R\equiv\ddot{\bar{\Phi}}_2+[(N-2)\dot{\alpha}+\gamma\dot{\beta}_{(k)}]\dot{\bar{\Phi}}_2$$
$$+\left[(\gamma-1)\dot{\beta}_{(k)}{}^2-\tfrac{1}{2}(1-c_s{}^2)\sum_c\dot{\beta}_{(c)}{}^2+(N-2)^2(\gamma/2-1)\dot{\alpha}^2+(N-2)(2-\gamma)\dot{\alpha}\dot{\beta}_{(k)}\right.$$
$$\left.+\gamma\dot{\beta}_{(k)}\lambda(1-c_s{}^2)/[(N-2)\dot{\alpha}-\dot{\beta}_{(k)}]\right]\bar{\Phi}_2$$
$$-i\gamma\dot{\beta}_{(k)}[(N-2)\dot{\alpha}-\dot{\beta}_{(k)}]^{-1}\{-3(N-2)\dot{\alpha}+2\dot{\beta}_{(k)}+[\lambda(1-c_s{}^2)-(N-2)(\ddot{\alpha}+\dot{\alpha}\dot{\beta}_{(k)})]/[(N-2)\dot{\alpha}-\dot{\beta}_{(k)}]\}\xi\ . \qquad \text{(B24)}$$

Substituting Eqs. (5.24) and (5.25) into Eq. (B22) we get

$$k(1-c_s{}^2)\{\bar{\Phi}_{21}-i\bar{\xi}_1/[(N-2)\dot{\alpha}-\dot{\beta}_{(k)}]\}-[(N-2)\dot{\alpha}-\dot{\beta}_{(k)}]^{-1}$$
$$\times\left\{2(N-2)\dot{\alpha}+2\frac{(N-2)(\ddot{\alpha}+\dot{\alpha}\dot{\beta}_{(k)})}{(N-2)\dot{\alpha}-\dot{\beta}_{(k)}}-\gamma\dot{\beta}_{(k)}+\tfrac{1}{2}(1-c_s{}^2)\left[\left[1+2\frac{N-1}{N-2}\right]\dot{\beta}_{(k)}-(N-2)\dot{\alpha}\right]\right\}\bar{\xi}_0=0\ . \qquad \text{(B25)}$$

Eliminating $\bar{\Phi}_{21}$ from Eqs. (B21) and (B25), we get

$$\bar{\xi}_1=\tfrac{1}{2}i\bar{\xi}_0\int d\tau\, k^{-1}M\ ,$$
$$=\tfrac{1}{2}i\bar{\xi}_0\left[-\frac{c_1}{\mu+1}\tau^{-\mu-1}-\frac{c_1\mu+c_2}{\mu+1-\gamma}\tau^{-\mu-1+\gamma}\right]\ , \qquad \text{(B26)}$$

where

$$M\equiv-\tfrac{1}{2}(N-2)\ddot{\alpha}+\tfrac{1}{4}\dot{\beta}_{(k)}{}^2-\tfrac{1}{2}(N-2)\dot{\alpha}\dot{\beta}_{(k)}-\tfrac{1}{4}(N-2)^2\dot{\alpha}^2+\tfrac{1}{2}(1-c_s{}^2)\lambda$$
$$-\frac{\lambda}{(N-2)\dot{\alpha}-\dot{\beta}_{(k)}}\left\{-2(N-2)\dot{\alpha}+\gamma\dot{\beta}_{(k)}-\frac{2(N-2)(\ddot{\alpha}+\dot{\alpha}\dot{\beta}_{(k)})}{(N-2)\dot{\alpha}-\dot{\beta}_{(k)}}-(1-c_s{}^2)\left[\left[\frac{1}{2}+\frac{N-1}{N-2}\right]\dot{\beta}_{(k)}-\tfrac{1}{2}(N-2)\dot{\alpha}\right]\right\}$$
$$\equiv(c_1+c_2\tau^\gamma)\tau^{-2}\ . \qquad \text{(B27)}$$

It is found that c_1 and c_2 are reduced to

$$c_1=\tfrac{1}{4}(\mu+1)^2\ , \qquad \text{(B28)}$$

$$c_2/\gamma=-\tfrac{1}{2}[\gamma(\gamma-\mu)+\mu(\mu+1)]-\gamma(\mu^2-1)+\frac{N-1}{N-2}(\mu-1)\left[-2\gamma+2\frac{\mu-1}{\mu+1}(\gamma-1)-2(\gamma-\mu)/(\mu+1)\right]\ . \qquad \text{(B29)}$$

Then from Eq. (B25)

$$\overline{\Phi}_{21}=-\frac{1}{2}\left[1-\frac{c_1}{(\mu+1)^2}\right]\tau^{-3\mu/2-3/2}[1+O(\tau^\gamma)] \tag{B30}$$

and from Eq. (4.20)

$$\overline{\Phi}_{11}=-\frac{i}{2}\frac{c_1}{(\mu+1)^2}\tau^{-3\mu/2-3/2}[1+O(\tau^\gamma)] \,. \tag{B31}$$

For ϵ_m we use $\overline{\epsilon}_m \equiv \epsilon_m/\exp(i\int k\,d\tau)$. Then from Eq. (4.4)

$$e^{2\alpha}\kappa E\overline{\epsilon}_m=[(N-2)\dot{\alpha}-\dot{\beta}_{(k)}]k\overline{\Phi}_2-\left[(N-1)\dot{\alpha}\dot{\beta}_{(k)}-\sum_c \dot{\beta}_c{}^2\right]\overline{\Phi}_1-[ik\overline{\xi}+\dot{\overline{\xi}}+(2N-3)\dot{\alpha}\overline{\xi}] \,. \tag{B32}$$

If we eliminate $k\overline{\Phi}_2$ in Eq. (B32) by use of Eq. (B22), we obtain

$$e^{2\alpha}\kappa E\overline{\epsilon}_m=-\left[(N-1)\dot{\alpha}\dot{\beta}_{(k)}-\sum_c \dot{\beta}_{(c)}{}^2\right]\overline{\Phi}_1-\left[(N-1)\dot{\alpha}-\frac{N-1}{N-2}\dot{\beta}_{(k)}\right]\overline{\xi}$$
$$+\frac{(N-2)\dot{\alpha}-\dot{\beta}_{(k)}}{1-c_s{}^2}i\{2\dot{\overline{\Phi}}_2-\dot{\beta}_{(k)}\overline{\Phi}_2+[(N-2)\dot{\alpha}+\gamma\dot{\beta}_{(k)}]\overline{\Phi}_2\}+\frac{(N-2)\dot{\alpha}-\dot{\beta}_{(k)}}{k(1-c_s{}^2)}R \,. \tag{B33}$$

Then substituting Eq. (B26), (B30), and (B31), we find that

$$\epsilon_m=h_1\tau^{-3\mu/2-3/2-\gamma}+h_2\tau^{-3\mu/2-3/2} \,, \tag{B34}$$

where

$$h_1=0$$

and (B35)

$$h_2=\text{const}(\neq 0) \,.$$

The counterpart in the paper of Perko, Matzner, and Shepley can easily be derived by assuming the solution in the form

$$\delta=\overline{\delta}\exp\left[iK\int F^{1/2}dt\right] \,,$$

$$\eta=\overline{\eta}\exp\left[iK\int F^{1/2}dt\right]$$

in the case $K^2Ft^2 \gg 1$. Here ϵ_m and $\overline{\epsilon}_m$ are equal to δ and $\overline{\delta}$ in the synchronous gauge. From their Eqs. (5.4) and (5.5) we obtain

$$\overline{\delta}=-\frac{i}{3}(s_1-s_2)\overline{\eta}/(KF^{1/2}t)$$

and

$$2\partial\overline{\eta}/\partial t+\left[\frac{1}{2}\frac{\partial F/\partial t}{F}+\frac{1}{t}\right]\overline{\eta}=0 \,.$$

Then the solution is

$$\overline{\eta}\propto t^{-1/2}F^{-1/4},\quad \overline{\delta}\propto t^{-3/2}F^{-3/4} \,.$$

Noticing that $F\propto t^{-2/3}(1+s_k)$, $dt=e^\alpha d\tau$, and $2\mu=-s_k$, it is found that

$$\overline{\epsilon}_m=\overline{\delta}\propto t^{-(s_k-2)/2}$$
$$\propto t^{-(\mu+1)}\propto\tau^{-3(\mu+1)/2} \,. \tag{B36}$$

The final expression (B36) is consistent with (B34) and (B35).

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