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Truncation of the operator-product expansion for the $\langle q\bar{q} \rangle$ -condensate component of the quark mass

V. Elias and T. Steele

Department of Applied Mathematics, University of Western Ontario, London, Ontario, Canada N6A5B9

M. D. Scadron

Physics Department, University of Arizona, Tucson, Arizona 85721

R. Tarrach

Department de Física Teòrica, Universitat Autònoma de Barcelona, Bellaterra, Barcelona, Spain

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Quark-condensate contributions to m_{dyn} , the quark mass generated by the dynamical breakdown of chiral symmetry, are shown to arise from only the first two terms of the operator-product expansion (OPE) for the appropriate $O(g^2)$ nonperturbative vacuum expectation value. Since the $m/|p|$ expansion parameter of the OPE is unity on the gauge-invariant $p = m_{\text{dyn}}$ propagator pole, such truncation of OPE contributions is an essential support to the calculation of m_{dyn} .

A persistent problem in elementary-particle physics is to find field-theoretical support for the discrepancy between the small nonstrange "current"-quark masses, motivated by PCAC (partial conservation of axial-vector current) and deep-inelastic scattering, and the relatively large ($\sim m_{\text{nucleon}}/3$) constituent-quark masses, motivated most convincingly by baryonic magnetic moments. To account for this discrepancy, a large component of the nonstrange-quark mass must be dynamical in origin, dominating the quark mass near the $m_{\text{nucleon}}/3$ "mass shell" and falling off sharply with p^2 away from this mass shell.¹

When expressed in terms of condensates arising from nonperturbative vacuum expectation values of quark and gluon fields, dynamical contributions to the quark mass necessarily involve inserting *all* orders of the operator-product expansion (expansion parameter $m/|p|$) into

each order of perturbation theory (expansion parameter g). For example, order- g^2 quark-condensate contributions to the dynamical quark mass¹⁻³ are obtained by substituting the short-distance operator-product expansion (OPE) for the nonperturbative vacuum expectation value $\langle 0 | \psi(y) \bar{\psi}(z) : | 0 \rangle^{\text{NP}}$ into the quark self-energy (Fig. 1). These contributions are expressed in terms of gauge-invariant condensates through use of the Fock-Schwinger gauge (FSG) condition⁴

$$x_\mu B^\mu(x) = 0, \quad (1)$$

in which case derivatives within the Taylor-series expansion about $y = z = 0$ for the nonperturbative matrix element may be replaced with covariant derivatives evaluated at zero:⁵

$$\begin{aligned} \langle 0 | : \psi_n^\alpha(y) \bar{\psi}_p^\beta(z) : | 0 \rangle &= \langle 0 | : \psi_n^\alpha(0) \bar{\psi}_p^\beta(0) : | 0 \rangle + y^\omega \langle 0 | : [\partial_\omega \psi_n^\alpha(0)] \bar{\psi}_p^\beta(0) : | 0 \rangle \\ &+ z^\omega \langle 0 | : \psi_n^\alpha(0) [\partial_\omega \bar{\psi}_p^\beta(0)] : | 0 \rangle + \frac{1}{2} y^{\omega_1} y^{\omega_2} \langle 0 | : [\partial_{\omega_1} \partial_{\omega_2} \psi_n^\alpha(0)] \bar{\psi}_p^\beta(0) : | 0 \rangle \\ &+ \frac{1}{2} z^{\omega_1} z^{\omega_2} \langle 0 | : \psi_n^\alpha(0) [\partial_{\omega_1} \partial_{\omega_2} \bar{\psi}_p^\beta(0)] : | 0 \rangle + y^{\omega_1} z^{\omega_2} \langle 0 | : [\partial_{\omega_1} \psi_n^\alpha(0)] [\partial_{\omega_2} \bar{\psi}_p^\beta(0)] : | 0 \rangle + \dots \\ &\xrightarrow[\text{FSG}]{\partial^\alpha \rightarrow D^\alpha(0)} -\delta^{\alpha\beta} \left\{ \frac{\delta_{nr} \langle 0 | : \bar{\psi}_a^\rho(0) \psi_a^\rho(0) : | 0 \rangle}{12} + \frac{[\gamma \cdot (y-z)]_{nr}}{48} \langle 0 | : \bar{\psi}_a^\rho(0) \not{D}_{ab}^\rho(0) \psi_b^\rho(0) : | 0 \rangle \right. \\ &\quad \left. + \frac{1}{2} (y^{\omega_1} y^{\omega_2} + z^{\omega_1} z^{\omega_2} - 2y^{\omega_1} z^{\omega_2}) \langle 0 | : \bar{\psi}_p^\rho(0) D_{\omega_1}^{\rho\sigma}(0) D_{\omega_2}^{\sigma\tau}(0) \psi_n^\tau(0) : | 0 \rangle + \dots \right\} \\ &= -\delta^{\alpha\beta} \{ [\delta_{nr}/12 - im_{\text{dyn}} [\gamma \cdot (y-z)]_{nr}/48 + [-m_{\text{dyn}}^2 (y-z)^2 \delta_{nr}/32 + O(m_{\text{dyn}}^3)] \} \langle \bar{\psi} \psi \rangle \\ &\quad + [-i(y-z)^2 \delta_{nr}/64 - y_\alpha z_\beta \sigma_{nr}^{\alpha\beta}/96 + O(m_{\text{dyn}})] \langle 0 | : \bar{\psi}_a^\alpha(0) F_{\mu\nu}^I(0) (ig\lambda_{ab}^I/2) \sigma_{ab}^{\mu\nu} \psi_b^\beta(0) : | 0 \rangle \\ &\quad + \text{higher-dimensional condensates} \} \end{aligned} \quad (2)$$

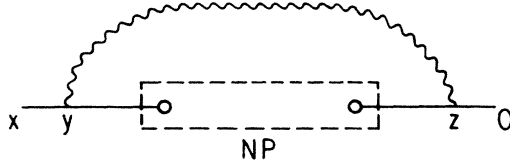


FIG. 1. The graph for $O(g^2\langle\bar{\psi}\psi\rangle)$ contributions to the dynamical component of the quark mass.

Insertion into Fig. 1 of the OPE terms in (2), involving at most one covariant derivative (i.e., those terms proportional to $\langle\bar{\psi}\psi\rangle$ containing at most one power of m_{dyn}), leads to an order- g^2 correction to the fermion propagator which is independent of the gauge parameter (a) at the $\not{p} = m_{\text{dyn}}$ pole position:³

$$\Delta S_F(p) = \frac{g^2|\langle\bar{\psi}\psi\rangle|}{9(p^2)^2} [(3+a) - am_{\text{dyn}}\not{p}/p^2]. \quad (3)$$

This on-shell gauge independence is not merely a coincidence; it is also seen to occur (trivially) in gluon-gluon-condensate contributions³ to $\Delta S_F(p)$, and (nontrivially) in mixed quark-antiquark-gluon-condensate contributions⁶ to $\Delta S_F(p)$.

Equation (3) can therefore be used to show that the quark mass obtained from chiral-symmetry breakdown is

$$m_{\text{dyn}} = |g^2\langle\bar{\psi}\psi\rangle/3|^{1/3}, \quad (4)$$

$$\begin{aligned} [\Delta S_F(p)]_{ij}^{\sigma} &= -i \int d^4x e^{ip \cdot x} \left\langle T \psi_i^{\sigma}(x) \bar{\psi}_j^{\sigma}(0) \left[-\frac{1}{2} \int d^4y \int d^4z \mathcal{L}_{\text{int}}(y) \mathcal{L}_{\text{int}}(z) \right] \right\rangle_0 \\ &= \frac{ig^2}{4} \int d^4(x-y) e^{ip \cdot (x-y)} \langle T \psi_i^{\rho}(x) \bar{\psi}_k^{\lambda}(y) \rangle_0 \lambda_{\tau\alpha}^a \gamma_{kn}^{\mu} \left[\int d^4(y-z) e^{ip \cdot (y-z)} \langle : \psi_n^{\alpha}(y) \bar{\psi}_r^{\beta}(z) : \rangle_0^{\text{NP}} (TB_{\mu}^a(y) B_{\nu}^b(z)) \right]_0 \\ &\quad \times \gamma_{rs}^{\nu} \lambda_{\beta\eta}^b \int d^4z e^{ip \cdot z} \langle T \psi_s^{\eta}(z) \bar{\psi}_j^{\sigma}(0) \rangle_0. \end{aligned} \quad (5)$$

The nonperturbative vacuum expectation value within the large parentheses on the right-hand side (RHS) of (5) is precisely the same as the expression (2). Let us first consider within (5) the contributions to (2) arising from two covariant derivatives:

$$- \left[\frac{y^{\omega_1} y^{\omega_2} + z^{\omega_1} z^{\omega_2} - 2y^{\omega_1} z^{\omega_2}}{2} \right] \langle 0 | : \bar{\psi}_r(0) [D_{\omega_1}(0) D_{\omega_2}(0) \psi_n(0)] : | 0 \rangle = m_{\text{dyn}}^2 \delta_{nr} (y-z)^2 \langle \bar{\psi}\psi \rangle / 32 + \dots \quad (6)$$

[The dots in (6) represent a mixed quark-antiquark-gluon condensate contribution; color indices in (6) are suppressed.] Upon insertion into the large parentheses on the RHS of (5), the quark condensate component of (6) leads to a contribution proportional to the following integral over $(y-z)$:

$$\begin{aligned} &\left[\int \frac{d^4(y-z)}{(2\pi)^4} e^{ip \cdot (y-z)} (y-z)^2 \int \frac{d^4k}{(k^2)^2} e^{-ik \cdot (y-z)} [-g_{\mu\nu} k^2 + (1-a) k_{\mu} k_{\nu}] \right] \\ &= - \int \frac{d^4k}{(k^2)^2} [-g_{\mu\nu} k^2 + (1-a) k_{\mu} k_{\nu}] \frac{\partial}{\partial p^{\omega}} \frac{\partial}{\partial p^{\omega}} \int \frac{d^4(y-z)}{(2\pi)^4} e^{i(y-z) \cdot (p-k)} \\ &= - \frac{\partial}{\partial p} \cdot \frac{\partial}{\partial p} \left[\frac{[-g_{\mu\nu} p^2 + (1-a) p_{\mu} p_{\nu}]}{(p^2)^2} \right] = \frac{2(1-a)}{(p^2)^3} (g_{\mu\nu} p^2 - 4p_{\mu} p_{\nu}). \end{aligned} \quad (7)$$

The final expression of (7) vanishes upon contraction into the γ_{μ} and γ_{ν} adjacent to the large parentheses on the RHS in (5). Consequently, $O(m^2/p^2)$ contributions to $\Delta S_F(p)$ proportional to $\langle\bar{\psi}\psi\rangle$ do not occur.³

We now consider the contributions to (2) arising from terms involving three covariant derivatives:

$$\begin{aligned} &\left[- \left[\frac{y^{\omega_1} y^{\omega_2} y^{\omega_3} - z^{\omega_1} z^{\omega_2} z^{\omega_3} + 3z^{\omega_1} z^{\omega_2} y^{\omega_3} - 3z^{\omega_1} y^{\omega_2} y^{\omega_3}}{6} \right] \langle 0 | : \bar{\psi}_r(0) [D_{\omega_1}(0) D_{\omega_2}(0) D_{\omega_3}(0) \psi_n(0)] : | 0 \rangle \right] \\ &\quad \sim m_{\text{dyn}}^3 (y-z)^2 \gamma_{nr}^{\omega} (y-z)_{\omega} \langle \bar{\psi}\psi \rangle + \dots \quad (8) \end{aligned}$$

regardless of gauge.^{3,7} True confidence in this result (for which reasonable phenomenological support exists⁷), however, is contingent on there being no further alterations from OPE terms in (2) arising from two or more covariant derivatives. Note that the “one-covariant-derivative term” in (2) led to a contribution to $\Delta S_F(p)$ involving a factor of $m_{\text{dyn}}\not{p}/p^2$ relative to the lead term in (3). This factor exhibits $m/|p|$ as the expansion parameter in momentum space of the OPE given by (2), a standard OPE property. The second-derivative terms in (2) are therefore expected to yield contributions to $\Delta S_F(p)$ comparable to (m_{dyn}^2/p^2) times the lead term. Moreover, third-derivative contributions should differ in magnitude from the lead term by $m_{\text{dyn}}^3\not{p}/(p^2)^2$, etc.

This $m\not{p}/p^2$ expansion parameter is necessarily unity on the $\not{p} = m_{\text{dyn}}$ mass shell. Consequently, the only way to extract a meaningful dynamical quark mass from (2) is to show that the full set of higher-order OPE contributions to $\Delta S_F(p)$ truncates to a finite number of terms. We prove below that such truncation does indeed occur. Specifically, we show that no nonvanishing $O(g^2\langle\bar{\psi}\psi\rangle)$ contributions to $\Delta S_F(p)$ occur from OPE terms in (2) arising from two or more covariant derivatives.

To obtain this result, we first note that Fig. 1 corresponds to the second-order interaction-picture corrections to the fermion propagator generated from the quark-quark-gluon interaction of QCD:

Substitution of (8) into the nonperturbative vacuum expectation value within the large parentheses on the RHS of (5) leads to the following integral over $(y - z)$:

$$\left[\gamma_{nr}^{\omega} \int \frac{d^4(y-z)}{(2\pi)^4} e^{ip \cdot (y-z)} (y-z)^2 (y-z)_{\omega} \int \frac{d^4 k}{(k^2)^2} e^{-ik \cdot (y-z)} [-g_{\mu\nu} k^2 + (1-a) k_{\mu} k_{\nu}] \right. \\ \left. - i \left[\gamma \cdot \frac{\partial}{\partial p} \right]_{nr} \left[\frac{\partial}{\partial p} \cdot \frac{\partial}{\partial p} \right] \left[\frac{-g_{\mu\nu} p^2 + (1-a) p_{\mu} p_{\nu}}{(p^2)^2} \right] - 8i(1-a)(p^2)^{-3} (-\not{p} g_{\mu\nu} - \gamma_{\mu} p_{\nu} - \gamma_{\nu} p_{\mu} + 6p_{\mu} p_{\nu} \not{p} / p^2)_{nr} \right] \quad (9)$$

The result (9) vanishes when contracted into the γ^{μ} and γ^{ν} on opposite sides of the large parentheses on the RHS in (5):

$$\gamma^{\mu} \left[\gamma \cdot \frac{\partial}{\partial p} \right] \left[\frac{\partial}{\partial p} \cdot \frac{\partial}{\partial p} \right] \left[\frac{[-g_{\mu\nu} p^2 + (1-a) p_{\mu} p_{\nu}]}{(p^2)^2} \right] \gamma^{\nu} = 0. \quad (10)$$

Consequently, the OPE terms arising from three covariant derivatives within (2) also fail to contribute to $\Delta S_F(p)$, and $O(m^3 \not{p} / p^4)$ contributions to $\Delta S_F(p)$ are similarly absent.

One can now see easily that no higher-than-first-derivative terms within (2) contribute to $\Delta S_F(p)$. Consider, for example, the n th-derivative contribution to (2), where $n \geq 2$:

$$\left[- \left[\frac{y^{\omega_1} y^{\omega_2} \cdots y^{\omega_n} - n z^{\omega_1} y^{\omega_2} \cdots y^{\omega_n} + \cdots + (-1)^n z^{\omega_1} z^{\omega_2} \cdots z^{\omega_n}}{n!} \right] \langle 0 | : \bar{\psi}_r(0) [D_{\omega_1}(0) D_{\omega_2}(0) \cdots D_{\omega_n}(0) \psi_n(0)] : | 0 \rangle \right] \\ \sim [\gamma \cdot (y-z)]^n (-im_{\text{dyn}})^n \langle \bar{\psi} \psi \rangle + \cdots \quad (11)$$

[Equation (11) is easily obtained as a straightforward generalization of (6) and (8).] Upon substituting (11) for the nonperturbative vacuum expectation value in (5), one obtains a result proportional to

$$\gamma^{\mu} \left[\int \frac{d^4(y-z)}{(2\pi)^4} e^{ip \cdot (y-z)} [\gamma \cdot (y-z)]^n \int d^4 k e^{-ik \cdot (y-z)} \left[\frac{-g_{\mu\nu} k^2 + (1-a) k_{\mu} k_{\nu}}{(k^2)^2} \right] \right] \gamma^{\nu} \\ \sim \gamma^{\mu} \left[\gamma \cdot \frac{\partial}{\partial p} \right]^n \left[\frac{-g_{\mu\nu} p^2 + (1-a) p_{\mu} p_{\nu}}{(p^2)^2} \right] \gamma^{\nu}. \quad (12)$$

If n is even, the right-hand side of (12) is seen to vanish via the last line of (7):

$$\gamma^{\mu} \left[\gamma \cdot \frac{\partial}{\partial p} \right]^n \left[\frac{-g_{\mu\nu} p^2 + (1-a) p_{\mu} p_{\nu}}{(p^2)^2} \right] \gamma^{\nu} = \left[\frac{\partial}{\partial p} \cdot \frac{\partial}{\partial p} \right]^{(n-2)/2} \left[\gamma^{\mu} \left[\frac{\partial}{\partial p} \cdot \frac{\partial}{\partial p} \right] \left[\frac{-g_{\mu\nu} p^2 + (1-a) p_{\mu} p_{\nu}}{(p^2)^2} \right] \gamma^{\nu} \right] = 0. \quad (13)$$

If n is odd, the right-hand side of (12) is seen to vanish via (10):

$$\gamma^{\mu} \left[\gamma \cdot \frac{\partial}{\partial p} \right]^n \left[\frac{-g_{\mu\nu} p^2 + (1-a) p_{\mu} p_{\nu}}{(p^2)^2} \right] \gamma^{\nu} = \left[\frac{\partial}{\partial p} \cdot \frac{\partial}{\partial p} \right]^{(n-3)/2} \left[\gamma^{\mu} \left[\gamma \cdot \frac{\partial}{\partial p} \right] \left[\frac{\partial}{\partial p} \cdot \frac{\partial}{\partial p} \right] \left[\frac{-g_{\mu\nu} p^2 + (1-a) p_{\mu} p_{\nu}}{(p^2)^2} \right] \gamma^{\nu} \right] = 0. \quad (14)$$

Thus, there are no $O(g^2 \langle \bar{\psi} \psi \rangle)$ contributions to $\Delta S_F(p)$ from the terms within (2) arising from two or more covariant derivatives. Consequently, (3) is correct to all orders of the OPE, and the resulting expression for the on-shell quark mass [Eq. (4)] does not suffer successive order-unity corrections from higher-order OPE contributions.

It is worth noting that there appear to be no $O(g^3 \langle \bar{\psi} \psi \rangle)$ contributions to $\Delta S_F(p)$. Three-vertex self-energy graphs, as considered in Ref. 6, necessarily involve the nonperturbative vacuum expectation value $\langle 0 | : \psi(x) \bar{\psi}(y) B(z) : | 0 \rangle$. When this quantity is expanded [analogous to Eq. (2)] we see that there are no $\langle \bar{\psi} \psi \rangle$ components for lead and first-derivative OPE terms; i.e., the lowest-order condensate contributions are mixed quark-gluon condensates.

We further note that truncation of dynamical contributions to the quark mass in (2) does not appear to be restricted to the quark-antiquark condensate. In Ref. 6, the mixed quark-antiquark-gluon condensate [as in the last

line of Eq. (2)] is shown to contribute in part to the dynamical quark mass through second-order (i.e., two-derivative) OPE contributions to the nonperturbative portion of Fig. 1. However, both third- and fourth-order OPE contributions proportional to this mixed condensate vanish explicitly when inserted into Fig. 1, suggesting a truncation of mixed-condensate contributions beyond two-derivative terms of the OPE.

As a final remark concerning the $\lambda = m_{\text{dyn}} \not{p} / p^2$ expansion parameter, truncation of the OPE, upon substitution into the quark self-energy (Fig. 1), justifies application of the OPE in this $\lambda \sim 1$ region. Such "precocious asymptotic freedom" is reminiscent of the known precocious scaling in deep-inelastic processes at low q^2 ($\sim 1 \text{ GeV}^2$). Indeed, in earlier work⁷ we have regarded the renormalization point $M^2 \cong 1 \text{ GeV}^2$ as deep Euclidean, and have extrapolated down to $\lambda = 1$, $p^2 \cong (m_N/3)^2$, by dimensional arguments alone to obtain (4) from (3).

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