Lagrangian dynamics of particles and fluids with intrinsic spin in Einstein-Cartan space-time

Wojciech Kopczyński
Institute of Theoretical Physics, Warsaw University, Pl-00-681 Warsaw, Poland
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A Lagrangian formulation of equations of motion for spinning particles and fluids interacting with torsion and curvature of space-time is given. Our approach is general in the sense that it does not specify either the form of Lagrangian or the auxiliary condition. The derived energy-momentum and spin tensors have the form, which is usually postulated phenomenologically.

I. INTRODUCTION

The aim of this paper is to give a Lagrangian formulation of equations of motion for point particles and perfect fluids both carrying intrinsic spin and interacting with a gravitational field. The gravitational field is described in the framework of the Riemann-Cartan geometry; i.e., space-time is equipped with torsion and curvature. This means we are using a general approach of the Poincaré gauge theory of gravitation[1] without a specific choice of the gravitation Lagrangian (the choice of the Einstein-Cartan theory is the best known example).

The problem of these equations and their Lagrangian formulation has a long history. The list of contributors begins with Frenkel,[2] who derived the equations for a classical model of the electron in special relativity from a variational principle. As we know today, his Lagrangian has a special form and his auxiliary condition $S^{ij}u_j=0$ may not be the best choice. The perfect fluid consisting of particles carrying an internal but classical spin was introduced by Weyssenhoff and Raabe.[3] Its extensive investigation was carried out by Halbwachs,[4] who gave a Lagrangian formulation for the "Weyssenhoff fluid" equations. Meanwhile, Mathisson[5] and Papapetrou[6] derived equations for extended bodies in general relativity in the pole-dipole approximation,[7] which in the special-relativistic limit coincided with those of Frenkel. A Lagrangian formulation for particles and fluids in general relativity was given by Bailey and Israel.[8,9] Their treatment is general in two respects. First, they do not specify the functional form of the Lagrangian, but only its variables. Second, they do not specify any auxiliary condition (as, e.g., the Frenkel condition or the Tulczyjew[7,10] condition $S^{ij}P_j=0$).

In the 1970s the Weyssenhoff fluid became a basic phenomenological tool in the Einstein-Cartan theory.[11–13] Nearly all investigations of the Einstein-Cartan field equations were done under the assumption that their right-hand side is that of the Weyssenhoff fluid; i.e., the energy-momentum tensor and spin tensor have the form

$$t^i{}_j=N^iP_j+p(u^iu_j-\delta^i_j)\,, \tag{1}$$

$$s^i{}_{jk}=N^iS_{jk}\,, \tag{2}$$

respectively. The Lagrangian description of the Weyssenhoff fluid was given in 1982 by Ray and Smalley.[14] But their approach, based on a special-relativistic approach of Halbwachs, is far less general than that of Bailey and Israel in general relativity. In particular, they imposed the Frenkel auxiliary condition and postulated a relation betwen spin and angular velocity involving a nondynamical function. In effect, their matter tensors $t^i{}_j$ and $s^i{}_{jk}$ are of a more specific form than those given by the formulas (1) and (2).

In Sec. II, I consider point particles in the presence of curvature and torsion. I develop here part of the methods used in the next section, which is the core of the paper.

In Sec. III, I attempt a general Lagrangian approach to Weyssenhoff fluids in Einstein-Cartan space-time. The one-particle Lagrangian of Sec. II is an analogue of the specific Lagrangian of the perfect fluid in Sec. III. The specific entropy and the concentration of particles are included in its list of variables. It is *a priori* assumed that the entropy is constant along world lines of the fluid and the number of particles is conserved. To solve these constraints I employ the method of Lagrange coordinates.[8,9,15] Dynamical equations for the fluid are obtained from variation of world lines and orthonormal tetrads in a fixed geometry. Variation of geometrical variables leads to the Einstein-Cartan field equations (or to their generalization of a Poincaré gauge theory) with right-hand sides given by (1) and (2). Fluid equations are contained in gravitational field equations due to a set of identities valid in any Poincaré gauge theory of gravitation.

II. POINT PARTICLES

A. The Lagrangian and its variables

The variables which describe motion of a particle are (1) the world line $t\to x^i(t)$, (2) the frame $t\to\theta^a_i(t)$ defined over the world line and satisfying the orthonormality conditions

$$\eta_{ab}\theta^a_i\theta^b_j=g_{ij}\ . \tag{3}$$

Throughout the paper the indices $a,b,\ldots=0,1,2,3$ and are moved vertically by means of $\eta_{ab}=\mathrm{diag}(+1,-1,$

$-1,-1)$ and its inverse η^{ab}. The holonomic indices are $i,j,\ldots=0,1,2,3$.

The Lagrangian L is an unspecified function of

$$x^i, v^i=\frac{dx^i}{dt}, \theta^a_i, \dot\theta^a_i .$$

Above, we have introduced

$$\dot\theta^a_i = v^j\nabla_j\theta^a_i=\frac{d\theta^a_i}{dt}-\Gamma^k{}_{ij}\theta^a_k v^j ,$$

where $\Gamma^k{}_{ij}$ are connection coefficients of a metrical (i.e., $\nabla_k g_{ij}=0$) but nonsymmetric connection.

We require that L be invariant under (1) an arbitrary transformation of coordinates, and (2) a stiff transformation of the frame $\theta^a_i \to \Lambda^a_b\theta^b_i$, where Λ^a_b is a constant Lorentz matrix. These requirements lead to a simplification of the Lagrangian. Because of requirement (1), L does not depend explicitly on x^i. Moreover, instead of $\dot\theta^a_i$, we can use the angular velocity tensor

$$\omega^i_j=\dot\theta^i{}_a\theta^a_j$$

which, due to Eq. (3), is skew symmetric $\omega_{ij}=-\omega_{ji}$. Now, since two variables of the Lagrangian v^i and $\omega^i{}_j$ do not carry an a index, in order to satisfy requirement (2), θ^a_i can enter into L in the combination (3) only. Therefore

$$L=L(v^i,\omega^i{}_j,g_{ij}) .$$

We require (3) that the action $A=\int_{t_0}^{t_1} L\,dt$ be invariant under arbitrary transformation of the parameter t. This leads to the last restriction on the Lagrangian:

$$L(\alpha v^i,\alpha\omega^i{}_j,g_{ij})=\alpha L(v^i,\omega^i{}_j,g_{ij}) \tag{4}$$

for any $\alpha>0$. Therefore

$$A=\int L(u^i,\tau^i{}_j,g_{ij})ds ,$$

where u^i and $\tau^i{}_j$ are linear and angular velocities, respectively, in terms of a parametrization by the proper time s.

The variation

$$\delta L=-P_i\delta v^i+\tfrac{1}{2}S_i{}^j\delta\omega^i{}_j-\tfrac{1}{2}I^{ij}\delta g_{ij} \tag{5}$$

gives a definition of the momentum P_i, spin $S_i{}^j$, and the additional quantity I^{ij}. An infinitesimal coordinate transformation induces the following variations: $\delta L=0$, $\delta v^i=\epsilon^i{}_j v^j$, $\delta\omega^i{}_j=\epsilon^i{}_k\omega^k{}_j-\epsilon^k{}_j\omega^i{}_k$, $\delta g_{ij}=-\epsilon_{ij}-\epsilon_{ji}$. Substituting them into (5) and taking into account the arbitrariness of $\epsilon^i{}_j$, we obtain the identity

$$P_i v_j = I_{ij}+\tfrac{1}{2}(\omega_{kj}S^k{}_i-\omega_{ki}S^k{}_j) \tag{6}$$

which is a prototype of the decomposition of the energy-momentum tensor onto its symmetric and skew-symmetric parts.

On the other hand, the homogeneity of L [Eq. (4)] leads to the Euler identity

$$L=-E\,|v|+\tfrac{1}{2}S_i{}^j\omega^i{}_j , \tag{7}$$

where $|v|$ is defined by $v^i=|v|u^i$ and $E=P_iu^i$ is the rest energy of a particle.

B. The variational procedure

In formula (5) we have variations δ of functions and their arguments. In general, the variation $\delta f(x)=f'(x')-f(x)$ can be decomposed as

$$\delta f=\delta_0 f+\delta_1 f , \tag{8}$$

where $\delta_0 f(x)=f'(x)-f(x)$ and $\delta_1 f=\delta x^i\partial_i f$. We can substitute this decomposition of δ into (5). Moreover, because of identity (6), instead of δ_1 we can use its covariant analogue $\Delta=\delta x^i\nabla_i$. Note that an arbitrary covariant derivative ∇_i can be used here; nevertheless, we shall use that one determined by $\Gamma^i{}_{jk}$. Note also that the replacement $\delta_1\to\Delta$ is generally valid for scalar functions of tensor variables.

Following these remarks, we can write (5) as

$$\delta L=-P_i\Delta v^i+\tfrac{1}{2}S_i{}^j(\delta_0+\Delta)\omega^i{}_j-\tfrac{1}{2}p^iv^j\delta_0 g_{ij} . \tag{9}$$

To calculate the first term in (9), we need the formula

$$\Delta v^i=(\delta x^i)^{\cdot}+Q^i{}_{jk}\delta x^j v^k \tag{10}$$

which can be used as the definition of the torsion tensor $Q^i{}_{jk}$. The complementary formula

$$\Delta\dot\theta^i{}_a=(\Delta\theta^i_a)^{\cdot}+R^i{}_{jkl}\theta^j_a\delta x^k v^l$$

is needed in order to calculate

$$\begin{aligned}\Delta\omega^i{}_j&=\Delta\dot\theta^i{}_a\theta^a_j+\dot\theta^i{}_a\Delta\theta^a_j\\&=\theta^a_j(\Delta\theta^i_a)^{\cdot}+R^i{}_{jkl}\delta x^k v^l+\omega^i{}_k\theta^k_a\Delta\theta^a_j .\end{aligned} \tag{11}$$

We also need

$$\delta_0\omega^i{}_j=\theta^a_j(\delta_0\theta^i_a)^{\cdot}+\delta_0\Gamma^i{}_{jk}v^k+\omega^i{}_k\theta^k_a\delta_0\theta^a_j . \tag{12}$$

Substituting (10), (11), and (12) into (9), and rearranging the terms, we get

$$\begin{aligned}\delta L=&A_i\delta x^i+\tfrac{1}{2}B_i{}^j\theta^i_a(\delta_0+\Delta)\theta^a_j+\tfrac{1}{2}v^kS_i{}^j\delta_0\Gamma^i{}_{jk}\\&-\tfrac{1}{2}P^iv^j\delta_0 g_{ij}+\frac{dF}{dt} ,\end{aligned} \tag{13}$$

where

$$F=(-P_i+\tfrac{1}{2}S_k{}^j\theta^a_j\nabla_i\theta^k_a)\delta x^i ,$$
$$B_i{}^j=\dot S_i{}^j-\omega^j{}_kS_i{}^k+\omega^k{}_iS_k{}^j , \tag{14}$$

$$A_i=\dot P_i-(Q^k{}_{ij}P_k-\tfrac{1}{2}R^k{}_{lij}S_k{}^l)v^j . \tag{15}$$

In fact, in order to derive equations for test particles from the principle of least action we need much simpler formulas. In this case the geometry involved is kept fixed so $\delta_0\Gamma^i{}_{jk}=0$ and $\delta_0 g_{ij}=0$; thus the first two terms in (13) play an essential role. The condition

$$\delta_0 g_{ij}\equiv\theta_{ai}\delta_0\theta^a_j+\theta_{aj}\delta_0\theta^a_i=0 \tag{16}$$

gives a restriction on variation of frame; note, however, that $B_{ij}=-B_{ji}$.

The variation of frame leads therefore to the dynamical equation

$$\dot S_i{}^j-\omega^j{}_kS_i{}^k+\omega^k{}_iS_k{}^j=0 \tag{17}$$

which can be represented in a more familiar form

$$\dot{S}^{ij}=P^i v^j - P^j v^i \tag{17'}$$

if we use the identity (6). Another interesting form of these equations is

$$\frac{dS^{ab}}{dt}=0 \tag{17''}$$

which shows that the spin tensor is constant relative to the frame θ_i^a.

The variation of world line leads to the equation

$$A_i=0 \; . \tag{18}$$

The term in (13) containing $\Delta\theta_j^a$ vanishes if either we impose the condition of parallel transport[8,9] or we use the dynamical equation (17).

Equations (17) and (18) are fully deterministic. In order to solve them, we do not need an auxiliary condition, but rather a form of the Lagrangian. If it is given, the initial conditions should completely determine the time development of the particle.

III. PERFECT FLUIDS

A. Test fluids

To describe motion of a perfect spinning fluid, we shall use its Lagrange description. The dynamical variables are (1) the three-dimensional family of world lines $(t,y^\alpha)\equiv y^\mu \to x^i(y^\mu)$ and (2) the orthonormal frames $y^\mu \to \theta_i^a(y^\mu)$. The mapping $y^\mu \to x^i$ is assumed to be a diffeomorphism. The indices of the Lagrange-type coordinates are $\mu,\nu,\ldots=0,1,2,3$ and $\alpha,\beta,\ldots=1,2,3$. The velocity field is now $v^i=\partial x^i(t,y^\alpha)/\partial t$.

In addition to the dynamical variables and their derivatives, the Lagrange density function $\mathcal{L}$ will depend on the concentration of particles n and the specific entropy S (i.e., entropy per particle). The number of particles is conserved,

$$dN=0 \tag{19}$$

and the specific entropy is constant along each world line,

$$dS\wedge N=0 \, , \tag{20}$$

where we have introduced

$$N=N_i * dx^i = n u_i * dx^i$$

as a three-form of a particle current.

The Lagrange coordinates y^μ are useful for solving the constraints (19) and (20). Namely, in these coordinates

$$N=\frac{n}{|v|}J\sqrt{-g}\,dy^1\wedge dy^2\wedge dy^3 \, ,$$

where $J=\det(\partial_\mu x^i)$ and $g=\det(g_{ij})$. This gives

$$n=\frac{|v|}{J\sqrt{-g}}n_0(y^\alpha) \, , \tag{21}$$

$$S=S_0(y^\alpha) \; . \tag{22}$$

The functions n_0 and S_0 will be treated as given. The function n_0 determines a volume four-form in the space of Lagrange variables

$$\eta_0=dt\wedge N=n_0(y^\alpha)dt\wedge dy^1\wedge dy^2\wedge dy^3 \; .$$

According to Eq. (21), it is related to the standard space-time volume four-form $\eta=*1$ by the relation

$$\eta=\frac{|v|}{n}\eta_0 \; . \tag{23}$$

The action integral is

$$\int_\Omega L\eta_0=\int \mathcal{L}\eta \, ,$$

where the Lagrange density function $\mathcal{L}$ is connected to the specific Lagrangian L by the relation

$$\mathcal{L}=\frac{n}{|v|}L$$

consistently with (23).

The specific Lagrangian L will be a generalization of the one-particle Lagrangian of Sec. II. The list of its variables will be extended to include the specific entropy S and the concentration of particles n:

$$L=L(S,n,v^i,\omega^i{}_j,g_{ij}) \; .$$

We introduce temperature T and pressure p by

$$\delta L=-|v|\left[T\delta S+\frac{p}{n^2}\delta n\right]-P_i\delta v^i+\tfrac{1}{2}S_i{}^j\delta\omega^i{}_j + \tfrac{1}{2}I^{ij}\delta g_{ij} \; . \tag{24}$$

Since S and n are scalars, the identity (6) remains unchanged, in particular $I^{ij}=v^{(i}P^{j)}$. Identity (7) also holds true.

According to (22), $\delta S=0$. Thus, in order to calculate the new terms in (24), we need only

$$\frac{\delta n}{n}=\frac{\delta|v|}{|v|}-\frac{\delta J}{J}-\frac{1}{2}\frac{\delta g}{g}$$

as follows from (21). A direct calculation gives

$$\frac{\delta n}{n}=\tfrac{1}{2}h^{ij}\delta g_{ij}+u_i\frac{\delta v^i}{|v|}-\partial_i y^\mu\delta\partial_\mu x^i \, , \tag{25}$$

where

$$h^{ij}=u^i u^j-g^{ij}$$

is the metric tensor in the space perpendicular to u^i. Now, we can substitute $\delta=\delta_0+\delta_1$ into (25) and use Δ instead of δ_1 as done in Sec. II. This leads to

$$\frac{\delta n}{n}=\tfrac{1}{2}h^{ij}\delta_0 g_{ij}+u_i\frac{\delta v^i}{|v|}-\partial_i y^\mu\Delta\partial_\mu x^i \; .$$

The last term in the expression above is

$$-\nabla_i\delta x^i-Q^j{}_{ij}\delta x^i \; .$$

Thus, taking (10) into account we obtain

$$\frac{\delta n}{n}=\tfrac{1}{2}h^{ij}\delta_0 g_{ij}+h_i^j\nabla_j\delta x^i+Q^j{}_{ik}h_j^k\delta x^i \; . \tag{26}$$

In order to calculate

$$\delta(\mathscr{L}\eta)=\delta L\eta_0=\frac{n}{|v|}\delta L\eta\,, \tag{27}$$

we should now substitute (13) and (26) into (24). To write the whole expression in a relatively compact form, we introduce the three-forms

$$t_i=t_{ji}*dx^j \tag{28}$$

and

$$s_{ij}=s_{kij}*dx^k\,, \tag{29}$$

where t_{ji} and s_{kij} are given by the formulas (1) and (2). The variation (27) has the form

$$\delta(\mathscr{L}\eta)=C_i\delta x^i+\frac{n}{2}B_i{}^j\eta\theta^i_a(\delta_0+\Delta)\theta^a_j + \tfrac{1}{2}s^k{}_i{}^j\eta\delta_0\Gamma^i{}_{jk}-\tfrac{1}{2}t^{ij}\eta\delta_0 g_{ij}+d\Phi\ . \tag{30}$$

Above, we have introduced

$$\Phi=(-t_i+\tfrac{1}{2}s_k{}^j\theta^a_j\nabla_i\theta^k_a)\delta x^i$$

and

$$C_i=Dt_i-Q^j{}_i\wedge t_j+\tfrac{1}{2}R^k{}_{ji}\wedge s_k{}^j\,, \tag{31}$$

where

$$Q^j{}_i=Q^j{}_{ik}dx^k,\quad R^k{}_{ji}=R^k{}_{jil}dx^l\ .$$

We are ready now to formulate the principle of least action for test fluids. It reads $\delta\int_\Omega L\eta_0=0$ under the conditions $\delta_0 g_{ij}=0=\delta_0\Gamma^i{}_{jk}$ and

$$\delta_0\theta^a_i\,|_{\partial\Omega}=0=\delta x^i\,|_{\partial\omega}\ .$$

The Euler-Lagrange equations resulting from (30) are the equation for spin (17) which is formally the same as in the case of a single particle and the equation

$$C_i=0\ . \tag{32}$$

The last equation reduces to the one-particle Eq. (18) in the case of dust, i.e., if $p=0$. One has to draw attention to the presence of $\Delta=\delta x^i\nabla_i$ in the second term of (30). It gives a contribution to C_i, which vanishes if the equation for spin (17) is satisfied. In the case of fluids we cannot postulate $\Delta\theta^a_j=0$ as done in the case of particles, since it might be inconsistent with dynamical equations.

One can observe that Eq. (17) can be written also in the form

$$nB_i{}^j\eta\equiv Ds_i{}^j+dx_i\wedge t^j+dx^j\wedge t_i=0 \tag{33}$$

because of identity (6) and the conservation law (19).

B. The gravitational interaction

To describe the interaction of the fluid with gravity we shall introduce the total Lagrangian four-form[16]

$$\mathbf{K}+\mathscr{L}\eta\,,$$

where $\mathbf{K}$ is the gravitation Lagrangian four-form

$$\mathbf{K}=\mathbf{K}(\theta^a,D\theta^a,\Gamma^a{}_b,D\Gamma^a{}_b)\,,$$

$\theta^a=\theta^a_i dx^i$ are orthonormal frame one-forms, $\Gamma^a{}_b=\Gamma^a{}_{bc}\theta^c$ are connection one-forms, $D\theta^a=d\theta^a+\Gamma^a{}_b\wedge\theta^b$ are torsion two-forms, and $D\Gamma^a{}_b=d\Gamma^a{}_b+\Gamma^a{}_c\wedge\Gamma^c{}_b$ are curvature two-forms. The best known choice for $\mathbf{K}$ is that of the Einstein-Cartan theory $\mathbf{K}=\frac{1}{2}*(\theta_a\wedge\theta^b)\wedge D\Gamma^a{}_b$. We shall not use this particular form, however.

The variation

$$\delta_0\mathbf{K}=-\delta_0\theta^a\wedge e_a+\tfrac{1}{2}\delta_0\Gamma^a{}_b\wedge c^b_a+\text{ an exact form} \tag{34}$$

introduces the Einstein three-form e_a and the Cartan three-form c^b_a.

The total variation δ of an exterior form ω [which does not depend explicitly on derivatives of $y^\mu\rightarrow x^i(y^\mu)$] is

$$\delta\omega=\delta_0\omega+\mathscr{L}_{\delta x}\omega\,, \tag{35}$$

where $\mathscr{L}_{\delta x}$ is the Lie derivative with respect to the vector field $\delta x=\delta x^j\partial_j$. Equation (35) is a generalization of (8). In the case of $\mathbf{K}$ it gives

$$\delta\mathbf{K}=\delta_0\mathbf{K}+d(\delta x\,\lrcorner\,\mathbf{K})$$

since $\mathbf{K}$ is a four-form. In effect the gravitation Lagrangian does not contribute to (31).

The variation of the fluid Lagrangian which corresponds to (34), according to (30) and (33), is equal to

$$\delta_0(\mathscr{L}\eta)=\theta^i_a\delta_0\theta^a_j(-t_{(i}{}^{j)}\eta+Ds_i{}^j+dx_i\wedge t^j-dx^j\wedge t_i) +\tfrac{1}{2}\delta_0\Gamma^i{}_j\wedge s_i{}^j\,, \tag{36}$$

where $\Gamma^i{}_j=\Gamma^i{}_{jk}dx^k$. The transformation formula

$$\Gamma^i{}_j=\theta^i_a\theta^b_j\Gamma^a{}_b+\theta^i_a d\theta^a_j$$

allows us to replace $\delta_0\Gamma^i{}_j$ by $\delta_0\Gamma^a{}_b$ and in effect to get rid of the $Ds_i{}^j$ term in (36). The result is

$$\delta_0(\mathscr{L}\eta)=-\delta_0\theta^a\wedge t_a+\tfrac{1}{2}\delta_0\Gamma^a{}_b\wedge s^b_a+d(s_i{}^j\theta^i_a\delta_0\theta^a_j)\,,$$

where $t_a=\theta^i_a t_i$ and $s^b_a=\theta^i_a\theta^b_j s_i{}^j$. The last formula completes the interpretation of $t^j{}_i$ and $s^k{}_{ij}$ as energy-momentum and spin tensors, respectively.

The gravitation field equations resulting from the Hamilton principle $\delta_0\int(\mathbf{K}+\mathscr{L}\eta)=0$ are

$$e_i=-t_i\,, \tag{37}$$

$$c_i{}^j=-s_i{}^j\ . \tag{38}$$

The equations of motion (32) and (33) are contained in the gravitational Eqs. (37) and (38) due to the differential identities[13]

$$De_i=Q^j{}_i\wedge e_j-\tfrac{1}{2}R^j{}_{ki}\wedge c_j{}^k\,,$$

$$Dc^{ij}=dx^j\wedge e^i-dx^i\wedge e^j\,,$$

which follow from covariance of $\mathbf{K}$ under diffeomorphisms and invariance under the Lorentz transformations, respectively.

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