Vector and scalar mesons in the Skyrmion model

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It is shown how the Skyrme and the non-Skyrme terms arise naturally from an effective chiral Lagrangian with the vector and scalar mesons. Some errors in the literature concerning the use of the Skyrme and the non-Skyrme terms are pointed out.

I. INTRODUCTION

A nice feature of the Skyrmion model[1–4] is the possibility of predicting the nucleon mass and its static properties in terms of a few parameters determined from the low-energy hadron physics.[5,6] Recent advance in this direction, as reported in our previous works,[6] makes it possible to compute the nucleon mass in terms of the coupling of vector mesons to the pseudoscalar mesons.[7] The low-lying vector mesons (ρ,ω, etc.) are found to play an important role in stabilizing the Skyrmion while the scalar meson σ destabilizes it. To the extent that the local approximation is valid, that is, using the resulting chiral Lagrangian involving higher derivatives of pion fields when the heavy fields ρ, ω, and σ are eliminated, it was previously found[7] that the nucleon mass is too large by a factor of 2. This was done without the flexibility of adjusting parameters such as the pion decay constant, and the ρ and ω widths which are directly measured by experiments. One exception is the $I=0$ scalar meson σ whose existence is not well established.

To proceed further we must first clarify the important role of the vector mesons in the Skyrme Lagrangian by expressing the model-independent dispersion relation result for the quartic terms in the language of chiral Lagrangian. In recent works[8,9] a number of people have chosen to ignore the role of the ρ meson in Skyrmion physics and treated the strength of the Skyrme term and non-Skyrme terms as parameters to be fitted with the computed nucleon mass. Many workers[9] in this field even suggested that the Skyrme term has to be added to the contribution of the vector meson ρ Lagrangian (and similarly the non-Skyrme term to be added to the scalar meson σ Lagrangian). Progress in the low- and high-energy physics in the last 25 years does not allow us to have these flexibilities. It is the purpose of this article to straighten out this situation.

II. ρ MESON AND THE SKYRME TERM

In a previous work,[6] two of us (T.N.P. and T.N.T.) have identified and evaluated the Skyrme term in a model-independent way using the forward dispersion relation and the Froissart bound. This result was subsequently verified in special models of local and global chiral symmetry,[10] or hidden symmetry;[11,12] we construct here a chiral Lagrangian for ρ which has the virtue of having the Skyrme term appear naturally and explicitly, and want to emphasize once more the generality of the sum rule discussed in our previous work.[6]

To analyze the nucleon properties in the Skyrmion model, we need only the SU(2)$\times$SU(2) chiral Lagrangian. The following expressions, although written for SU(3)$\times$SU(3) as usually done for soft meson processes, is of course valid for SU(2)$\times$SU(2) without modification. Now the simplest way to include the vector mesons in the chiral Lagrangian is to define a vector meson field with a nonlinear transformation law which depends only on the pseudoscalar meson field and to construct a chiral-invariant Lagrangian with these vector meson fields. Given a representation for vector mesons (i.e., octet representation) under the diagonal SU(3), then in the standard nonlinear realization chiral symmetry, it is possible to define a transformation law for the ρ-meson field as[13]

$$\rho_\mu \to U\rho_\mu U^{-1} \tag{1}$$

under chiral SU(3)$\times$SU(3). U is a function of the pseudoscalar-meson field. In this nonlinear realization the meson coupling matrix M is defined as[13,14]

$$M=\xi^2\to LMR^\dagger,\;\; \xi\to L\xi U^\dagger = U\xi R^\dagger \;,$$

where L and R are, respectively, elements of $SU(3)_L$ and $SU(3)_R$. In terms of the pseudoscalar meson field ϕ,

$$M=\exp(2if\phi),\;\; \phi=\sum_i \frac{1}{\sqrt{2}}\lambda_i\phi_i,\;\; f=f_\pi{}^{-1}\;,$$

and

$$\xi=\exp(if\phi)\;.$$

A covariant derivative for ρ_μ is then[13]

$$D_\nu\rho_\mu=\partial_\nu\rho_\mu+[v_\nu,\rho_\mu] \tag{2}$$

with

$$v_\mu=\tfrac{1}{2}[\xi^\dagger,\partial_\mu\xi],\;\; p_\mu=\tfrac{1}{2}\{\xi^\dagger,\partial_\mu\xi\}\;. \tag{3}$$

Note that v_μ transforms as a gauge field and p_μ as a covariant derivative

$$v_\mu\to Uv_\mu U^{-1}-\partial_\mu U U^{-1}\;,$$
$$p_\mu\to Up_\mu U^{-1}\;.$$

The usual left-handed and right-handed currents are defined as

$$L_\mu = 2\xi p_\mu \xi^\dagger = M\partial_\mu M^\dagger ,$$
$$R_\mu = 2\xi^\dagger p_\mu \xi = M^\dagger \partial_\mu M . \quad (4)$$

From (2) we can define a covariant field strength tensor for ρ_μ as

$$F_{\mu\nu} = D_\nu \rho_\mu - D_\mu \rho_\nu - \frac{ig}{2}[\rho_\nu, \rho_\nu] . \quad (5)$$

The last term is needed to ensure the covariance of $F_{\mu\nu}$ under the local SU(3)×SU(3) transformation.[13,15]

We now add to the chiral Lagrangian the vector-meson terms. For vector mesons with the transformation law given by (1), the mass term is automatically invariant under global SU(3)×SU(3) transformation. The interactions of vector mesons with pseudoscalar mesons are contained in a chiral-invariant part with at least two field derivatives:

$$\mathscr{L}_\rho = -\tfrac{1}{4}\mathrm{Tr}\left[F_{\mu\nu} + \frac{2i}{g}[p_\mu, p_\nu]\right]^2 + \tfrac{1}{2}m_\rho^{\,2}\mathrm{Tr}\rho_\mu^{\,2} + \frac{1}{g'2}\mathrm{Tr}[p_\mu, p_\nu]^2 , \quad (6)$$

where for generality the last term, which could be considered as a contact term, is added to the right-hand side (RHS) of Eq. (6). (It will be shown below $1/g' \simeq 0$.) The ρ meson-pion interaction terms

$$\mathscr{L}_I = -\left[\frac{i}{g}\right]\mathrm{Tr}(\rho_{\mu\nu}[p_\mu, p_\nu] + \text{covariant derivative})$$

can be easily brought into the following form (ignoring a total divergence term):

$$\mathscr{L}_I = \mathrm{Tr}\rho_\mu J_\mu ,$$
$$J_\mu = -\left[\frac{2i}{g}\right]\xi^\dagger(\partial_\nu[L_\mu, L_\nu] + \tfrac{1}{2}[L_\nu, [L_\mu, L_\nu]])\xi . \quad (7)$$

In terms of L_μ we have

$$\mathscr{L}_\rho = \frac{1}{32e^2}\mathrm{Tr}[L_\mu, L_\nu]^2 + \mathscr{L}_I + \rho \text{ kinetic terms} , \quad (8)$$

where the first term on the RHS of Eq. (8) is the Skyrme term with

$$\frac{1}{e^2} = 2\left[\frac{1}{g^2} + \frac{1}{g'^2}\right] . \quad (9)$$

On the other hand, the sum rule derived in Ref. 6 reads

$$\frac{1}{e^2} = \frac{1}{\pi}f_\pi^{\,4}\int_{4m_\pi^{\,2}}^{\infty}\frac{[s(s-4m_\pi^{\,2})]^{1/2}\sigma^{\pi^+\pi^0}(s)ds}{(s-2m_\pi^{\,2})^3} . \quad (10)$$

Saturating the RHS with the ρ resonance and making the δ-function approximation for the cross section (in order to compare it directly with the tree graph Lagrangian approach), we find

$$\frac{1}{e^2} = 48\pi\left[\frac{\Gamma_\rho}{m_\rho}\right]\left[\frac{f_\pi}{m_\rho}\right]^4 \quad (11)$$

which determines e in terms of the mass and the width of the ρ meson. The sum rule (10) is rigorous in the sense that its general validity depends only on the Froissart bound for the scattering amplitudes. Therefore by comparing the result (9) and (10), g' can be calculated. Now from (7), the $\rho\to 2\pi$ decay amplitude is given in terms of the usual on-shell $g_{\rho\pi\pi}$ coupling constant as

$$m(\rho\to 2\pi) = g_{\rho\pi\pi}\epsilon\cdot(p_1 - p_2)$$

with

$$g_{\rho\pi\pi} = \frac{\sqrt{2}}{g}\left[\frac{m_\rho^{\,2}}{f_\pi^{\,2}}\right] . \quad (12)$$

Putting this into Eq. (11), we find

$$\frac{1}{e^2} = \frac{2}{g^2} . \quad (13)$$

Consistency between (9) and (13) requires that

$$\frac{1}{g'} = 0$$

which tells us that the Skyrme term receives its most important contribution from the ρ meson. Higher-mass vector mesons or continuum can modify slightly this result which is neglected here. For the ρ-meson contribution, from (12) and using the Kawarabayashi-Suzuki-Riazzuddin-Fayyazuddin (KSRF) relation,[16] i.e.,

$$f_\pi^{\,2} g_{\rho\pi\pi}^{\,2} = m_\rho^{\,2}$$

we find

$$\frac{1}{e^2} = \frac{2}{g^2} = \frac{f_\pi^{\,2}}{m_\rho^{\,2}}$$

which is the result obtained in Ref. 6. This shows the equivalence between the effective Lagrangian approach and the dispersion relation for the ρ-meson contribution. We have here a very rare situation in physics where some general properties of the S matrix, e.g., the forward dispersion relation and the Froissart bound, can be used to give strict constraints on the low-energy physics. In fact the sum rules given in Ref. 6 for both the Skyrme and the non-Skyrme terms are even more general than the well-known Goldberger-Miyazawa-Oehme sum rule[17] or the Adler-Weisberger sum rule[18] whose validity depend on the assumption of the absence of the subtraction constant which grows linearly with the laboratory energy for the crossing-odd amplitudes; while this assumption seems reasonable, it cannot be proved from first principles.

The interaction term $\mathscr{L}_I$ gives rise to terms with six or more power of field derivatives. In the local approximation they are expected to make a small contribution to the nucleon mass. In the following we shall make this approximation for simplicity and ignore these higher derivative terms.

In the Weinberg[15] and Wess-Zumino[19] effective Lagrangian treatment of current algebra, the ρ mesons transform as gauge fields under chiral transformation. If we normalize ρ_μ such that its transformation under SU(3)×SU(3) is that of $(2i/g)v_\mu$, then in terms of the new $\tilde{\rho}_\mu$, the minimal chiral Lagrangian for ρ meson is

$$\tilde{\mathscr{L}}_\rho=-\tfrac{1}{4}\mathrm{Tr}\tilde{F}_{\mu\nu}{}^2+\tfrac{1}{2}m_\rho{}^2\mathrm{Tr}\left[\tilde{\rho}_\mu-\frac{2i}{g}v_\mu\right]^2 , \tag{14}$$

where $\tilde{F}_{\mu\nu}$ is the covariant field strength tensor for $\tilde{\rho}_\mu$:

$$\tilde{F}_{\mu\nu}=\partial_\mu\tilde{\rho}_\nu-\partial_\nu\tilde{\rho}_\mu-\frac{ig}{2}[\tilde{\rho}_\mu,\tilde{\rho}_\nu] . \tag{15}$$

If we now identify the linear combination

$$\tilde{\rho}_\mu-\frac{2i}{g}v_\mu$$

as our ρ_μ field with the transformation law given by (1),

$$\rho_\mu=\tilde{\rho}_\mu-\frac{2i}{g}v_\mu , \tag{16}$$

then after some algebra, $\tilde{\mathscr{L}}_\rho$ can be shown to be the expression (6). This equivalence proof is based on our freedom to redefine the ρ field to simplify computation with tree diagrams for physical processes. In this way, either choice of the ρ field will lead to the Skyrme term with the strength given by (13) in agreement with dispersion relation.

III. THE ROLE OF THE ω MESON

The ω meson through its coupling with the topological baryon currents, stabilizes the Skyrmion and makes a positive contribution to its mass.[7,20] The strength of this coupling can be determined from the $\omega\to3\pi$ decay rate as follows: Within SU(2)×SU(2), the most general chiral-invariant coupling of the ω with ρ and π is of the form

$$\mathscr{L}'_I=ic\epsilon_{\mu\nu\rho\sigma}\omega_\mu\mathrm{Tr}p_\nu\left[\left[F_{\rho\sigma}+\frac{2i}{g}[p_\rho,p_\sigma]\right]+\frac{2i}{g''}[p_\rho,p_\sigma]\right] . \tag{17}$$

As with the Skyrme term, the effective Lagrangian for the process $\omega\to3\pi$ with the lowest power of pion field derivative is given by the direct terms. To obtain the strength of these direct terms, we compute the $\omega\to3\pi$ decay rate using (7) and (17). We find that the ρ-exchange graphs produce an amplitude of the form $s_{ij}/(m_\rho{}^2-s_{ij})$ (s_{ij} being the dipion-invariant mass), which when added to the direct g term, the usual ρ-dominance amplitude of the form $m_\rho{}^2/(m_\rho{}^2-s_{ij})$ is obtained. Hence in the absence of the g'' term, the low-energy effective ω to 3π coupling is obtained from the $\omega\to3\pi$ decay rate computed with the usual ρ-dominance graph [i.e., of the form $m_\rho{}^2/(m_\rho{}^2-s_{ij})$ mentioned above]. As mentioned previously,[7] recent theoretical analyses and the good agreement with experiment for the ratio $\Gamma(\omega\to\pi^0\gamma)/\Gamma(\omega\to3\pi)$ obtained with the Gell-Mann–Sharp–Wagner model indicates that the g'' term is small and therefore will be neglected for our purpose. The total Lagrangian for the ω meson is then

$$\mathscr{L}_\omega=-\tfrac{1}{4}\omega_{\mu\nu}{}^2+\tfrac{1}{2}m_\omega{}^2\omega_\mu{}^2+\beta\omega_\mu B_\mu+\cdots , \tag{18}$$

where

$$B_\mu=\left[\frac{1}{24\pi^2}\right]\epsilon_{\mu\nu\alpha\beta}\mathrm{Tr}(L_\nu L_\alpha L_\beta) \tag{19}$$

and

$$\beta=-6\pi^2\left[\frac{c}{g}\right]\simeq17$$

as determined from the $\omega\to3\pi$ decay rate.

IV. THE SCALAR-MESON AND THE NON-SKYRME TERM

It has been suggested that the $I=0$, S-wave $\pi\pi$ scattering amplitude in the 500–1000-MeV region can be fitted with a broad resonance, the σ meson with mass $m_\sigma\simeq700$ MeV. In this case, the main contribution to the non-Skyrme term comes from the σ meson[21] and its strength parameter γ/e^2 is given by the sum rules derived in Ref. 6 for $\pi^0\pi^0$ scattering (using the Froissart bound). In the δ-function approximation for the $\pi^0\pi^0$ cross section, γ/e^2 depends on the mass and width of the σ meson. This result can also be obtained directly from the tree-graph Lagrangian by assuming a chiral-invariant $\sigma\pi\pi$ coupling of the form

$$\mathscr{L}_\sigma=g_\sigma\sigma\,\mathrm{Tr}(\partial_\mu M\partial_\mu M^\dagger) \tag{20}$$

from which we find

$$\frac{\gamma}{e^2}=4\left[\frac{g_\sigma{}^2}{m_\sigma{}^2}\right] . \tag{21}$$

The on-mass shell $\sigma\pi\pi$ coupling constant ($p_\sigma{}^2=m_\sigma{}^2$) is given by

$$g_{\sigma\pi\pi}=2\left[\frac{g_\sigma m_\sigma{}^2}{f_\pi{}^2}\right] .$$

If we now identify $g_{\sigma\pi\pi}$ with the nonderivative $\sigma\pi\pi$ vertex in the linear σ model, we have

$$g_{\sigma\pi\pi}=\frac{m_\sigma{}^2}{\sqrt{2}f_\pi} \tag{22}$$

similar to the KSRF relation for $g_{\rho\pi\pi}$. From (21) we get

$$\frac{\gamma}{e^2}=\frac{1}{2}\left[\frac{f_\pi{}^2}{m_\sigma{}^2}\right]$$

which is the expression obtained with the tree-graph linear σ model. With the parameters for the quartic and higher derivative terms determined from the low-energy data as shown above, we now present our result for the nucleon mass as a test of the Skyrmion model.

V. THE NUCLEON MASS IN THE SKYRMION MODEL

To the extent that the local approximation is valid, i.e., for a Skyrmion radius R much bigger that the Compton

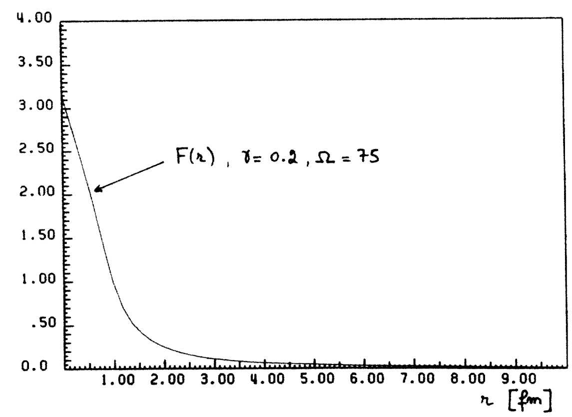

FIG. 1. Numerical solution for the chiral angle $F(r)$ as function of the radial distance with $\gamma=0.20$, $\Omega=75$.

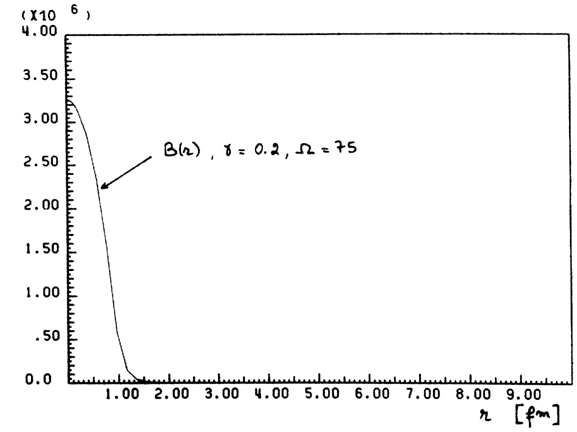

FIG. 2. The computed baryon current density $B_0(r)$ as a function of the radial distance.

wavelength of the vector and scalar meson ($1/m_\omega{}^2R^2 \ll 1$), the effects of heavy particles can be represented by the two quartic terms and the six power of derivative terms (the $B_\mu B_\mu$ terms). In this approximation the Skyrmion mass is given by[7]

$$M=H_0+H_\rho+H_\omega+H_\sigma\ , \tag{23}$$

where

$$\begin{aligned} H_0&=\frac{C}{4}\int_0^\infty dx\,x^2\left[F'^2+\frac{2\sin^2F}{x^2}\right],\\ H_\rho&=C\int_0^\infty dx\left[2F'^2+\frac{\sin^2F}{x^2}\right]\sin^2F\ ,\\ H_\sigma&=-\gamma C\int_0^\infty dx^2\left[F'^2+\frac{2\sin^2F}{x^2}\right]^2,\\ H_\omega&=\frac{\Omega C}{2}\int_0^\infty dx\frac{F'^2\sin^4F}{x^2}\ ,\end{aligned} \tag{24}$$

with[22]

$$C=2\sqrt{2}\pi f_\pi/e,\ \ \Omega=\beta^2e^4f_\pi{}^2/\pi^4m_\omega{}^2\ .$$

The expression for the ω contribution (H_ω) agrees with that of Adkins and Nappi[20] in the local approximation.

We now look for a stable soliton by a numerical solution of the Euler-Lagrange equation for the chiral angle $F(r)$ with the appropriate boundary conditions chosen for $B=1$ soliton. Since the non-Skyrme term destabilizes the Skyrmion, we find a stable soliton only for γ below a critical value γ_c. For $\Omega=75$, we find

$$\gamma_c=0.21$$

which is below the value $\gamma\simeq\frac{1}{3}$ determined from the $I=0$, S-wave $\pi\pi$ cross section. Because of large uncertainties in the determination of γ, we shall for our calculation take

$$\gamma\simeq\gamma_c$$

as the largest allowed value.

We find

$$m_0=1751\ \text{MeV}$$

as the static soliton mass before quantum correction. The computed nucleon and Δ mass are

$$m_N=1772\ \text{MeV},\ \ m_\Delta=1858\ \text{MeV}$$

larger than the measured values by more than 50%.

To have an idea of the Skyrmion as an extended object, we give in Figs. 1 and 2 the shape of $F(r)$ and the baryon current density $B_0(r)$ as a function of the radial distance.

Finally as a check of our numerical method, we compute separately $H_0,H_\rho,H_\omega,H_\sigma$ and find that the relation

$$H_0=\tfrac{1}{2}(H_\rho+H_\sigma+3H_\omega) \tag{25}$$

obtained from the Derrick theorem[23] is satisfied to a great accuracy.

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