Solving momentum-space integral equations for quarkonia spectra with confining potentials

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Singular integral equations for quarkonia ($q\bar{q}$) spectra are solved in momentum space for different choices of confining potentials by introducing a regularization procedure. The method is sufficiently general to treat nonlocal potentials and combinations of singular potentials. Through nonrelativistic model applications we demonstrate the stability and accuracy of the method. The method works in all partial waves. A first-order correction to the eigenenergies brings calculated results for soluble model problems into remarkable agreement with exact results. Extensions of the method to solve the nonrelativistic spectra of three-quark systems and to solve the relativistic Bethe-Salpeter equation are discussed.

I. INTRODUCTION

As interest has grown in developing a covariant description of the mass spectra of the elementary particles based on QCD, there has emerged a need for stable methods to solve the Bethe-Salpeter[1] integral equation in momentum space with nonlocal and singular kernels. In the case of the charmonium and b-quarkonium ("heavy"-quarkonia) spectra the Schrödinger equation has been widely used with local phenomenological interactions since the nonrelativistic approximation is reasonable and the techniques for solution in coordinate space are elementary. We adopt the quarkonia problem to illustrate a method for treating singular potentials in integral equations that is also applicable to more general nonlocal interactions as will arise in applications of the Bethe-Salpeter equation to problems in QCD.

We begin by considering separately the potentials $v_i(r)=\alpha_i r^i$ $(i=-1,1,2)$, which represent the Coulomb, linear confining, and quadratic confining potentials, respectively, for the heavy-quarkonia problem, to illustrate the singular structure of the momentum-space two-body Lippmann-Schwinger equation. We then introduce a regularization (screening) procedure and examine the implications for the low-lying spectra of these potentials. A major effect of this regularization procedure is to change the long-range behavior of the potential. The long-range effects, such as the confining part of the potential if $i>0$, are then treated as a perturbation. A correction is evaluated and encouraging results are obtained. We then consider the more interesting and more challenging problem of the Coulomb-plus-linear confinement potential and obtain remarkable agreement with the exact results.

Since the method is applicable to nonlocal and singular potentials, it is interesting to compare our results with other approximate results. The method of Shifman, Vainshtein, and Zakharov[2] (SVZ) was introduced to draw connections between the empirical heavy-quarkonia spectra and fundamental parameters of QCD. In the process of critically reviewing the limitations of the SVZ method, Durand, Durand, and Whitenton[3] (DDW) have quoted results of the SVZ method for the same potentials we consider here.

II. INTEGRAL EQUATIONS

The two-body Hamiltonian is $h=h_0+v$, where h_0 is the kinetic energy operator and v is the potential. In momentum space the kinetic energy is $h_0=q^2/(2m)$, where m is the reduced mass.

The aim of this paper is to find low-lying eigensolutions of h when v is a confining potential. For such potentials h has only a discrete spectrum. We shall, however, also consider the Coulomb potential, which has a continuous spectrum, in addition to an infinite number of bound states that accumulate at zero energy. In general, therefore, we shall assume that h has a discrete spectrum on the left of the real energy axis, and a continuous spectrum on the right extending to $+\infty$.

We focus on the discrete spectrum of h. Let $-\epsilon_b$ be an eigenvalue of the Schrödinger equation

$$h\,|\psi_b\rangle=-\epsilon_b\,|\psi_b\rangle\ . \tag{1}$$

Here $|\psi_0\rangle$ is the bound-state wave function and ϵ_b is the binding energy. We denote internal quantum numbers by the label b.

Equation (1) may be rewritten as an integral equation, which in operator form is

$$|\psi_b\rangle=-g_0(-\epsilon_b)v\,|\psi_b\rangle\ . \tag{2}$$

Here we have introduced the free-resolvent operator

$$g_0(z)=(h_0-z)^{-1}\ , \tag{3}$$

where, for convenience, we restrict the energy parameter z to lie along the real negative line, $z\in(-\infty,0]$. In general, for $z\in(-\infty,0]$, the equation

$$\eta_\alpha\,|\,\psi_\alpha\rangle=-g_0(z)v\,|\,\psi_\alpha\rangle \tag{4}$$

has an infinite number of solutions $[\,|\,\psi_\alpha\rangle\}$ corresponding to the numbers $\{\eta_\alpha\}$. For $z=-\epsilon_b$ we may regard Eq. (4) as a kind of Schrödinger equation where each $|\,\psi_\alpha\rangle$ is an eigenstate of the Hamiltonian $h_\alpha=h_0+\eta_\alpha{}^{-1}v$, and the interaction strength $\eta_\alpha{}^{-1}$ is adjusted to give the binding energy ϵ_b. The physical solution corresponds to $\alpha=b$, and $\eta_b=1$. The functions $[\,|\,\psi_\alpha\rangle\}$ are usually referred to as Sturmians because they are solutions of a particular form of the Sturm-Liouville equation.

We may symmetrize Eq. (4). Let

$$|\,\psi_\alpha\rangle=\sqrt{g_0(z)}\,|\,\chi_\alpha\rangle\ . \tag{5}$$

The $\{\,|\,\chi_\alpha\rangle\}$ are now eigenstates of the equation

$$\eta_\alpha\,|\,\chi_\alpha\rangle=-\sqrt{g_0(z)}v\sqrt{g_0(z)}\,|\,\chi_\alpha\rangle\ . \tag{6}$$

If v is self-adjoint, then the eigenvalues $\{\eta_\alpha\}$ are real. In order for the Sturmians $\{\,|\,\psi_\alpha\rangle\}$ to be normalized to unity it is convenient to use the normalization condition

$$\langle\chi_\alpha\,|\,g_0(z)\,|\,\chi_\alpha\rangle=1\ . \tag{7}$$

After the usual partial-wave decomposition, Eq. (6) in momentum space reads

$$\eta_\alpha\chi_\alpha(q)=-\int_0^\infty K(q,q')\chi_\alpha(q')q'^2dq'\ , \tag{8}$$

where K is the symmetric kernel

$$K(q,q')=2m\,(q^2-2mz)^{-1/2}v(q,q')\times(q'^2-2mz)^{-1/2}\ . \tag{9}$$

Here we have suppressed the subscript angular momentum l. Equations (8) and (9) are true for all angular momentum l. The potential $v(q,q')$ is the Fourier transform of $v(r)$, and a description of this Fourier transform is given in the Appendix.

III. APPROXIMATION METHOD

The problem is to find an approximate solution to Eq. (8) when v is a confining potential. For our test problem, the potential is chosen to be local in r space, that is $\langle\mathbf{r}\,|\,v\,|\,\mathbf{r}'\rangle=v(r)\delta(\mathbf{r}-\mathbf{r}')$. We shall, however, take the Fourier transform of $V(r)$ and solve Eq. (8) in momentum space. After partial-wave projection, no advantage is taken of the local r-space behavior of v. Furthermore, the long-range behavior of $v(r)$ leads to moving singularities in the kernel of Eq. (8). General moving singularities in integral equations are well known to provide major challenges to obtaining accurate solutions.[4]

The approach we shall adopt is to replace the potential $v(r)$ by a screened potential $e^{-\mu r}v(r)$, where μ is a free parameter. In the limit $\mu\to0$ the Fourier transform of a confining potential is a distribution.[5] However, we do not take this limit. The idea is that μ can be chosen small enough that the screened potential gives a binding energy $\epsilon_b^{(0)}$ that is close to that of the confining potential binding energy ϵ_b. The approximation $\epsilon_b^{(0)}$ can be further improved by using perturbative methods, as we shall show.

In the case of confining potentials the above screening procedure will result in continuum-state contributions to the binding energy. In order to avoid this complication we introduce an additional constant $c\geq0$, and write the potential as $v(r)-c$; the screened potential becomes $e^{-\mu r}[v(r)-c]$. In practice, c is chosen so that the calculated binding energies are real. All eigenenergies we quote below have been "corrected" by adding c to the values $\epsilon_b^{(0)}$ obtained in solving Eq. (8).

A. Zero-order solution

We seek a zero-order solution for Eq. (8). Let $K^{(0)}(q,q')$ be the kernel given by the screened potential. Equation (8) becomes

$$\eta_\alpha^{(0)}\chi_\alpha^{(0)}(q)=-\int_0^\infty K^{(0)}(q,q')\chi_\alpha^{(0)}(q')q'^2dq'\ . \tag{10}$$

We briefly describe our numerical treatment of Eq. (10). We first map q onto a finite interval. For this purpose we use the mapping

$$q(x)=\xi\left[\frac{1+x}{1-x}\right],\quad x\in[-1,+1]\ , \tag{11}$$

where ξ is a constant scale parameter. The interval $[-1,+1]$ is partitioned by n knots $\{t_\nu\}$, such that $t_{\nu-1}<t_\nu$. Although we shall restrict our numerical method to uniformly spaced knots on $[-1,1]$, the parameter ξ may be viewed as a means of concentrating these knots in a selected region of the semi-infinite domain. On this mesh we define basis functions $\{B_\nu\}$, which are chosen to be cubic B splines.[6] Next, the function $\chi_\alpha^{(0)}$ is approximated as

$$\chi_\alpha^{(0)}(q(x))\approx\sum_{\nu=0}^{n+1}\beta_\nu^\alpha B_\nu(x)\ . \tag{12}$$

The spline coefficients $\{\beta_\nu^\alpha;\nu=0,\ldots,n+1\}$ are determined from a Galerkin technique for second-kind Fredholm integral equations.[7] Finally, the energy $\epsilon_b^{(0)}$ is obtained by ensuring that the eigenvalue $\eta_b^{(0)}=1$. This procedure also yields the wave function in momentum space, $\psi_b^{(0)}(q)$.

B. First-order correction

Let $\tilde{v}$ be the correction term to the screened potential. In r space we write

$$\tilde{v}(r)=(1-e^{-\mu r})(v(r)-c)\ . \tag{13}$$

This potential has all of the long-range behavior of the original power-law potential $v(r)$. Using first-order perturbation theory, we find

$$\epsilon_b^{(1)}=\epsilon_b^{(0)}+\langle\psi_b^{(0)}\,|\,\tilde{v}\,|\,\psi_b^{(0)}\rangle\ . \tag{14}$$

The correction term in Eq. (14) is finite. This is because the bound-state wave function $\psi_b^{(0)}$ falls off exponentially

as $r \to \infty$. The zero-order solution is close to the correct result, and we find that the first-order corrections produce satisfactory results in our numerical calculations so that higher-order terms will not be considered.

IV. APPLICATION TO THE BETHE-SALPETER EQUATION

A full relativistic equation for two interacting particles was developed originally by Bethe and Salpeter. Central to the derivation is the infinite series of irreducible graphs for the kernel of the integral equation. Only in simple models with certain approximations, such as the Wick-Cutkosky model,[8] can the problem be converted easily to a differential equation with local interaction form. For the general problem of relativistic bound states one faces the full integral equation in four-momentum space.[9] In light of the interest in strong-interaction problems such as $q\bar{q}$ systems one would also like to assess the convergence of the expansion for the kernel.

In order to demonstrate the applicability of the methods we have discussed above, we shall treat the Wick-Cutkosky model in its integral equation form—that is, without exploiting the specific limit (mass of scalar field exchanged $\mu \to 0$) where it can be rewritten as a differential equation. Furthermore, the more general problem of $\mu \neq 0$ is no more difficult than $\mu = 0$ in our method. We also remark that it is possible, within our framework, to go beyond the ladder approximation. However, we reserve that effort for a later work.

The integral Bethe-Salpeter (BS) equation for two complex scalar fields of masses m_1 and m_2 interacting through a real scalar field of mass μ can be written in the center-of-momentum system, after a Wick rotation has been performed, as

$$(q^4+2aq^2+1)\psi(q) = \int d^4q' v(q,q')\psi(q') , \tag{15}$$

where

$$a = \frac{1}{m_1 m_2}[(m_1+m_2)B - \tfrac{1}{4}B^2] - 1 , \tag{16}$$

and B represents the (positive) binding energy. Let us restrict ourselves to the case $a \geq 0$, where manipulations analogous to those performed above may also be performed. In practice, this limits solutions to the range

$$0 \leq m_1+m_2-(m_1^2+m_2^2)^{1/2} \leq B \leq m_1 + m_2 + (m_1^2+m_2^2)^{1/2} , \tag{17}$$

which certainly covers a large range of relativistic bound-state solutions. In the case $m_1 = m_2$ this implies

$$0.59 m_2 \leq B \leq 3.41 m_2 , \tag{18}$$

and in the case $m_1 \to 0$ this implies

$$0 \leq B \leq 2m_2 . \tag{19}$$

Thus one may achieve a highly relativistic composite system of zero mass.

A framework for a simple study of the convergence of the BS binding energies with higher-order contributions to the kernel can be obtained by restricting the discussion to two space and one time dimensions. The problem that then emerges may be treated easily by a partial-wave decomposition, just as we had done in the Schrödinger problem. If we restrict ourselves to the lowest-order contribution to $v(q,q')$, we have a Yukawa-type form:

$$v(q,q') \to \frac{\lambda}{2\pi^2} \frac{1}{(q-q')^2+\mu^2} , \tag{20}$$

where the metric is Euclidean due to the Wick rotation ($q^2 = q_1^2+q_2^2+q_0^2$). The partial-wave decomposition of (15) renders the integral equation similar to Eq. (8), but with $g_0(z)$ in Eq. (7) replaced by

$$g_0(z) \to (q^4+2aq^2+1)^{-1} , \tag{21}$$

which introduces no additional complexity to our method.

Thus we see that it is possible to study interesting questions of convergence for bound states of the kernel expansion in the BS framework directly with our method. At present this is limited to the two space and one time dimensions, but we have also extended the discussion to the more general $\mu \neq 0$ situation.

V. RESULTS AND DISCUSSION

We solve the momentum-space integral Eq. (8) using the method described in Sec. III. Our results are compared with exact solutions. We note that we are presenting Schrödinger, not BS, solutions.

A basis of 20 splines is used to solve Eq. (10) with simple power-law potentials. A basis of 40 splines is used for the Coulomb-plus-linear potential. The parameters μ and c are chosen so that the solution does not change when more basis functions are used. In practice, the zero-order approximation is sensitive to the value of these parameters, but the first-order approximation is not. The scaling parameter ξ is adjusted separately for each potential. The values of the parameters chosen for each model problem are given in Table I.

Our first results concern the simple power-law potentials. We restrict our discussion to the case $l=0$. We calculate the eigenvalues ϵ_n and eigenfunctions squared $|\psi_n(r)|^2$ at $r=0$. The value of $|\psi_n(0)|^2$ is of some interest because it is related to leptonic widths.[10] Exact eigenvalues for the Coulomb, linear, and harmonic-oscillator potential are given in Table II. Table III gives our numerical results. We find that for these soluble model problems a first-order correction to the eigenenergies is sufficient in most cases to yield accurate numerical solutions. The least accurate results seem to occur for the

TABLE I. Choice of parameters. The screening parameter μ and subtraction constant c for the screened potential $e^{-\mu r}[v(r)-c]$. Scale parameter ξ in Eq. (11).

Potential	μ	c	ξ
Coulomb	0.01		1.0
Linear	0.05	10.0	2.0
Harmonic oscillator	0.1	20.0	5.0
Coulomb plus linear	0.05	2.0	1.0

TABLE II. Exact values for the simple power-law potential $V(r)=\alpha_i r^i$. The a_n is the nth zero of the Airy function $\mathrm{Ai}(-a_n)=0$. Energies are in units of $\alpha_i^{2/(2+i)} m_q^{-i/(2+i)}$ and the square of the wave function is in units of $(m_q\alpha_i)^{3/(2+i)}$ with $\hbar=c=1$.

Potential	ϵ_n	$\lvert\psi_n(0)\rvert^2$
Coulomb	$(4n^2)^{-1}$	$(8\pi n^3)^{-1}$
Linear	a_n	$(4\pi)^{-1}$
Harmonic oscillator	$4n-1$	$\frac{2}{\pi}\frac{\Gamma(n+\frac{1}{2})}{\Gamma(n)}$

Coulomb potential over values of the principal quantum number $n>1$. In this case the eigenenergies accumulate at zero and are therefore difficult to reproduce accurately. For $n=4$ we find an error of about 50% in the binding energy. We remark that the value of ξ in Eq. (11) is adjusted to optimize the stability of the solution when different mesh points are used. Our result therefore represents an accurate solution of the integral Eq. (8) plus the first-order perturbation. The eigenenergies of the harmonic oscillator are uniformly spaced, whereas those of the linear potential are more closely packed. For these confining potentials the method works well for principal quantum numbers $n\lesssim 4$. The eigenenergies are calculated with an accuracy of about 6% for the harmonic-oscillator potential, and about 0.5% for the linear potential. We can expect less precise results for the harmonic oscillator because this potential has stronger confinement than the linear potential. It is noteworthy that the present method works well for the lowest-lying eigenstate, even in the case of the Coulomb potential.

Next we consider a combination of simple power-law potentials. The Coulomb-plus-linear potential $v(r)=-ar^{-1}+br$ has been used as a model for the heavy quark-antiquark system.[11] We choose $a=0.49$ and $b=0.17$ GeV^2. The masses of the charm and bottom quarks are taken as $m_c=1.35$ GeV and $m_b=4.77$ GeV, respectively. These parameters produce a good fit to the average quarkonia spectra.[11] Table IV shows our numerical results for the spectra of charmonium ($c\bar{c}$) and b quarkonium ($b\bar{b}$), which are predicted by this Coulomb-plus-linear potential.

From these results we may conclude that accurate solutions to momentum-space integral equations can be obtained in the case of confining potentials, and that a first-order correction to the eigenenergies is sufficient to bring most of the calculated results for the model problems considered in this paper into close agreement with exact results.

Faddeev[12] has shown how the integral equation approach may be used in a nonrelativistic three-body problem. Faddeev equations have been used as a model for the heavy three-quark system.[13] One possible application of the present method for treating confining potentials is in the solution of three-body integral equations. Indeed a simple method that uses Sturmians as a basis for solving Faddeev equations exists[14] and could be extended to the confining potential case.

Finally, the numerical method may be used to treat relativistic integral equations such as the Bethe-Salpeter equation. An extension of the present method to treat integral equations that arise from the Wick-Cutkosky model is currently being investigated.

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APPENDIX: Fourier transform of the power-law potential

The momentum-space partial-wave projection of the potential is defined in our notation by

TABLE III. Comparison of $l=0$ solutions for the power-law potentials $v(r)=r^i$ obtained by our method with the exact results labeled ϵ_n and $\lvert\psi_n(0)\rvert^2$. The units are the same as in Table II.

i	n	ϵ_n	$\epsilon_n^{(0)}$	$\epsilon_n^{(1)}$	$\lvert\psi_n(0)\rvert^2$	$\lvert\psi_n^{(0)}(0)\rvert^2$
−1	1	−0.2500	−0.240	−0.250	0.039 79	0.039 74
	2	−0.0625	−0.053	−0.060	0.004 97	0.004 92
	3	−0.0278	−0.019	−0.021	0.001 47	0.001 41
	4	−0.0156	−0.009	−0.010	0.000 62	0.000 66
1	1	2.338	2.92	2.35	0.079 58	0.105
	2	4.088	4.93	4.10	0.079 58	0.096
	3	5.521	6.45	5.51	0.079 58	0.088
	4	6.787	7.71	6.80	0.079 58	0.081
2	1	3.0	4.83	3.09	0.179 59	0.277
	2	7.0	9.35	7.05	0.269 38	0.329
	3	11.0	13.40	11.11	0.336 73	0.358
	4	15.0	17.18	15.93	0.392 85	0.381

TABLE IV. Ground-state energies in MeV for Coulomb-plus-linear potential $v(r)=-ar^{-1}+br$, with $a=0.49$ and $b=0.17$ GeV^2. The masses of the charm and bottom quarks are $m_c=1.35$ GeV and $m_b=4.77$ GeV. The values of ϵ_{1l} are taken from Table II of Ref. 3.

State	ϵ_{1l}	$\epsilon_{1l}^{(0)}$	$\epsilon_{1l}^{(1)}$
Charmonium			
$1S$	364	535	368
$1P$	772	1001	775
$1D$	1060	1315	1062
b quarkonium			
$1S$	−98	10	−97
$1P$	349	519	353
$1D$	585	792	588

$$\langle \mathbf{q} \mid v \mid \mathbf{q}' \rangle = 4\pi \sum_l (2l+1) P_l(\hat{\mathbf{q}}\cdot\hat{\mathbf{q}}') v_l(q,q') , \tag{A1}$$

$$v_l(q,q') = \frac{2}{\pi} \int_0^\infty j_l(qr) v(r) j_l(q'r) r^2 dr , \tag{A2}$$

where P_l is the first-kind Legendre polynomial and j_l is the spherical Bessel function.

The screened power-law potential is

$$v_i(r) = a_i r^i e^{-\mu r} . \tag{A3}$$

For the case $i=-1$, the interaction is a Yukawa function which can be written in momentum space as a second-kind Legendre polynomial, namely,

$$v_l^{-1}(q,q') = \frac{2}{\pi} \frac{a}{2qq'} Q_l \left[\frac{q^2+q'^2+\mu^2}{2qq'} \right] . \tag{A4}$$

Screened potentials for $i > -1$ can be obtained from the relationship

$$v_l^i(q,q') = (-1)^{i+1} \frac{\partial^{i+1}}{\partial \mu^{i+1}} v_l^{-1}(q,q') . \tag{A5}$$

We give examples for $l=0$:

$$v_0^{-1}(q,q') = \frac{2}{\pi} \frac{a}{2qq'} \frac{1}{2} \ln \left[\frac{(q+q')^2+\mu^2}{(q-q')^2+\mu^2} \right] , \tag{A6}$$

$$v_0^0(q,q') = \frac{2}{\pi} \frac{a}{2qq'} \left[\frac{1}{[(q-q')^2+\mu^2]} - \frac{1}{[(q+q')^2+\mu^2]} \right] , \tag{A7}$$

$$v_0^1(q,q') = \frac{2}{\pi} \frac{a}{2qq'} \left[\frac{[(q+q')^2-\mu^2]}{[(q+q')^2+\mu^2]^2} - \frac{[(q-q')^2-\mu^2]}{[(q-q')^2+\mu^2]^2} \right] , \tag{A8}$$

$$v_0^2(q,q') = \frac{2}{\pi} \frac{a}{2qq'} 2\mu \left[\frac{[\mu^2-3(q-q')^2]}{[(q-q')^2+\mu^2]^3} - \frac{[\mu^2-3(q+q')^2]}{[(q+q')^2+\mu^2]^3} \right] . \tag{A9}$$

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