Neutrino oscillations in nonuniform matter

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We examine the effect of flavor-dependent elastic scattering upon neutrino flavor vacuum oscillations. The results are derived from a first-principles Dirac equation for the phenomenon. The conjecture of Wolfenstein is validated (with a correction $G \rightarrow \sqrt{2}G$ as noted in more recent literature) for a homogeneous medium. While the differential equation obtained for the extension to a nonhomogeneous case appears to differ significantly from that which has been used in the literature heretofore, we are able to show that they are equivalent for densities that vary little over the neutrino's de Broglie wavelength.

I. INTRODUCTION

Mikheyev and Smirnov[1] have recently pointed out that neutrino oscillations in a vacuum associated with very small mass differences and mixing angles will be appreciably enhanced when passing through matter as dense as that within the Sun or other stars. Indeed, such an effect [the Mikheyev-Smirnov-Wolfenstein (MSW) effect] may ultimately prove to be responsible for the observed solar-neutrino deficit.[2] There has therefore been considerable discussion of the consequences of the MSW effect in connection with neutrino propagation in the Sun and in other dense bodies as well as the early Universe.[3-9] The foundation for the MSW effect is an interesting paper by Wolfenstein.[10] In that paper a prescription is given for combining the spatial phase shift due to the refractive effect of the medium with the temporal phase shift arising from the mass matrix in vacuum. Wolfenstein's prescription is given for a homogeneous medium, and an appropriate extension to the nonhomogeneous case is not really obvious. Mikeyev and Smirnov have discussed the nonhomogeneous case in terms of an effective local Hamiltonian as seen by an observer moving with the neutrino wave packet and in this way have recognized that what is called an e neutrino inside a dense medium will be called by a different name when it gets outside the medium.

In this paper we provide a more rigorous justification of Wolfenstein's conjecture for pasting together spatial and temporal phase shifts, which allows us to deal more completely with the nonhomogeneous case.

We begin with an unconventional discussion of vacuum oscillations in Sec. II which is more adaptable to the more general case. In Sec. III we introduce into the Dirac equation for the neutrino, a source term arising from charged-current neutrino scattering in a uniform, electron-rich medium and derive Wolfenstein's conjecture in an appropriate limit. In Sec. IV we generalize to a nonuniform medium and obtain an expression for the electron transition probability in the two-flavor model for very slowly varying density. The results are illustrated by a simple application and are in accord with known results in the adiabatic limit.

II. OSCILLATIONS IN VACUUM

Neutrino oscillations between two families, say e and μ, are described in vacuum by the Lagrangian density

$$\mathscr{L} = \bar{\nu}(i\partial - M)\nu , \tag{1}$$

where M is a 2×2 matrix in flavor space. This implies the field equation

$$(i\partial - M)\nu = 0 , \tag{2}$$

from which there follows a Klein-Gordon equation and hence the energy-momentum relation

$$E^2 - P^2 - M^2 = 0 . \tag{3}$$

The rotation in flavor space which diagonalizes M^2 is characterized by the angle θ_v defined through the relations

$$\begin{aligned} |\nu_1\rangle &= |\nu_e\rangle \cos\theta_v - |\nu_\mu\rangle \sin\theta_v , \\ |\nu_2\rangle &= |\nu_e\rangle \sin\theta_v + |\nu_\mu\rangle \cos\theta_v , \end{aligned} \tag{4}$$

where the states labeled 1 and 2 are the simultaneous eigenstates of E and p; i.e., they are the eigenstates of M.

The $\nu_e \rightarrow \nu_\mu$ transition probability is usually discussed in terms of an electron neutrino of definite momentum, which is then expressed as a superposition of the corresponding energy eigenstates associated with the mass eigenvalues discussed above. The temporal evolution of ν_e can then be followed trivially. We choose, instead, to consider an initial ν_e state which is an energy eigenstate. Its temporal evolution is given by

$$|\nu_e(x,t)\rangle = |\nu_e(x,0)\rangle e^{-iEt} . \tag{5}$$

In order to determine its spatial development, we must first write the spatial part as a superposition of wave functions with definite momenta $p_{1,2}{}^2 = E^2 + m_{1,2}{}^2$. The full temporal and spatial development of the electron state is thus given by

$$|\nu_e(x,t)\rangle = e^{-iEt}[\,|\nu_1(0,0)\rangle e^{ip_1x}\cos\theta_v + |\nu_2(0,0)\rangle e^{ip_2x}\sin\theta_v] . \tag{6}$$

The probability of finding this state to be an electron at $x=R$ is given by

$$P(\nu_e \to \nu_e;R) = |\langle \nu_e(0,0)|\nu_e(R,t)\rangle|^2 . \tag{7}$$

Evaluating this yields the standard result for relativistic energies

$$P(\nu_e \to \nu_e;R) = 1 - \sin^2 2\theta_v \sin^2(\pi R/l_v) , \tag{8a}$$

$$l_v = 4\pi E/(m_2{}^2 - m_1{}^2) . \tag{8b}$$

III. OSCILLATIONS IN UNIFORM MATTER

As Wolfenstein has argued, the relevant effective interaction for neutrino scattering in a medium which contains electrons but not muons, and is therefore not diagonal in flavor space, arises from the charged current and is given by

$$\mathscr{L}_W = -(G/\sqrt{2}) j^\mu \bar{\nu}\gamma_\mu(1+\gamma_5)N\nu , \tag{9}$$

where j is the electron density

$$j^\mu = \bar{e}\gamma^\mu(1+\gamma_5)e \tag{10}$$

and N is a 2×2 matrix which *in flavor basis* is given by

$$N = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} . \tag{11}$$

The field equation for the neutrino field that follows from $\mathscr{L}_0 + \mathscr{L}_W$ is

$$(i\partial\!\!\!/ - M)\nu = (G/\sqrt{2}) j^\mu \gamma_\mu (1+\gamma_5)\nu . \tag{12}$$

The energy-momentum relation is most easily established by introducing the chiral projections $\nu_{L/R} = (1\pm\gamma_5)\nu/2$. Equation (12) is then equivalent to the following pair of equations:

$$i\partial\!\!\!/ \nu_L - M\nu_R = \sqrt{2} G j_\mu \gamma^\mu N \nu_L , \tag{13a}$$

$$i\partial\!\!\!/ \nu_R - M\nu_L = 0 . \tag{13b}$$

We avoid the temptation to include Majorana mass terms, i.e., $M_L\nu_L$ and $M_R\nu_R$ in (13a) and (13b), respectively. The right-handed field can be eliminated to give

$$(\partial_\mu \partial^\mu + M^2)\nu_L = -i\partial\!\!\!/(\sqrt{2} G j_\mu \gamma^\nu N \nu_L) . \tag{14}$$

We now assume the electron distribution is static and unpolarized, so that

$$j_\mu = \rho(x)\delta_{\mu 0} , \tag{15}$$

where ρ is a constant equal to the number of electrons/volume. We can now seek stationary energy eigenfunctions of the form

$$|\nu(x,t)\rangle = |L(x)\rangle e^{-iEt} . \tag{16}$$

The problem is thereby reduced to the time-independent equation for $L(x)$

$$(E^2 - M^2 + \nabla^2)L(x) = \sqrt{2} G[E - (-i\gamma_0 \boldsymbol{\gamma}\cdot\nabla)]\rho N L(x) . \tag{17}$$

Limiting ourselves to the case of uniform density and noting that for high energies the left-handed chiral projection satisfies

$$\gamma^0 \boldsymbol{\gamma}\cdot(-i\nabla)L = -EL \tag{18}$$

allows a great simplification on the right-hand side of Eq. (17). Equation (17) then becomes

$$(F + \nabla^2)L = 0 , \tag{19}$$

where

$$F = E^2 - M^2 - 2\sqrt{2} G\rho E N . \tag{20}$$

The one-dimensional solution is therefore of simple plane-wave form with the momentum of the right-moving solution satisfying

$$(p - \sqrt{F})L(0) = 0 . \tag{21}$$

Even though M and N do not commute, to lowest order in M^2/E^2 and $G\rho E/E^2$ we can write

$$\sqrt{F} = E - \frac{M^2}{2E} - \sqrt{2} G\rho N . \tag{22}$$

This result is identical to Wolfenstein's conjecture [Ref. 10, Eq. (22) with $G \to -\sqrt{2}G$]. Consequently, we obtain the same expressions for the mixing angle in matter.

Calculating the probability of finding a state which was originally ν_e to be ν_e at $x=R$ as discussed in Sec. II we obtain a modification of Eq. (8)

$$P(\nu_e \to \nu_e;R) = 1 - \sin^2 2\theta_m \sin^2\left[\frac{\pi R}{l_m}\right] , \tag{23}$$

where the mixing angle and the oscillation length in matter are given in terms of the vacuum mixing angle as

$$\tan 2\theta_m = (1 - \eta \sec 2\theta_v)^{-1} \tan 2\theta_v , \tag{24a}$$

$$l_m = l_v (1 - 2\eta \cos 2\theta_v + \eta^2)^{-1/2} , \tag{24b}$$

and

$$\eta = \sqrt{2} G\rho E/(m_2{}^2 - m_1{}^2) . \tag{25}$$

This is in complete accord with the prescription of Wolfenstein.

IV. OSCILLATIONS IN NONUNIFORM MATTER

The discussion of Sec. II is valid for nonuniform matter through Eq. (17). Furthermore, as long as $1/E$ is small compared to the distance scale over which the density varies, we can still use the previous results through Eq. (20). Moreover, the same criteria, $(\rho E)^{-1} d\rho/dx \ll 1$, allows us to factorize the wave operator in Eq. (19) even for a spatially varying density. We are therefore led to the following differential equation:

$$\left[\sqrt{F} - i\frac{d}{dx}\right]\left[\sqrt{F} + i\frac{d}{dx}\right] L(x) = 0 . \tag{26}$$

To the extent that nonforward scattering can be ignored, we can omit the locally reflected solution, say the left-moving solution, and finally arrive at the first-order differential equation

$$\left[E-\frac{M^2}{2E}-\sqrt{2}G\rho N+i\frac{d}{dx}\right]L(x)=0 . \tag{27}$$

Since E is a multiple of the identity matrix in flavor space, the E term in Eq. (27) does not affect the oscillation phenomenon and can be dropped. Dropping this term and replacing x by t produces the time-evolution equation based on Wolfenstein's work that has been widely used in the literature. We therefore conclude that Wolfenstein's time-evolution equation is fully justified provided only that the density varies little on a length scale equal to the de Broglie wavelength of the neutrino.

For completeness we consider the solution to Eq. (27) in the limit that the density varies slowly on a distance scale equal to the local oscillation length. We can then use the approximate local plane-wave form,

$$L(x)=A(x)\exp\left[i\int_0^x p(y)dy\right] , \tag{28}$$

in which $A(x)$ is a slowly varying vector in flavor space.

The local momentum as determined by Eq. (27) must therefore satisfy

$$[p(x)-\sqrt{F}\,]A(x)=0 \tag{29}$$

which is identical to Eq. (21) except for the x dependence of p and of ρ. We therefore immediately have a solution to this flavor space eigenvalue problem in terms of a local mixing angle defined by Eq. (24) through the x dependence of η as given in Eq. (25).

We can now calculate $P(\nu_e\rightarrow\nu_e;R)$ in the same manner and find in this case

$$P(\nu_e\rightarrow\nu_e;R)=\cos^2(\alpha-\beta)-\sin2\alpha\sin2\beta\sin^2\left[\frac{f}{2}\right] , \tag{30a}$$

where

$$\alpha=\theta(0) , \tag{30b}$$

$$\beta=\theta_m(R) , \tag{30c}$$

$$f=2\pi\int_0^R l_m{}^{-1}(x)dx . \tag{30d}$$

This expression obviously reduces to the previous result for uniform density, since then $\alpha=\beta$ and $f=2\pi R/l_m$. It is also clear from this expression that the ν_e probability will be very small outside of the matter distribution if the mixing angle at the point of origin approaches its maximum value of 90°.

As a simple application we consider oscillations in a medium with a linear density distribution out to some distance R_0; more specifically for

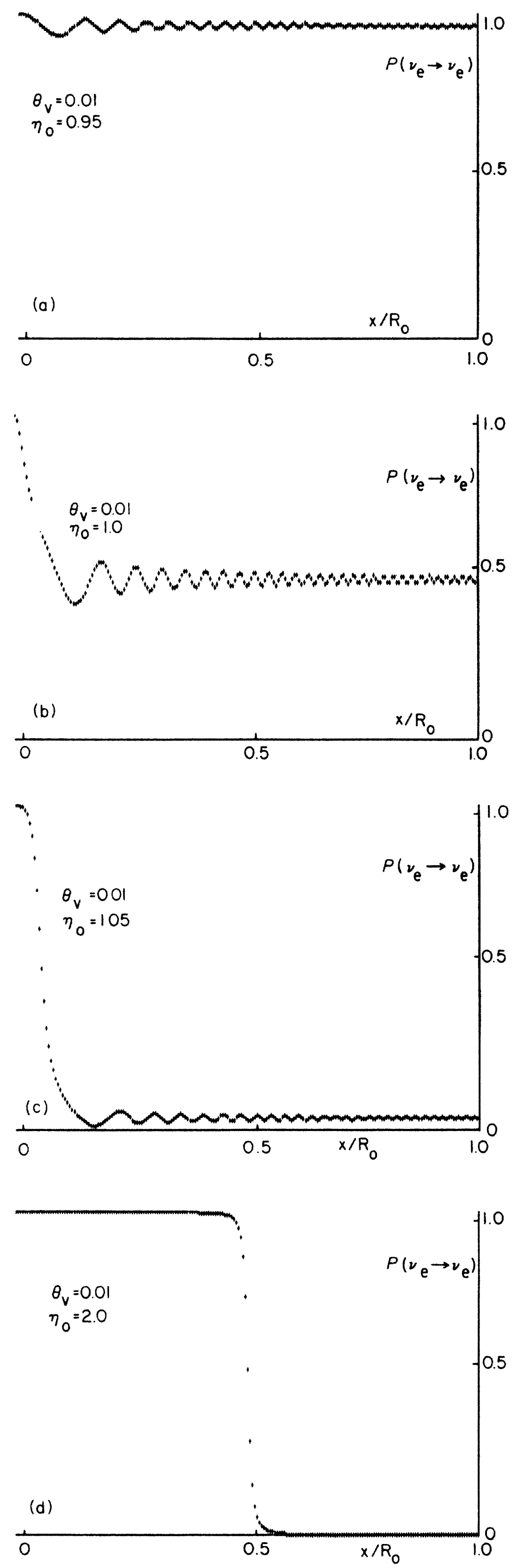

FIG. 1. ν_e appearance probability as a function of distance from the point of creating for the linear density distribution. All curves are for a vacuum mixing angle of 0.01. (a)—(d) illustrate the shape change as one progresses from core densities below the resonance value to core densities above this critical value.

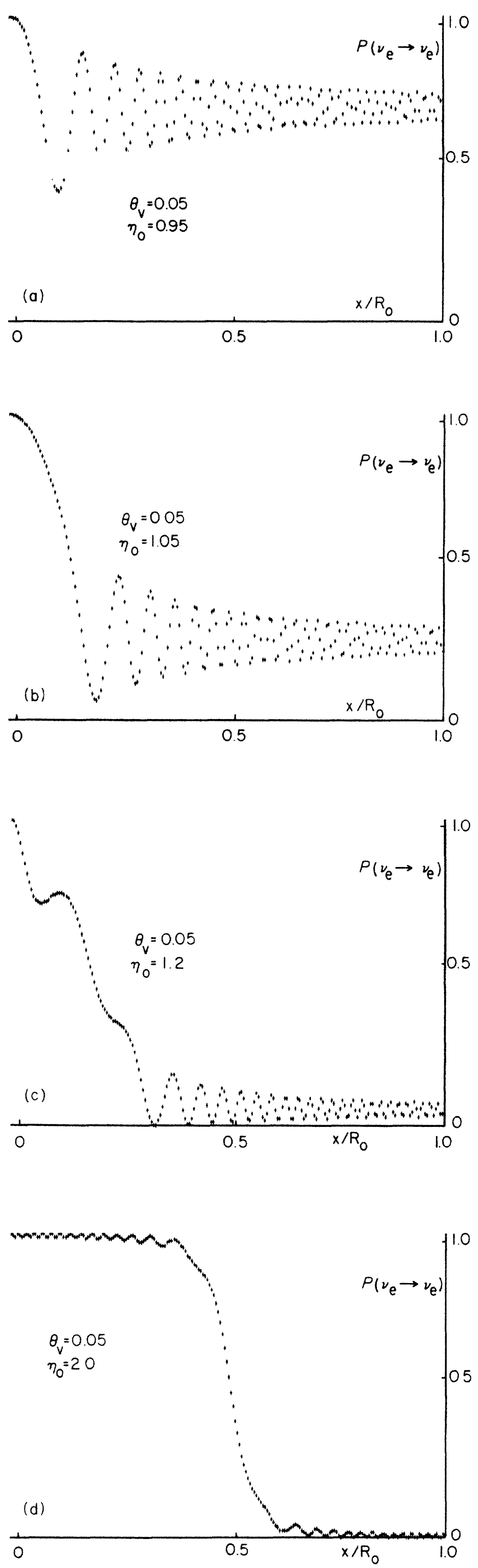

FIG. 2. Same as Fig. 1 for the much larger mixing angle of 0.05.

$$\eta = \begin{cases} \eta_0 \left[1 - \dfrac{x}{R_0}\right], & 0 < x < R_0, \\ 0 & \text{otherwise.} \end{cases} \tag{31}$$

In this case we can evaluate f in terms of elementary functions and find

$$f = \frac{-2\pi R_0}{\eta_0}[I(\eta(R)) - I(\eta_0)] , \tag{32a}$$

where

$$I(\eta) = \frac{2\eta+\beta}{4} g(\eta) + \left[\frac{4-\beta^2}{8}\right] \ln(\beta + 2\eta + 2g\eta) \tag{32b}$$

and

$$\beta = -2\cos 2\theta_v, \quad g(\eta) = (1 + \beta\eta + \eta^2)^{1/2} . \tag{32c}$$

Figures 1 and 2 illustrate the results for two different vacuum mixing angles and several different core densities.

V. SUMMARY

We reformulated the theory of neutrino oscillations in terms of the mixing of different momentum eigenstates having equal energy. This allowed us to reduce the problem of oscillations in static matter to the solution of a time independent second-order differential equation derived from an appropriate Dirac equation with source term. With neglect of nonforward scattering, this effective Schrödinger equation was reduced to a first-order differential equation which was shown to be equivalent to the time-evolution equation for variable matter density that has been used in previous applications. The procedure was illustrated by application to a linear density distribution, and analytical as well as numerical results were given in the limiting case of very slowly varying density.

While we have not been able to discover any new physical phenomenon by this approach, we hope it will serve a pedagogical purpose in clarifying some of the issues surrounding this exciting effect.

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