Anisotropic fluids with multifluid components

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The anisotropic-fluid interpretation of a stress-energy tensor formed from the sum of three tensors, each of which is the energy-momentum tensor of a perfect fluid or a null fluid in the special case that the fluids four-velocities are linearly dependent, is studied. The anisotropic-fluid model formed by an arbitrary number of perfect fluids and null fluids is also studied in the particular case that all the fluids' four-velocities lay on a timelike two-plane. The anisotropic-fluid interpretation of the Bondi model of self-gravitating spheres is presented. The particular case of an anisotropic-fluid model formed with three perfect fluids with a stiff equation of state, and the particular case of two null and one perfect fluid with a $p=\rho$ equation of state, are used as sources of the Einstein equations for a cylindrically symmetric spacetime, and these last equations are solved. Also, for these particular cases the generalization for an arbitrary number of fluid components is indicated.

I. INTRODUCTION

Recently, we studied a model of anisotropic fluid constructed with two perfect fluids, one perfect and one null fluid, and two null fluids.[1] This model has been used (i) to describe anisotropic spheres in general relativity, especially to study the effects of anisotropy on the structure of neutron stars,[2] (ii) to find exact solutions representing self-gravitating anisotropic matter with different symmetries,[1,3-5] and (iii) to generate a particular model of matter that can propagate as soliton waves.[6] Also, the special form of the anisotropic energy-momentum tensor (EMT) that appears in this model has been used in the study of conformally flat, anisotropic spheres.[7]

The purpose of this paper is to generalize the anisotropic-fluid model constructed with two fluid components by the inclusion of an arbitrary number of fluid components such that all the fluids' four-velocities lay on a timelike two-plane. A physical motivation to make such a generalization is that in the study of self-gravitating anisotropic spheres one of the most used models is the Bondi model[8] that has an EMT formed by three fluids: two perfect fluids and one null fluid with linearly dependent four-velocities. Also, the generalization of the Bondi model of Herrera and co-workers[9] can be considered as arising from the superposition of at least four fluids. Furthermore, one of the most simple and interesting solutions to the self-gravitating two-fluid model is the particular example of anisotropic fluid formed with two irrotational perfect fluids obeying the stiff equation of state ($p=\rho$) in a cylindrically symmetric spacetime.[4] This solution admits an almost trivial generalization for the case of a fluid model formed with an arbitrary number of perfect-fluid components as the ones already described.

In Sec. II we study the anisotropic-fluid interpretation of a stress-energy tensor formed from the sum of three tensors each of which is the EMT of a perfect fluid or a null fluid, in the special case that the four-velocities are linearly dependent. In Sec. III the anisotropic-fluid interpretation of the Bondi model of self-gravitating spheres is presented together with a brief discussion of the other already existent models for anisotropic fluid spheres.[10-13] In Sec. IV the particular case of anisotropic fluid model formed with three perfect fluids with a stiff equation of state and the particular case of two null and one perfect fluids with a $p=\rho$ equation of state are studied. In the next section (Sec. V) the Einstein equations coupled to particular cases of anisotropic fluids studied in Sec. IV are solved for a cylindrically symmetric spacetime. Finally, in Sec. VI the generalization of the model for an arbitrary number of fluid components in the particular case that the four-velocities of each fluid component lay on a two-plane is presented. Also, an application of the generalized model is outlined.

II. THE MULTIFLUID MODEL

In this section we study some of the algebraic properties of a stress-energy tensor formed from the sum of three tensors, each of which is the EMT of a perfect fluid or a null fluid, in the special case that the four-velocities associated to these three fluids are *linearly dependent.*

Let us start by analyzing the EMT (Ref. 14)

$$T^{\mu\nu}=\sum_{i=1}^{3} t^{\mu\nu}_{(i)} , \tag{2.1}$$

where each $t^{\mu\nu}_{(i)}$ is either the usual EMT for a perfect fluid, i.e.,

$$t^{\mu\nu}_{(i)}=(p_i+\rho_i)u^{\mu}_{(i)}u^{\nu}_{(i)}-p_i g^{\mu\nu} , \tag{2.2}$$

$$u^{\mu}_{(i)}u_{(i)\mu}=1 , \tag{2.3}$$

or the usual EMT for a null fluid, i.e.,

$$t^{\mu\nu}_{(i)}=\rho_i u^{\mu}_{(i)}u^{\nu}_{(i)} , \tag{2.4}$$

$$u^{\mu}_{(i)}u_{(i)\mu}=0 ; \tag{2.5}$$

$u^{\mu}_{(i)}$, p_i, and ρ_i represent the four-velocity, the pressure, and the rest energy density of the fluid, respectively. Furthermore, we shall assume that there exist functions

b_1, b_2, and b_3, not all of them zero, such that

$$\sum_{i=1}^{3} b_i u^{\mu}_{(i)} = 0 . \tag{2.6}$$

To study the physical meaning of the EMT (2.1) we need to cast it in the general form of the EMT for a single fluid,[15] i.e.,

$$T^{\mu\nu} = \rho U^{\mu} U^{\nu} + S^{\mu\nu} \tag{2.7}$$

with

$$S^{\mu\nu} U_{\nu} = 0 , \tag{2.8}$$

$$U^{\mu} U_{\mu} = 1 , \tag{2.9}$$

$$\rho > 0 ; \tag{2.10}$$

$\rho U^{\mu} U^{\nu}$ represents the EMT kinetic part and $S^{\mu\nu}$ the stress tensor. From (2.1)–(2.5) we get

$$T^{\mu\nu} = \alpha u^{\mu}_{(1)} u^{\nu}_{(1)} + 2\beta u^{(\mu}_{(1)} u^{\nu)}_{(2)} + \gamma u^{\mu}_{(2)} u^{\nu}_{(2)} - \left[\sum_{i=1}^{3} p_i \right] g^{\mu\nu} , \tag{2.11}$$

where the parentheses enclosing the indices μ and ν indicate symmetrization, and

$$\alpha \equiv (p_1 + \rho_1) + (p_3 + \rho_3) a_1{}^2 , \tag{2.12a}$$

$$\gamma \equiv (p_2 + \rho_2) + (p_3 + \rho_3) a_2{}^2 , \tag{2.12b}$$

$$\beta \equiv (p_3 + \rho_3) a_1 a_2 , \tag{2.12c}$$

$$a_1 \equiv -\frac{b_1}{b_3} = \frac{\epsilon_{13}\epsilon_{22} - \epsilon_{23}\epsilon_{12}}{\epsilon_{11}\epsilon_{22} - \epsilon_{12}{}^2} , \tag{2.13a}$$

$$a_2 \equiv -\frac{b_2}{b_3} = \frac{\epsilon_{11}\epsilon_{23} - \epsilon_{12}\epsilon_{13}}{\epsilon_{11}\epsilon_{22} - \epsilon_{12}{}^2} , \tag{2.13b}$$

with

$$\epsilon_{ij} = \epsilon_{ji} \equiv u^{\mu}_{(j)} u_{(i)\mu} . \tag{2.14}$$

Note that $\epsilon_{ii} = 1$ for a perfect fluid and $\epsilon_{ii} = p_i = 0$ for a null fluid. Also, we shall always assume that $u^{0}_{(i)} > 0$, $i = 1,2,3$ i.e., fluids traveling from the past to the future (bona fide fluids). From these assumptions we have $\alpha > 0$, $\gamma > 0$, $\alpha\gamma > \beta^2$, and $\epsilon_{ij} \geq 0$.

By letting

$$u^{\mu}_{(1)} \rightarrow \cos\phi\, u^{*\mu}_{(1)} - \left[\frac{\beta\epsilon_{11} + \gamma\epsilon_{12}}{\alpha\epsilon_{12} + \beta\epsilon_{22}} \right]^{1/2} \sin\phi\, u^{*\mu}_{(2)} , \tag{2.15a}$$

$$u^{\mu}_{(2)} \rightarrow \left[\frac{\alpha\epsilon_{12} + \beta\epsilon_{22}}{\beta\epsilon_{11} + \gamma\epsilon_{12}} \right]^{1/2} \sin\phi\, u^{*\mu}_{(1)} + \cos\phi\, u^{*\mu}_{(2)} , \tag{2.15b}$$

in (2.11) we find

$$T^{\mu\nu} = \alpha^* u^{*\mu}_{(1)} u^{*\nu}_{(1)} + \gamma^* u^{*\mu}_{(2)} u^{*\nu}_{(2)} - \left[\sum_{i=1}^{3} p_i \right] g^{\mu\nu} , \tag{2.16}$$

where

$$\tan(2\phi) = \frac{2[(\alpha\epsilon_{12} + \beta\epsilon_{22})(\beta\epsilon_{11} + \gamma\epsilon_{12})]^{1/2}}{\alpha\epsilon_{11} - \gamma\epsilon_{22}} , \tag{2.17}$$

and

$$\alpha^* \equiv \alpha \cos^2\phi + \beta \left[\frac{\alpha\epsilon_{12} + \beta\epsilon_{22}}{\beta\epsilon_{11} + \gamma\epsilon_{12}} \right]^{1/2} \sin 2\phi + \gamma \left[\frac{\alpha\epsilon_{12} + \beta\epsilon_{22}}{\beta\epsilon_{11} + \gamma\epsilon_{12}} \right] \sin^2\phi , \tag{2.18a}$$

$$\gamma^* \equiv \alpha \left[\frac{\beta\epsilon_{11} + \gamma\epsilon_{12}}{\alpha\epsilon_{12} + \beta\epsilon_{22}} \right] \sin^2\phi - \beta \left[\frac{\beta\epsilon_{11} + \gamma\epsilon_{12}}{\alpha\epsilon_{12} + \beta\epsilon_{22}} \right]^{1/2} \sin 2\phi + \gamma \cos^2\phi , \tag{2.18b}$$

$$u^{*}_{(1)} = \cos\phi\, u^{\mu}_{(1)} + \left[\frac{\beta\epsilon_{11} + \gamma\epsilon_{12}}{\alpha\epsilon_{12} + \beta\epsilon_{22}} \right]^{1/2} \sin\phi\, u^{\mu}_{(2)} , \tag{2.19a}$$

$$u^{*\mu}_{(2)} = -\left[\frac{\alpha\epsilon_{12} + \beta\epsilon_{22}}{\beta\epsilon_{11} + \gamma\epsilon_{12}} \right]^{1/2} \sin\phi\, u^{\mu}_{(1)} + \cos\phi\, u^{\mu}_{(2)} . \tag{2.19b}$$

A direct verification shows that

$$u^{*\mu}_{(1)} u^{*}_{(2)\mu} = 0 . \tag{2.20}$$

The range of the "angle" ϕ is $-\frac{1}{4}\pi \leq \phi \leq \frac{1}{4}\pi$. When $\phi \geq 0$ we have $u^{*\mu}_{(1)} u^{*}_{(1)\mu} > 0$ and $u^{*\mu}_{(2)} u^{*}_{(2)\mu} < 0$; i.e., $u^{*\mu}_{(1)}$ is a future-oriented timelike vector and $u^{*\mu}_{(2)}$ a spacelike vector. And, when $\phi \leq 0$, we have the opposite situation,[16] i.e., $u^{*\mu}_{(1)} u^{*}_{(1)\mu} < 0$ and $u^{*\mu}_{(2)} u^{*}_{(2)\mu} > 0$. This fact can be easily proved taking into account (2.20) and that a timelike vector can only be orthogonal to a spacelike one. Let us first assume $u^{*\mu}_{(1)} u^{*}_{(1)\mu} > 0$ and $u^{*}_{(2)} u^{*}_{(2)\mu} < 0$. In this case, defining

$$U^{\mu} \equiv u^{*\mu}_{(1)} / (u^{*\alpha}_{(1)} u^{*}_{(1)\alpha})^{1/2} , \tag{2.21a}$$

$$\chi^{\mu} = u^{*\mu}_{(2)} / (-u^{*\alpha}_{(2)} u^{*}_{(2)\alpha})^{1/2} , \tag{2.21b}$$

$$\rho = T^{\mu\nu} U_{\mu} U_{\nu} = \alpha^* u^{*\nu}_{(1)} u^{*}_{(1)\nu} - \sum_{i=1}^{3} p_i , \tag{2.22}$$

$$\sigma = T^{\mu\nu} \chi_{\mu} \chi_{\nu} = \sum_{i=1}^{3} p_i - \gamma^* u^{*\alpha}_{(2)} u^{*}_{(2)\alpha} , \tag{2.23}$$

$$\pi = p_1 + p_2 + p_3 , \tag{2.24}$$

we can cast (2.1) as

$$T^{\mu\nu} = (\rho + \pi) U^{\mu} U^{\nu} + (\sigma - \pi) \chi^{\mu} \chi^{\nu} - \pi g^{\mu\nu} . \tag{2.25}$$

From (2.8) and (2.25) we have

$$S^{\mu\nu} = (\sigma - \pi) \chi^{\mu} \chi^{\nu} - \pi (g^{\mu\nu} - U^{\mu} U^{\nu}) , \tag{2.26}$$

and also

$$U^{\mu} U_{\mu} = -\chi^{\mu} \chi_{\nu} = 1 , \tag{2.27}$$

$$\chi^{\mu} U_{\mu} = 0 , \tag{2.28}$$

$$S^{\mu\nu} U_{\nu} = 0 , \tag{2.29a}$$

$$S^{\mu\nu} \chi_{\nu} = -\sigma \chi^{\mu} . \tag{2.29b}$$

A direct computation shows

$$\rho = +\tfrac{1}{2}(\alpha\epsilon_{11}+\gamma\epsilon_{22}+2\beta\epsilon_{12}-2\pi) + \tfrac{1}{2}[(\alpha\epsilon_{11}-\gamma\epsilon_{22})^2 + 4(\alpha\epsilon_{12}+\beta\epsilon_{22})(\beta\epsilon_{11}+\gamma\epsilon_{12})]^{1/2} , \quad (2.30)$$

$$\sigma = -\tfrac{1}{2}(\alpha\epsilon_{11}+\gamma\epsilon_{22}+2\beta\epsilon_{12}-2\pi) + \tfrac{1}{2}[(\alpha\epsilon_{11}-\gamma\epsilon_{22})^2 + 4(\alpha\epsilon_{12}+\beta\epsilon_{22})(\beta\epsilon_{11}+\gamma\epsilon_{12})]^{1/2} . \quad (2.31)$$

Note that the density ρ and the pressures σ and π are positive quantities. In the case that $u^{*\mu}_{(1)}u^{*}_{(1)\mu}<0$ and $u^{*\mu}_{(2)}u^{*}_{(2)\mu}>0$ we find that we must change $(1)\rightarrow(2)$ in (2.21) and that the expressions (2.24), (2.30), and (2.31) that give π, ρ, and σ, respectively, are kept as they are.

The EMT (2.25) describes an anisotropic fluid with a pressure $\sigma > \pi$ along the spacelike direction χ^μ and a pressure π on the plane defined by the two other spacelike directions that are perpendicular to both χ^μ and U^μ.

The expression (2.25) is identical to the one obtained in the two-fluid case, but the expressions for ρ, σ, and π are different. In general, the inclusion of the third fluid component is equivalent to having a different equation of state. We shall return to this point in the following sections. The quantities defined for the two-fluid model can be recovered from the corresponding ones of the multifluid model by letting $u^{\mu}_{(3)}=\rho_3=p_3=0$.

The multifluid model studied in this section presents a great number of parameters that need to be either specified or related via "equations of state" or other equations. The restrictions on these parameters will depend on the specific applications as we shall see in the following sections. A discussion of this point for the two-fluid model can be found in Refs. 1 and 6.

The EMT (2.25) is defined by eight functions: ρ, σ, π, and the five independent components of U^μ and χ^μ. Note that the EMT built out of two fluids is defined by ten functions ρ_1,ρ_2,p_1,p_2, and the six independent components of $u^{\mu}_{(1)}$ and $u^{\mu}_{(2)}$. Thus, given a particular EMT (2.25), we can always find a two-fluid model with an equivalent EMT. Since the number of independent functions that define the two-fluid model is greater than the one that defines the anisotropic one-fluid model with EMT (2.25), we have that the two-fluid model is not uniquely determined by the specification of (2.25), alone. Also, given a particular EMT (2.25), we can always find a three-fluid model with equivalent EMT. The problem of uniqueness already mentioned worsens in this case because the number of independent functions needed to specify the three-fluid model discussed in this section is 12: three densities ρ_i, three pressured p_i and the six independent components of the three linearly dependent $u^{\mu}_{(i)}$, $i=1,2,3$.

III. SELF-GRAVITATING ANISOTROPIC SPHERES

In this section—as an application of the multifluid model presented in Sec. II—we study the anisotropic-fluid interpretation of the Bondi approach to the contraction of self-gravitating spheres in general relativity. This approach consists in the supposition that the EMT associated with the fluid sphere when expressed in pure local Minkowski coordinates and viewed by an observer moving relative to these coordinates with velocity ω in the radial direction is formed by three parts[8]: (a) a perfect fluid of density ρ_1, pressure p_1, and four-velocity $u^{\mu}_{(1)}$ ($\epsilon_{11}=1$); (b) unpolarized radiation of energy density ρ_2 and four-velocity $u^{\mu}_{(2)}$ ($\epsilon_{22}=p_2=0$); (c) isotropic radiation of energy density $\rho_3=3p_3$, pressure p_3, and four-velocity $u^{\mu}_{(3)}$ ($\epsilon_{33}=1$).

When viewed by this moving observer, the covariant EMT is[8]

$$\begin{bmatrix} \rho_1+\rho_2+3p_3 & -\rho_2 & 0 & 0 \\ -\rho_2 & p_1+p_3+\rho_2 & 0 & 0 \\ 0 & 0 & p_1+p_3 & 0 \\ 0 & 0 & 0 & p_1+p_3 \end{bmatrix} . \quad (3.1)$$

To describe the contraction of the sphere the spacetime line element is usually written in either Schwarzschild coordinates

$$ds^2 = e^{\nu}dt^2 - e^{\lambda}dr^2 - r^2(d\theta^2+\sin^2\theta\, d\phi^2) , \quad (3.2)$$

or Bondi radiation coordinates[8]

$$ds^2 = e^{2\beta}[(V/r)du^2 + 2\,du\,dr] - r^2(d\theta^2+\sin^2\theta\, d\phi^2) , \quad (3.3)$$

where ν and λ are functions of t and r only and β and V are functions of u and r only. In Schwarzschild coordinates we find that the above described EMT reduces to[8]

$$T^{00} = \frac{e^{-\nu}}{1-\omega^2}[\rho_1+p_1+4p_3+(1+\omega)^2\rho_2 - (1-\omega^2)(p_1+p_3)] , \quad (3.4a)$$

$$T^{11} = \frac{e^{-\lambda}}{1-\omega^2}[(\rho_1+p_1+4p_3)\omega^2+(1+\omega)^2\rho_2 + (1-\omega^2)(p_1+p_3)] , \quad (3.4b)$$

$$T^{01} = \frac{e^{-(\nu+\lambda)/2}}{1-\omega^2}[(\rho_1+p_1+4p_3)\omega+(1+\omega)^2\rho_2] , \quad (3.4c)$$

$$T^{2}{}_{2} = T^{3}{}_{3} = -(p_1+p_3) , \quad (3.4d)$$

and in Bondi radiation coordinates to

$$T^{00} = \frac{r(1-\omega)e^{-2\beta}}{V(1+\omega)}(\rho_1+p_1+4p_3) , \quad (3.5a)$$

$$T^{01} = \frac{e^{-2\beta}}{1+\omega}[\omega(\rho_1+p_1+4p_3) - (1+\omega)(p_1+p_3)] , \quad (3.5b)$$

$$T^{11} = \frac{Ve^{-2\beta}}{r(1-\omega^2)}[\omega^2(\rho_1+p_1+4p_3)+(1+\omega)^2\rho_2 + (1-\omega^2)(p_1+p_3)] , \quad (3.5c)$$

$$T^{2}{}_{2} = T^{3}{}_{3} = -(p_1+p_3) . \quad (3.5d)$$

We notice that (3.4) and (3.5) have exactly the form

(2.1)–(2.5) and that, in each case, the fluids' four-velocities are given by

$$(u^\mu_{(1)})=(u^\mu_{(3)})=\left[\frac{e^{-\nu/2}}{(1-\omega^2)^{1/2}},\frac{\omega e^{-\lambda/2}}{(1-\omega^2)^{1/2}},0,0,\right], \quad (3.6a)$$

$$(u^\mu_{(2)})=\left[\frac{1+\omega}{(1-\omega^2)^{1/2}}e^{-\nu/2},\frac{1+\omega}{(1-\omega^2)^{1/2}}e^{-\lambda/2},0,0\right], \quad (3.6b)$$

and

$$(u^\mu_{(1)})=(u^\mu_{(3)})=\left[\frac{(1-\omega)(r/V)^{1/2}e^{-\beta}}{(1-\omega^2)^{1/2}},\frac{\omega(V/r)^{1/2}e^{-\beta}}{(1-\omega^2)^{1/2}},0,0\right], \quad (3.7a)$$

$$(u^\mu_{(2)})=\left[0,\frac{(1+\omega)(V/r)^{1/2}e^{-\beta}}{(1-\omega^2)^{1/2}},0,0\right], \quad (3.7b)$$

respectively.

Since the three four-velocities $u^\mu_{(i)}$, $i=1,2,3$, are linearly dependent we have that the Bondi EMT can be cast in the form (2.25), i.e., describes an anisotropic fluid with energy density and pressures given by

$$\rho=\tfrac{1}{2}(\rho_1-p_1+2p_3)+\tfrac{1}{2}[(\rho_1+p_1+4p_3)(\rho_1+p_1+4p_3+4\rho_2)]^{1/2}, \quad (3.8)$$

$$\sigma=\tfrac{1}{2}(p_1-\rho_1-2p_3)+\tfrac{1}{2}[(\rho_1+p_1+4p_3)(\rho_1+p_1+4p_3+4\rho_2)]^{1/2}, \quad (3.9)$$

$$\pi=p_1+p_3, \quad (3.10)$$

respectively. The anisotropic fluid four-velocity and the direction of anisotropy are given by (2.21a) and (2.21b) with

$$u^{*\mu}_{(1)}=\cos\phi u^\mu_{(1)}+\left[\frac{\rho_2}{\rho_1+p_1+4p_3}\right]^{1/2}\sin\phi u^\mu_{(2)}, \quad (3.11a)$$

$$u^{*\mu}_{(2)}=-\left[\frac{\rho_1+p_1+4p_3}{\rho_2}\right]^{1/2}\sin\phi u^\mu_{(1)}+\cos\phi u^\mu_{(2)}, \quad (3.11b)$$

respectively, where

$$\tfrac{1}{2}\tan(2\phi)=\left[\frac{\rho_2}{\rho_1+p_1+4p_3}\right]^{1/2}, \quad (3.12)$$

and $u^\mu_{(1)}$ and $u^\mu_{(2)}$ are given by (3.6) for Schwarzschild coordinates and by (3.7) for Bondi radiation coordinates.

A different approach to the problem of contracting spheres is Vaidya's[10] who, following an idea of Tolman,[17] considered a two-fluid model with a perfect- and a null-fluid component. In order to reduce the number of unknowns the perfect-fluid component is taken as being comoving to the system of coordinates. Hence, Bondi's approach can be considered as a generalization of Vaidya's. A completely different approach is the one due to Bowers and Liang[12] who consider *prima facie* the EMT for the fluid sphere as

$$T^\mu{}_\nu=\mathrm{diag}(\rho,-p_r,-p_\perp,-p_\perp), \quad (3.13)$$

where p_r and $p_\perp$ are the radial and the tangential pressure, respectively. Recently, Herrera and co-workers[9] considered a generalization of the Bondi model obtained by replacing in (3.1) the first fluid component by an anisotropic fluid component of EMT similar to (3.13).

The anisotropic fluid components associated with the Vaidya model can be easily computed using the expressions presented in Sec. II. Moreover, from the discussion presented at the end of Sec. II we conclude that the anisotropic fluid considered by Bowers and Liang can be considered as a multifluid model with at least two fluid components. Thus, the EMT considered by Herrera and co-workers can be considered as being formed by at least four fluids with EMT like (2.2) and (2.4). We shall come back to this point in the last section of this paper and in the future.

IV. ANISOTROPIC FLUIDS WITH IRROTATIONAL AND NULL FLUID COMPONENTS

In this section we study the model of anisotropic fluid presented in Sec. II in the particular cases that (a) each one of the three fluid components is irrotational and has equations of state $p_i=\rho_i$, $i=1,2,3$ and (b) one of the fluid components is irrotational with $p_1=\rho_1$ equation of state and the other two are null fluids. The condition of irrotationality for a fluid is guaranteed by the expression[18]

$$u^\mu_{(i)}=\phi^{;\mu}_{(i)}/(\lambda_{ii})^{1/2}, \quad (4.1)$$

where $\phi_{(i)}$ is the velocity potential, the comma denotes partial differentiation, as in $\phi_{(i),\mu}=\partial\phi_{(i)}/\partial x^\mu, \phi^{;\mu}_{(i)}=g^{\mu\nu}\phi_{(i),\nu}$, and

$$\lambda_{ij}=\lambda_{ji}\equiv g_{\alpha\beta}\phi^{;\alpha}_{(i)}\phi^{;\beta}_{(j)}. \quad (4.2)$$

For an irrotational fluid with $p_i=\rho_i$ equation of state we have[18]

$$\rho_i=p_i=\tfrac{1}{2}\lambda_{ii}. \quad (4.3)$$

The theory developed in Sec. II applies only in the case that the four-velocities of the three fluids are linearly dependent. In case (a) this condition is equivalent to the existence of functions $b_i'=b_i/(\lambda_{ii})^{1/2}$ such that

$$\sum_{i=1}^{3} b_i'\phi_{(i),\mu}=0 \; . \tag{4.4}$$

In the next section we shall give a particular example wherein (4.4) is automatically satisfied. From (2.12)–(2.14), (4.1), and (4.2) we get

$$\alpha=\lambda_{11}\left[1+\left[\frac{\lambda_{22}\lambda_{13}-\lambda_{12}\lambda_{23}}{\lambda_{11}\lambda_{22}-\lambda_{12}{}^2}\right]^2\right] , \tag{4.5a}$$

$$\gamma=\lambda_{22}\left[1+\left[\frac{\lambda_{11}\lambda_{23}-\lambda_{12}\lambda_{13}}{\lambda_{11}\lambda_{22}-\lambda_{12}{}^2}\right]^2\right] , \tag{4.5b}$$

$$\beta=(\lambda_{11}\lambda_{22})^{1/2}\times\frac{(\lambda_{22}\lambda_{13}-\lambda_{12}\lambda_{23})(\lambda_{11}\lambda_{23}-\lambda_{12}\lambda_{13})}{(\lambda_{11}\lambda_{22}-\lambda_{12}{}^2)^2} , \tag{4.5c}$$

$$\epsilon_{ij}=\lambda_{ij}/(\lambda_{ii}\lambda_{jj})^{1/2} \; . \tag{4.6}$$

The anisotropic fluid variables in this case reduce to

$$u^{*\mu}_{(1)}=\cos\phi u^{\mu}_{(1)}+\left[\frac{\gamma\lambda_{12}+\beta(\lambda_{11}\lambda_{22})^{1/2}}{\alpha\lambda_{12}+\beta(\lambda_{11}\lambda_{22})^{1/2}}\right]^{1/2}\sin\phi u^{\mu}_{(2)} , \tag{4.7a}$$

$$u^{*\mu}_{(2)}=-\left[\frac{\alpha\lambda_{12}+\beta(\lambda_{11}\lambda_{22})^{1/2}}{\gamma\lambda_{12}+\beta(\lambda_{11}\lambda_{22})^{1/2}}\right]^{1/2}\sin\phi u^{\mu}_{(1)}+\cos\phi u^{\mu}_{(2)} , \tag{4.7b}$$

and

$$\rho=\tfrac{1}{2}(\alpha+\gamma+2\beta\epsilon_{12}-2\pi)+\tfrac{1}{2}[(\alpha-\gamma)^2+4(\beta+\alpha\epsilon_{12})(\beta+\gamma\epsilon_{12})]^{1/2} , \tag{4.8}$$

$$\sigma=-\tfrac{1}{2}(\alpha+\gamma+2\beta\epsilon_{12}-2\pi)+\tfrac{1}{2}[(\alpha-\gamma)^2+4(\beta+\alpha\epsilon_{12})(\beta+\gamma\epsilon_{12})]^{1/2} , \tag{4.9}$$

$$\pi=\tfrac{1}{2}(\lambda_{11}+\lambda_{22}+\lambda_{33}) , \tag{4.10}$$

$$\tfrac{1}{2}\tan(2\phi)=\frac{[(\alpha\lambda_{12}+\beta(\lambda_{11}\lambda_{22})^{1/2}(\gamma\lambda_{12}+\beta(\lambda_{11}\lambda_{22})^{1/2}]^{1/2}}{(\alpha-\gamma)(\lambda_{11}\lambda_{22})^{1/2}} \; . \tag{4.11}$$

For the second case we have two null fluids that we shall take as fluid 1 and fluid 2; in other words, we take

$$p_1=p_2=\epsilon_{11}=\epsilon_{22}=0 , \tag{4.12}$$

and one irrotational perfect fluid with $p_3=\rho_3$ equation of state. In the present case relations (4.1)–(4.3) hold only for $i=3$. Now we shall assume

$$b_1u_{(1)\mu}+b_2u_{(2)\mu}+b_3'\phi_{(3),\mu}=0 , \tag{4.13}$$

instead of (4.4). From (2.12)–(2.14), (4.1), (4.2), and (4.12) we get

$$\alpha=\rho_1+\lambda_{33}(\epsilon_{23}/\epsilon_{12})^2 , \tag{4.14a}$$

$$\gamma=\rho_2+\lambda_{33}(\epsilon_{13}/\epsilon_{12})^2 , \tag{4.14b}$$

$$\beta=\lambda_{33}\epsilon_{23}\epsilon_{13}/\epsilon_{12}{}^2 \; . \tag{4.14c}$$

The anisotropic fluid variables in this case reduce to

$$2^{1/2}u^{*\mu}_{(1)}=u^{\mu}_{(1)}+(\gamma/\alpha)^{1/2}u^{\mu}_{(2)} , \tag{4.15a}$$

$$2^{1/2}u^{*}_{(2)}=-(\alpha/\gamma)^{1/2}u^{\mu}_{(1)}+u^{\mu}_{(2)} , \tag{4.15b}$$

$\phi=45°$, and

$$\rho=[\beta+(\alpha\gamma)^{1/2}]\epsilon_{12}-\tfrac{1}{2}\lambda_{33} , \tag{4.16}$$

$$\sigma=[-\beta+(\alpha\gamma)^{1/2}]\epsilon_{12}+\tfrac{1}{2}\lambda_{33} , \tag{4.17}$$

$$\pi=p_3=\tfrac{1}{2}\lambda_{33} \; . \tag{4.18}$$

It is instructive to compare Eqs. (4.16)–(4.18) with the corresponding relations for the two-fluid case. These are obtained from the former, in the limit $\phi_{(3)}=0$, and we get

$$\rho=(\alpha\gamma)^{1/2}\epsilon_{12} , \tag{4.19}$$

$$\sigma=(\alpha\gamma)^{1/2}\epsilon_{12} , \tag{4.20}$$

$$\pi=0 \; . \tag{4.21}$$

Note that (4.19) and (4.20) tell us $\rho=\sigma$. From the comparison of (4.16)–(4.18) with (4.19)–(4.21) it is clear that the addition of a third fluid to the two-fluid model has the effect of considering a more general equation of state for the anisotropic fluid pressures and density. Note that the previous remark is also valid in the general case.

V. SOLUTIONS TO THE EINSTEIN EQUATIONS

In this section we shall study solutions to the Einstein equations coupled to the two particular cases of the multifluid model presented in Sec. IV for a cylindrically symmetric spacetime. The Einstein equations coupled to (2.1) read

$$R_{\mu\nu}-\tfrac{1}{2}g_{\mu\nu}R=-\sum_{i=1}^{3}t^{\mu\nu}_{(i)} \; . \tag{5.1}$$

The integrability conditions for the system of equations (5.1) (Bianchi identity) give

$$\sum_{i=1}^{3}t^{\mu\nu}_{(i);\nu}=0 , \tag{5.2}$$

where the semicolon denotes a covariant derivative. The simplest way to implement (5.2) is to assume

$$t^{\mu\nu}_{(i);\nu}=0, \quad i=1,2,3 \; . \tag{5.3}$$

For the cases that we shall consider in this section this assumption is enough to give a complete set of equations. Note that for some particular cases (5.3) may become too restrictive.[10,6]

The Einstein equations (5.1) together with their integrability conditions (5.3) for the model of three irrotational fluids with $p_i=\rho_i$ equation of state can be cast as

$$R_{\mu\nu}=-\sum_{i=1}^{3}\phi_{(i),\mu}\phi_{(i),\nu}\ , \tag{5.4}$$

$$\partial_\mu(\sqrt{-g}\,g^{\mu\nu}\phi_{(i),\nu})=0,\quad i=1,2,3\ . \tag{5.5}$$

We shall consider a spacetime with the cylindrically symmetric metric[19]

$$ds^2=e^{\omega}(dt^2-dr^2)-\gamma_{ab}dx^a dx^b\ , \tag{5.6}$$

where the sum convention is assumed in the indices a and b that take the values 2 and 3, $(x^0,x^1,x^2,x^3)=(t,r,\theta,z)$, γ_{ab} and ω are functions of t and r only, and

$$\gamma_{ab}=\gamma_{ba}\ , \tag{5.7a}$$

$$\det(\gamma)=\tau^2>0\ . \tag{5.7b}$$

From (5.4)–(5.7) and the fact that cylindrical symmetry implies that the $\phi_{(i)}$ are functions of t and r only, we get

$$R_{00}+R_{11}=(\ln\tau)_{,00}+(\ln\tau)_{,11}-\tfrac{1}{4}(\gamma_{ab,0}\gamma^{ab}_{,0}+\gamma_{ab,1}\gamma^{ab}_{,1})-\omega_{,0}(\ln\tau)_{,0}-\omega_{,1}(\ln\tau)_{,1}$$
$$=-\sum_{i=1}^{3}[(\phi_{(i),0})^2+(\phi_{(i),1})^2]\ , \tag{5.8a}$$

$$2R_{01}=2(\ln\tau)_{,01}-\tfrac{1}{2}\gamma_{ab,0}\gamma^{ab}_{,1}-\omega_{,0}(\ln\tau)_{,1}-\omega_{,1}(\ln\tau)_{,0}$$
$$=-2\sum_{i=1}^{3}\phi_{(i),0}\phi_{(i),1}\ , \tag{5.8b}$$

and

$$(\tau\gamma_{ab,0}\gamma^{bc})_{,0}-(\tau\gamma_{ab,1}\gamma^{bc})_{,1}=0\ , \tag{5.9}$$

$$(\tau\phi_{(i),0})_{,0}-(\tau\phi_{(i),1})_{,1}=0\ , \tag{5.10}$$

where γ^{ab} is defined by

$$\gamma^{ab}\gamma_{ac}=\delta^b_c\ . \tag{5.11}$$

We have not listed the equation that results from $R_{00}-R_{11}$ due to the fact that it is a consequence of the other field equations. By taking the trace of (5.9) we get

$$(\tau)_{,00}-(\tau)_{,11}=0\ . \tag{5.12}$$

From (5.8) we find that

$$d\omega=(\tau/\Delta)[(A\tau_{,0}-B\tau_{,1})dt+(B\tau_{,0}-A\tau_{,1})dr]\ , \tag{5.13}$$

where

$$A\equiv\sum_{i=1}^{3}[(\phi_{(i),0})^2+(\phi_{(i),1})^2]+(\ln\tau)_{,00}$$
$$+(\ln\tau)_{,11}-\tfrac{1}{4}(\gamma_{ab,0}\gamma^{ab}_{,0}+\gamma_{ab,1}\gamma^{ab}_{,1})\ , \tag{5.14a}$$

$$B\equiv 2\sum_{i=1}^{3}\phi_{(i),0}\phi_{(i),1}+2(\ln\tau)_{,01}-\tfrac{1}{2}\gamma_{ab,0}\gamma^{ab}_{,1}\ , \tag{5.14b}$$

$$\Delta\equiv\tau_{,0}{}^2-\tau_{,1}{}^2\neq 0\ . \tag{5.15}$$

The integrability of the differential form (5.13) is guaranteed by Eqs. (5.9), (5.10), and (5.12). Moreover, we have that ω can be cast as

$$\omega=\ln(\Delta/\tau)+\Omega+\sum_{i=1}^{3}\Xi_i\ , \tag{5.16}$$

where $d\Omega$ and $d\Xi_i$ are the exact forms

$$d\Omega=-[\tau/(4\Delta)]\{[(\gamma_{ab,0}\gamma^{ab}_{,0}+\gamma_{ab,1}\gamma^{ab}_{,1})\tau_{,0}-2\gamma_{ab,0}\gamma^{ab}_{,1}\tau_{,1}]dt+[2\gamma_{ab,0}\gamma^{ab}_{,1}\tau_{,0}-(\gamma_{ab,0}\gamma^{ab}_{,0}+\gamma_{ab,1}\gamma^{ab}_{,1})\tau_{,1}]dr\}\ , \tag{5.17}$$

$$d\Xi_i=[\tau/(4\Delta)](\{[(\phi_{(i),0})^2+(\phi_{(i),1})^2]\tau_{,0}-2\phi_{(i),0}\phi_{(i),1}\tau_{,1}\}dt+\{2\phi_{(i),0}\phi_{(i),1}\tau_{,0}-[(\phi_{(i),0})^2+(\phi_{(i),1})^2\tau_{,1}\}dr)\ . \tag{5.18}$$

Thus, the solution of Eqs. (5.9) and (5.10) determine completely the solution of the Einstein equations (5.4) and (5.5) for the metric (5.6). Equation (5.10) is equivalent to the usual cylindrical wave equation when expressed in the coordinates defined by the solutions of (5.12), i.e.,

$$\tau=G_+(t-r)+G_-(t+r)\ , \tag{5.19a}$$

$$\xi=G_+(t-r)-G_-(t+r)\ , \tag{5.19b}$$

where $G_\pm(t\mp r)$ are arbitrary functions of the indicated arguments.[4,20] The system of Eqs. (5.9) is well known and has been studied by a variety of authors in different contexts.[21]

To interpret this solution as an anisotropic fluid, we first need to check if the condition (2.6) is satisfied. Indeed, this is the case since the four-velocities of the three fluids are on the plane (t,r). The anisotropic fluid variables are given by Eqs. (4.7)–(4.11) with

$$\lambda_{ij}=e^{-\omega}(\phi_{(i),0}\phi_{(j),0}-\phi_{(i),1}\phi_{(j),1}) , \tag{5.20}$$

$$\alpha=e^{-\omega}[(\phi_{(1),0})^2-(\phi_{(1),1})^2]\left[1+\left[\frac{\phi_{(2),0}\phi_{(3),1}-\phi_{(2),1}\phi_{(3),0}}{\phi_{(1),0}\phi_{(2),1}-\phi_{(1),1}\phi_{(2),0}}\right]^2\right] , \tag{5.21a}$$

$$\gamma=e^{-\omega}[(\phi_{(2),0})^2-(\phi_{(2),1})^2]\left[1+\left[\frac{\phi_{(1),1}\phi_{(3),0}-\phi_{(1),0}\phi_{(3),1}}{\phi_{(1),0}\phi_{(2),1}-\phi_{(1),1}\phi_{(2),0}}\right]^2\right] , \tag{5.21b}$$

$$\beta=e^{-\omega}\frac{\{[(\phi_{(1),0})^2-(\phi_{(1),1})^2][(\phi_{(2),0})^2-(\phi_{(2),1})^2]\}^{1/2}}{(\phi_{(1),0}\phi_{(2),1}-\phi_{(1),1}\phi_{(2),0})^2}(\phi_{(1),1}\phi_{(3),0}-\phi_{(1),0}\phi_{(3),1})(\phi_{(2),0}\phi_{(3),1}-\phi_{(2),1}\phi_{(3),0}) . \tag{5.21c}$$

The Einstein equations (5.1) together with their integrability conditions (5.3) for the model of one irrotational fluid with a $p_3=\rho_3$ equation of state and two null fluids can be cast as Eq. (5.5) for $i=3$ and

$$R_{\mu\nu}=-\rho_1 u_{(1)\mu}u_{(1)\nu}-\rho_2 u_{(2)\mu}u_{(2)\nu}-\phi_{(3),\mu}\phi_{(3),\nu} , \tag{5.22}$$

$$(\rho_i u^{\mu}_{(i)}u^{\nu}_{(i)})_{;\nu}=0, \quad i=1,2 . \tag{5.23}$$

For the particular metric (5.6) we choose the null vectors as $[u^{\mu}_{(1)}]=(1,1,0,0)$ and $[u^{\mu}_{(2)}]=(1,-1,0,0)$. In this case conditions (5.23) give the equations

$$\rho_{1,0}+\rho_{1,1}+\rho_1[2(\omega_{,0}+\omega_{,1})+(\tau_{,0}+\tau_{,1})/\tau]=0 , \tag{5.24a}$$

$$\rho_{2,0}-\rho_{2,1}+\rho_2[2(\omega_{,0}-\omega_{,1})+(\tau_{,0}-\tau_{,1})/\tau]=0 , \tag{5.24b}$$

that can be easily integrated yielding

$$\rho_1=\tau^{-1}e^{-2\omega}F_+(t-r) , \tag{5.25a}$$

$$\rho_2=\tau^{-1}e^{-2\omega}F_-(t+r) , \tag{5.25b}$$

where $F_\pm(t\mp r)$ are arbitrary functions of the indicated arguments. From (2.26), (5.23), and (5.6) we find, as in the preceding case, that the solution of the Einstein equations reduces to the integration of (5.9) and (5.10) with $i=3$ and to the computation of ω that is given by

$$\omega=\ln(\Delta/\tau)+\Omega+\Xi_3+\Lambda_++\Lambda_- , \tag{5.26}$$

where $\Lambda_\pm$ are the quadratures

$$\Lambda_\pm=\int\frac{2}{\tau_{,0}\mp\tau_{,1}}F_\pm(t\mp r)d(t\mp r) . \tag{5.27}$$

The vectors $u^{\mu}_{(1)}$ and $u^{\mu}_{(2)}$ defined above together with $u^{\mu}_{(3)}=\phi^{;\mu}_{(3)}/(\lambda_{33})^{1/2}$ are on the plane (t,r) and consequently they are linearly dependent. Hence, the solution to the Einstein equations (5.22) can also be interpreted as a solution for an anisotropic fluid with variables given by (4.16)–(4.18) with

$$\alpha=e^{-2\omega}[\tfrac{1}{4}(\phi_{(3),0}-\phi_{(3),1})^2+\tau^{-1}F_+] , \tag{5.28a}$$

$$\gamma=e^{-2\omega}[\tfrac{1}{4}(\phi_{(3),0}+\phi_{(3),1})^2+\tau^{-1}F_-] , \tag{5.28b}$$

$$\beta=\tfrac{1}{4}e^{-2\omega}[(\phi_{(3),0})^2-(\phi_{(3),1})^2] , \tag{5.28c}$$

$$\epsilon_{12}=2e^{\omega} , \tag{5.29}$$

and

$$\sqrt{2}[u^{*\mu}_{(1)}]=[1+(\gamma/\alpha)^{1/2},1-(\gamma/\alpha)^{1/2},0,0] , \tag{5.30a}$$

$$\sqrt{2}[u^{*\mu}_{(2)}]=[1-(\alpha/\gamma)^{1/2},-1-(\alpha/\gamma)^{1/2},0,0] . \tag{5.30b}$$

In other words, we found that a cylindrically symmetric irrotational fluid with stiff equation of state with polarized electromagnetic radiation going in and out of the symmetry axis is equivalent to the anisotropic fluid described by (4.16)–(4.18) and (5.28)–(5.30).

To end this section we want to point out that in both of the examples considered ω has the form

$$\omega=\omega_v+\omega_m , \tag{5.31}$$

where

$$\omega_v=\ln(\Delta/\tau)+\Omega , \tag{5.32a}$$

$$\omega_m=\begin{cases}\sum_{i=1}^{3}\Xi_i, & \text{three } p=\rho \text{ fluids} \quad (5.32b)\\ \Xi_3+\Lambda_++\Lambda_-, & \text{one } p=\rho \text{ and two null fluids} , \quad (5.32c)\end{cases}$$

i.e., in the coefficient ω the contributions of the vacuum and the matter are uncoupled. Particular cases of the solutions presented in this section have been studied by a number of authors.[4,22,18] Also, both of the solutions presented in this section can be easily generalized by adding to the respective EMT an arbitrary number of irrotational perfect fluids with a $p=\rho$ equation of state. As a matter of fact we only need to add the corresponding functions Ξ to the ω_m given by (5.32c) to have the solution of the Einstein equations coupled to an arbitrary number of irrotational perfect fluids with $p=\rho$ equations of state and the two null fluids already described. We shall come back to this point in the next section.

VI. DISCUSSION

The anisotropic-fluid model with three fluid components presented in Sec. II can be easily generalized to the case of an arbitrary number of fluid components in the particular case that all the fluids' four-velocities lay on a two-plane. Note that this is the case for the multifluid interpretation of the model of Herrera and coworkers[9] as well as for the cylindrically symmetric multifluid solution described at the end of Sec. V. If all the fluids' four-velocities are on a two-plane, we can choose two different four-velocities, say $u^{\mu}_{(1)}$ and $u^{\mu}_{(2)}$, as a basis to describe the rest, i.e., we can always set

$$u^{\mu}_{(i)}=a_{(i)1}u^{\mu}_{(1)}+a_{(i)2}u^{\mu}_{(2)}, \quad i=1,2,\ldots,n , \tag{6.1}$$

where n is the number of fluid components and

$$a_{(1)1}=a_{(2)2}=1\ , \tag{6.2a}$$

$$a_{(1)2}=a_{(2)1}=0\ . \tag{6.2b}$$

From (6.1) and (2.14) we have

$$a_{(j)1}=\frac{\epsilon_{1j}\epsilon_{22}-\epsilon_{2j}\epsilon_{12}}{\epsilon_{11}\epsilon_{22}-\epsilon_{12}^{2}}\ , \tag{6.3a}$$

$$a_{(j)2}=\frac{\epsilon_{2j}\epsilon_{11}-\epsilon_{1j}\epsilon_{12}}{\epsilon_{11}\epsilon_{22}-\epsilon_{12}^{2}},\ \ j=3,4,\ldots,n\ . \tag{6.3b}$$

The EMT for the n-fluid model

$$T^{\mu\nu}=\sum_{i=1}^{n} t_{(i)}^{\mu\nu} \tag{6.4}$$

can be cast as

$$T^{\mu\nu}=\alpha u_{(1)}^{\mu}u_{(1)}^{\nu}+2\beta u_{(1)}^{(\mu}u_{(2)}^{\nu)}+\gamma u_{(2)}^{\mu}u_{(2)}^{\nu}-\pi g^{\mu\nu}\ , \tag{6.5}$$

where

$$\alpha=\sum_{i=1}^{n}(p_i+\rho_i)(a_{(i)1})^2\ , \tag{6.6a}$$

$$\beta=\sum_{i=1}^{n}(p_i+\rho_i)a_{(i)1}a_{(i)2}\ , \tag{6.6b}$$

$$\gamma=\sum_{i=1}^{n}(p_i+\rho_i)(a_{(i)2})^2\ , \tag{6.6c}$$

$$\pi=\sum_{i=1}^{n}p_i\ . \tag{6.7}$$

Since expression (6.5) is formally equivalent to (2.11) we conclude that the EMT for the n-fluid model under consideration is equivalent to the EMT for a single anisotropic fluid (2.25), and that the anisotropic-fluid variables are given, as before, by (2.17), (2.19), (2.21), (2.30), and (2.31). Now the quantities α, β, γ, and π appearing in these formulas must be replaced by the respective expressions (6.6) and (6.7). Hence, we have that the EMT associated to the model of Herrera and co-workers[9] as well as the EMT associated to the cylindrically symmetric multifluid solution described at the end of the preceding section can be cast in the canonical form of a single anisotropic fluid.

To cast the EMT (6.4), with four-velocities satisfying (6.1), in the form (2.25) we have made use of the fact that one can easily guess two of its eigenvectors [cf. Eq. (2.18)]. To achieve the same result one usually solves the eigenvalue problem for the EMT (6.5) directly.[15] We have preferred the first method because in the present case it is simpler.

The model of two fluids with irrotational perfect-fluid components can be used to describe solitary waves of matter in general relativity. In particular we found a particular model in which the velocity-potentials were governed by an integrable system of equations.[6] In this case we have interaction between the fluid components, and each fluid component no longer obeys a "conservation law" like (5.3).

The multifluid model can also be used in this context to give a more general system of integrable equations. In particular we found an integrable system that can be described as the hyperbolic version of the integrable system studied to find instantons in the SU(N) gauge theory. In this case we also have interaction between the fluid components. Work along this line will soon be reported.

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