Cylindrical gravitational waves with two degrees of freedom: An exact solution

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The exact two-parameter solution of Einstein's equations described below represents ingoing and outgoing cylindrical gravitational waves with two degrees of polarization. It has been obtained from the Kerr metric by applying a well-known trick but, unlike the Kerr metric, it has no singularities.

There exists a well-known trick[1] to obtain cylindrical time-dependent solutions from axially symmetric stationary solutions of Einstein's equations. The trick is thus applicable to Kerr's[2] metric with mass m and angular momentum[3] $ma\neq 0$ in $G=c=1$ units. It gives an interesting new solution which comes about as follows. Start from Kerr's metric in Boyer-Lindquist[4] coordinates x^0,r,θ,ϕ. Go to "isotropic coordinates" $x^0,\tilde{R},\theta,\phi$ with

$$r=m+\tilde{R}+(m^2-a^2)/4\tilde{R}\ . \tag{1}$$

Next go to cylindrical coordinates $x^0,\tilde{\rho},\tilde{z},\phi$ with

$$\tilde{\rho}=\tilde{R}\sin\theta \quad \text{and} \quad \tilde{z}=\tilde{R}\cos\theta\ . \tag{2}$$

Now employ the well-known trick of setting

$$x^0=iz,\ \ \tilde{z}=i\tilde{t},\ \ a=i\tilde{a}\ , \tag{3}$$

and then introduce cylindrical coordinates T,R,Z,ϕ defined with

$$\begin{aligned} &M^2=m^2+\tilde{a}^2,\ \ T=M^{-1}\tilde{a}\tilde{t}(1+M^2/4\tilde{R}^2)\ , \\ &R=M^{-1}\tilde{a}\tilde{\rho}(1-M^2/4\tilde{R}^2),\ \ Z=z-2\tilde{a}^{-1}m(m+M)\phi\ , \end{aligned} \tag{4}$$

and $\phi=\tilde{a}^{-1}M\phi$.

As a result of these transformations we obtain from the Kerr solution a metric which has the Jordan-Ehlers-Kundt-Kompaneetz[5] (JEKK) form:

$$\begin{aligned} ds^2=&e^{2(\Gamma-\Psi)}(dT^2-dR^2)-e^{2\Psi}(dZ+\Omega\, d\Phi)^2 \\ &-e^{-2\Psi}R^2d\Phi^2\ , \end{aligned} \tag{5}$$

where Γ, Ψ, and Ω are functions of $U=T-R$ and $V=T+R$ that appear in the following combinations:

$$\begin{aligned} &\lambda_u=\tilde{a}^{-1}[(\tilde{a}^2+U^2)^{1/2}-U]\ , \\ &\lambda_v=\tilde{a}^{-1}[(\tilde{a}^2+V^2)^{1/2}+V]\ . \end{aligned} \tag{6}$$

Γ, Ψ, and Ω are given by

$$\begin{aligned} &e^{2\Gamma}=1+(\alpha^2-1)(1-\lambda_u\lambda_v)^2/[(1+\lambda_u{}^2)(1+\lambda_v{}^2)]\ , \\ &e^{2\Psi}=[\alpha^2(1-\lambda_u\lambda_v)^2+(\lambda_v+\lambda_u)^2]/[\alpha^2\Xi^2+(\lambda_v-\lambda_u)^2]\ , \\ &\Omega=\tilde{a}(\alpha^2-1)^{1/2}\{2[1+(1-\alpha^{-2})^{1/2}] \\ &\qquad -\Xi(\lambda_u\lambda_v)^{-1/2}(\lambda_v+\lambda_u)^2 \\ &\qquad \times[\alpha^2(1-\lambda_u\lambda_v)^2+(\lambda_v+\lambda_u)^2]^{-1}\}\ , \end{aligned} \tag{7}$$

in which

$$\Xi=1+\lambda_u\lambda_v+2[(1-\alpha^{-2})\lambda_u\lambda_v]^{1/2}\ . \tag{8}$$

We shall employ the chart $\{(T,Z,R,\Phi)\in R^4$: $-\infty<T,Z<\infty, 0\leq R<\infty, 0\leq\Phi<2\pi\}$, where spacetime points with $\Phi=2\pi$ are identified with those with $\Phi=0$ in the obvious way. The parameter $\alpha=M/a$ varies from 1 to ∞ and for $\alpha=1$, that is, $m=0$, the space is flat. α determines the total energy of the waves (see below). The other parameter $\tilde{a}$ ranges from 0 to ∞ and plays the role of a length scale.

The metric (7) is the only known analytic *cylindrical-wave* solution of the form (5) with two degrees of freedom or, what is the same, two polarizations. Solutions with one degree of freedom were found a long time ago.[6] A *plane-wave* solution with two degrees of freedom has also been found.[7] That solution can be obtained from the Kerr metric by a procedure[8] which is quite different from ours.

Our new metric is regular everywhere. On the axis $R=0$ there is no conical singularity. At past and future infinite $I^{\pm}(T=\pm\infty)$, the spacetime is Minkowski flat. At spatial infinity $I^0(R=\infty)$ it is conical:

$$ds^2=\alpha^2(dT^2-dR^2)-dz^2-R^2d\Phi^2 \tag{9}$$

[notice that z is the one defined in Eq. (3)]. At future null infinity $\mathscr{I}^+(U=\infty)$, $\Phi=0$, $\Omega=-\infty$, and

$$e^{2\Gamma}=(\alpha^2+\lambda_v{}^2)/(1+\lambda_v{}^2)\ . \tag{10}$$

At past null infinity $\mathscr{I}^-(V=-\infty)$ the metric is similar with λ_u instead of λ_v (Ref. 9). Einstein's energy flux per unit height through a surface (U,V)=const over the flat background is[10]

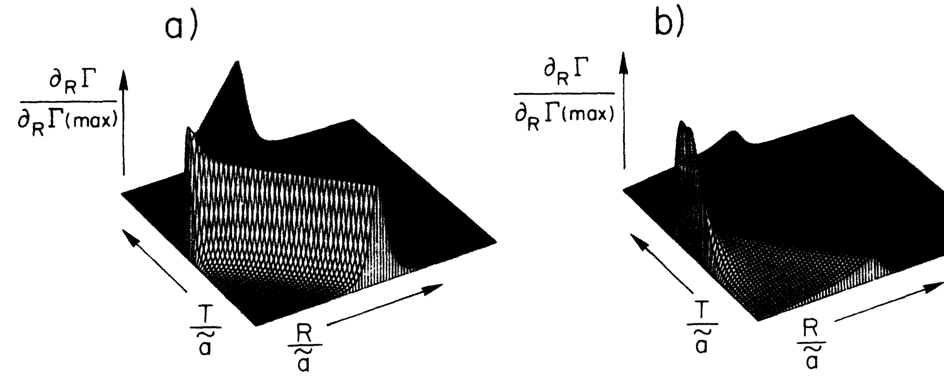

FIG. 1. $\partial_R \Gamma$ in units of its maximum value in terms of $R/\tilde{a}$ and $T/\tilde{a}$ in the ranges $0 \le R/\tilde{a} \le 20$ and $-10 \le T/\tilde{a} \le 10$. (a) is for $\alpha = 1.01$ and (b) for $\alpha = 10$. $\partial_R \Gamma(\max)$ scales like $(\alpha^2 - 1)/\tilde{a}$ for $\alpha \approx 1$ and like $1/\tilde{a}$ for $\alpha \gg 1$. The tendency towards a singular behavior at $R = 0$ for $\alpha \to \infty$ appears already for $\alpha = 10$.

$$E = \tfrac{1}{8}(e^{2\Gamma} - 1) \ . \tag{11}$$

The metric (7) thus describes a flow of ingoing-outgoing packets of waves on a flat background. A wave packet emerges at $\mathscr{I}^-$, reaches its highest concentration near the axis $R = 0$ at time $T = 0$, and is reflected out to $\mathscr{I}^+$. The total energy that comes in and goes out is $\frac{1}{8}(\alpha^2 - 1)$.

Figure 1 displays $\partial_R \Gamma$, which is related to the energy flux; it is positive definite:

$$\begin{aligned} \partial_R \Gamma &= 2R\,[(\partial_U \Psi)^2 + (\partial_V \Psi)^2] + \frac{e^{4\Psi}}{2R}[(\partial_U \Omega)^2 + (\partial_V \Omega)^2] \\ &= 2\tilde{a}^{-1}(\alpha^2 - 1)e^{-2\Gamma} \frac{(\lambda_u \lambda_v - 1)(\lambda_u + \lambda_v)}{(1+\lambda_u^2)(1+\lambda_v^2)} \left[\frac{\lambda_u^2}{(1+\lambda_u^2)^2} + \frac{\lambda_v^2}{(1+\lambda_v^2)^2} \right] . \end{aligned} \tag{12}$$

The behavior of $\partial_R \Gamma$ is representative of the behavior of other quantities in the metric.

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[9]In spite of the lack of asymptotic flatness in the z direction, we can define null infinity for *radially* going rays.

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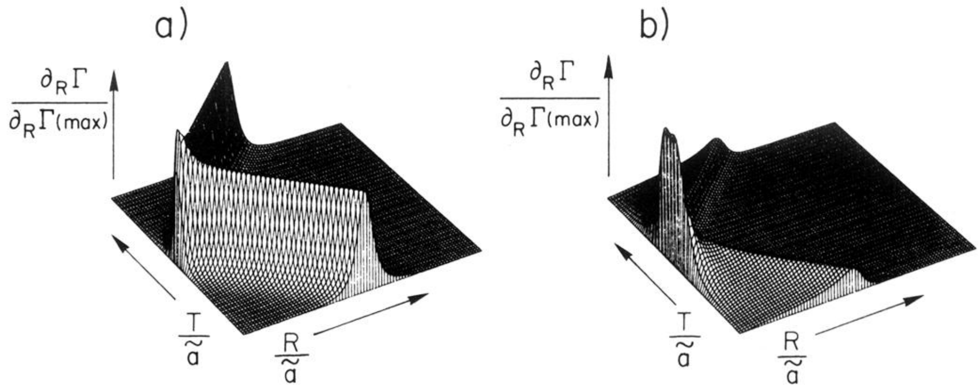

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