# Closed-form solutions for the modified potential

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Heretofore, the modified potential had to be described by approximation. Now, closed-form solutions may be developed for the modified potential. Closed-form solutions may now prove the hypotheses, which heretofore could only be shown to be plausible, that the quantum-action-variable quantization is precisely consistent with quantization of wave mechanics and that the Schrödinger wave function has microstates. An alternate representation of the set of nonlocal hidden variables for determining quantum continuous motion may be developed in terms of the closed-form solutions. The relationship between this alternate representation for the set of nonlocal hidden variables and Ermakov's invariant may now be developed.

The modified potential, which is the keystone for describing continuous quantum motion,[1,2] has been determined heretofore by either numerical computations,[3,4] perturbation expansion,[5] or power series.[1] The onerous numerical computations have obscured insight while the ponderous asymptotic expansions diverge in the classically "forbidden" region. Consequently, the Bohr-Sommerfeld quantization of the (effective) action variable for continuous quantum motion for a stationary bound state has been inferred numerically for only a particular set of potentials, and a general proof has yet to be offered.[3] The existence of microstates has only been inferred numerically and by power-series expansion also for only a particular set of potentials.[1]

Herein, we present closed-form solutions for the modified potential. These solutions are combinations of products of the pair of solutions to the time-independent Schrödinger equation. The closed-form solutions render insight into the foundations of quantum mechanics. With the closed-form solutions for the modified potential, we can prove that the action variable quantization for continuous quantum motion[1] is consistent with eigenvalue quantization of the time-independent Schrödinger equation. These solutions for the modified potential are not unique, which is a manifestation of hidden variables. We present an alternate set of hidden variables formed, in part, from the coefficients of the closed-form solutions. With the closed-form solutions, we can present a general proof that the Schrödinger wave function has microstates. We also present the relationship of these coefficients to the Ermakov invariant. One dimension suffices for this exposition.

For stationary states in one dimension, $x$, the Hamilton-Jacobi equation for continuous quantum motion is given by[2]

$$\frac{(\partial W/\partial x)^2}{2\mu}+U(x,\dot{x},\ddot{x},E)-E=0\ ,$$

where $W$ is Hamilton's characteristic function, $U$ is the modified potential where the dependence upon the entire set of hidden variables[1,2] $[x,\dot{x},\ddot{x}]$ is made explicit here, $E$ is energy, and $\mu$ is mass. The modified potential, $U$, is determined in one dimension, $x$, by the auxiliary equation[1]

$$U+\frac{\hbar^2}{8\mu}\frac{\partial^2U/\partial x^2}{E-U}+\frac{5\hbar^2}{32\mu}\left[\frac{\partial U/\partial x}{E-U}\right]^2=V\ , \tag{1}$$

where $\hbar$ is Planck's constant and $V$ is the potential. A closed-form solution for $U$ and the conjugate momentum $P$ may be given by

$$U=E-1/(a\phi^2+b\theta^2+c\phi\theta)^2 \tag{2}$$

and

$$P=\partial W/\partial x=(2\mu)^{1/2}/(a\phi^2+b\theta^2+c\phi\theta)\ ,$$

where $\phi$ and $\theta$ are the pair of independent solutions to the time-independent Schrödinger equation, where $\mathscr{W}(\phi,\theta)$ is the Wronskian [i.e., $\mathscr{W}(\phi,\theta)=\phi\partial\theta/\partial x-\partial\phi/\partial x\,\theta$], and where $a$, $b$, and $c$ are coefficients of the products[6] of the independent solutions of which $a$ and $b$ are positive definite. The independent solutions $\phi$ and $\theta$ are scaled such that the Wronksian satisfies $\mathscr{W}^2=2\mu/[\hbar^2(ab-c^2/4)]$ where $ab>c^2/4$. We shall show later that a pair of independent solutions to the Schrödinger equation can be chosen such that the coefficient $c$ may be set to zero.[7] This closed-form solution for $U$ may be confirmed by substituting Eq. (2) into Eq. (1) to generate the time-independent Schrödinger equation.

We can now prove that the bound-state action variable for quantum continuous motion is quantized. For bound state, the energy, $E$, is quantized and one solution, arbitrarily let it be $\phi$,is the eigenfunction that is bound, while the complimentary solution $\theta$ is unbound. As the vertex (i.e., turning) points for continuous quantum motion are at $\pm\infty$ for bound states,[2] the action variable $J$ is given by

$$J=\oint P\,dx'$$
$$=\hbar\oint\frac{(ab-c^2/4)^{1/2}\mathscr{W}(\phi,\theta)}{a\phi^2+b\theta^2+c\phi\theta}dx'$$

$$=2\hbar\int_{-\infty}^{\infty}\frac{(ab-c^2/4)^{1/2}d(\theta/\phi)/dx}{a+b(\theta/\phi)^2+c(\theta/\phi)}dx \ . \tag{3}$$

We shall now change the integration variable in Eq. (3) from $x$ to $(\theta/\phi)$, which renders the intergrand algebraic. The bound solution, $\phi$, for the $N$th eigenfunction has $(N-1)$ nodes ($\phi$ also has two additional zeros at $x\to+\infty,-\infty$, but these zeros are not nodes[8]). The corresponding unbound solution, $\theta$, has $N$ nodes, so that one solution has an odd number of nodes while the other solution has an even number of nodes. If we arbitrarily set the solution with an odd number of nodes to be negative and the other solution with an even number of nodes to be positive as $x\to-\infty$, then $\mathscr{W}(\phi,\theta)>0$. Hence, as $x\to-\infty$, $(\theta/\phi)\to-\infty$, and as $x\to\infty$, $(\theta/\phi)\to\infty$. In addition, we note that $(\theta/\phi)$ has singularities for finite points at the nodes of $\phi$ where the limiting value of $(\theta/\phi)$ depends upon which way the nodal point, $x_n$, of $\phi$ is approached. Let $\epsilon>0$, then we have that

$$\lim_{\epsilon\to0+}[\theta(x_n-\epsilon)/\phi(x_n-\epsilon)]=\infty$$

and

$$\lim_{\epsilon\to0+}[\theta(x_n+\epsilon)/\phi(x_n+\epsilon)]=-\infty$$

regardless of whether $\phi$ has an odd or even number of nodes. Thus the passing of $\phi$ through one of its nodes between $x_n-\epsilon$ and $x_n+\epsilon$ generates an additional Riemann sheet. Between adjacent nodes of $\phi$ or between a zero and its adjacent node of $\phi$, $(\theta/\phi)$ monotonically increases as $x$ increases because

$$d(\theta/\phi)/dx=\mathscr{W}(\phi,\theta)/\phi^2>0 \ .$$

As $x$ monotonically increases through a nodal point where $\phi(x_n)=0$, our new integration variable, $(\theta/\phi)$, completes one Riemann sheet at $(\theta/\phi)=\infty$ and immediately begins an additional sheet at $(\theta/\phi)=-\infty$. Thus, for the $N$th eigenfunction, changing the integration variable in Eq. (3) from $x$ to $(\theta/\phi)$ generates $N$ Riemann sheets, each with integration limits at $\pm\infty$, and poles at

$$(\theta/\phi)=\frac{c\pm(c^2-4ab)^{1/2}}{-2b}=\frac{c\pm i[(2\mu)^{1/2}/\hbar\mathscr{W}]}{-2b} \ .$$

We may now evaluate $J$ as

$$J=2N\hbar\int_{-\infty}^{\infty}\frac{(ab-c^2/4)^{1/2}}{a+c(\theta/\phi)+b(\theta/\phi)^2}d(\theta/\phi)$$

$$=Nh \ , \tag{3'}$$

where $h=2\pi\hbar$. Thus $J$ is quantized in accordance with the order of the eigenfunction. The quantization is independent of the coefficients $a$, $b$, and $c$, which manifest a unique microstate[1] of the Schrödinger wave function as specified by particular form of the modified potential, Eq. (2). Equation (3′) is general and consequently also specifies the quantization for cases for which closed-form solutions to the Schrödinger equation do not exist. Equation (3) now proves that the action variable quantization for different microstates is consistent in general with energy quantization—a hypothesis that heretofore had been shown[3] to be plausible for specific cases. We note that the quantization of the action variable for continuous quantum motion, which is a whole integer quantization of $h$ as shown by Eq. (3′), differs with the corresponding WKB half-integer quantization of the action variable, i.e., $J_{\mathrm{WKB}}=(N-\frac{1}{2})h$.

We note that our setting the solution with an odd number of nodes to be negative and the other solution with an even number of nodes to be positive [so that $(\theta/\phi)\to-\infty$ as $x\to-\infty$] was arbitrary and the quantization represented by Eq. (3′) is independent of our convention as long as we consistently specify $\mathscr{W}$. Also we could have changed variables to $(\phi/\theta)$.

Let us now show that the coefficient $c$ may be set equal to zero. We choose an independent set $(\zeta,\xi)$ of solutions such that $\zeta$ and $\xi$ are, respectively, the cosine and sine forms of the trigometric representation of the ansatz[1] for the time-independent Schrödinger wave function. Then the set $(\zeta,\xi)$ may be represented as a function of the set of closed-form solutions $(\phi,\theta)$ of the time-independent Schrödinger equation. We may reduce these functions whereas

$$\zeta=(a\phi^2+b\theta^2+c\phi\theta)^{1/2}\cos\left[\int^x(ab-c^2/4)^{1/2}\mathscr{W}(\phi,\theta)(a\phi^2+b\theta^2+c\phi\theta)^{-1}dx'\right]$$

$$=(a\phi^2+b\theta^2+c\phi\theta)^{1/2}\cos\left[\arctan\left[\frac{b(\theta/\phi)+c/2}{(ab-c^2/4)^{1/2}}\right]\right]=[a-c^2/(4b)]^{1/2}\phi \tag{4}$$

and

$$\xi=(a\phi^2+b\theta^2+c\phi\theta)^{1/2}\sin\left[\int^x(ab-c^2/4)^{1/2}\mathscr{W}(\phi,\theta)(a\phi^2+b\theta^2+c\phi\theta)^{-1}dx'\right]$$

$$=(a\phi^2+b\theta^2+c\phi\theta)^{1/2}\sin\left[\arctan\left[\frac{b(\theta/\phi)+c/2}{(ab-c^2/4)^{1/2}}\right]\right]=b^{1/2}\theta+c\phi/(2b^{1/2}) \ .$$

Thus, the set $(\zeta,\xi)$ is also another set of independent solutions for the time-independent Schrödinger equation. For this new set, the conjugate momentum $P$ for continuous quantum motion and the Wronksian $\mathscr{W}$ may be given, respectively, by

$$P=(2\mu)^{1/2}/(a\phi^2+b\theta^2+c\phi\theta)$$
$$=(2\mu)^{1/2}/(\zeta^2+\xi^2)\ .$$

and

$$\mathscr{W}(\zeta,\xi)=(ab-c^2/4)^{1/2}\mathscr{W}(\phi,\theta)\ .$$

Hence, we may always find another set of independent solutions in which the coefficient for the cross product of the new solutions is zero.

If we set $\zeta=\alpha\phi$ in Eq. (4) where $\alpha$ is a coefficient, then Eq. (4) is an identity with $\alpha=[a-c^2/(4b)]^{1/2}$. Therefore, the coefficients, $a$, $b$, and $c$ only effect the normalization of the Schrödinger wave function and do not change the predictions of quantum measurement of an observable (i.e., $\int\phi^\dagger O\phi/\int\phi^\dagger\phi$ where $O$ is the operator for an observable). Nevertheless, the coefficients determine different modified potentials, $U$, which induce different continuous quantum motion in phase space.[1] The different continuous quantum motions for the same energy eigenvalue determine different microstates of the same Schrödinger wave function.[1] Thus, Eq. (4) renders a general proof that the Schrödinger wave function does indeed have microstates and, therefore, is not an exhaustive description of natural phenomenon.

The set of hidden variables $[x_0,\dot{x}_0,\ddot{x}_0]$, which are necessary[1] and sufficient[2] to specify a unique microstate of the Schrödinger wave function, also determine[2] the particular modified potential associated with the unique microstate. In turn, the particular modified potential determines the coefficients for $a$ and $b$ [assuming a set of independent solutions have been chosen where $c=0$ and where $\phi(x_0)\neq 0$ and $\theta(x_0)\neq 0$]. Thus we may describe continuous quantum motion with an alternate set of hidden variables $[x_0,a,b]$. We have by Eq. (2) that the alternate hidden variables $a$ and $b$ are given by

$$a=\frac{1}{\phi\mathscr{W}(\phi,\theta)}\left[\frac{\partial\theta/\partial x}{(E-U)^{1/2}}-\frac{\theta\partial U/\partial x}{4(E-U)^{3/2}}\right]\Bigg|_{[x,\dot{x},\ddot{x}]=[x_0,\dot{x}_0,\ddot{x}_0]}$$

and

$$b=\frac{1}{\theta\mathscr{W}(\phi,\theta)}\left[\frac{\phi\partial U/\partial x}{4(E-U)^{3/2}}-\frac{\partial\phi/\partial x}{(E-U)^{1/2}}\right]\Bigg|_{[x,\dot{x},\ddot{x}]=[x_0,\dot{x}_0,\ddot{x}_0]}$$

We note that for $c=0$, then $\mathscr{W}(\phi,\theta)\propto(ab)^{-1/2}$. Had we arbitrarily chosen to double both $\phi$ and $\theta$, then in compensation both $a$ and $b$ would be quartered in accordance with the above equations.

We note that the coefficients, $a$, $b$, and $c$ may be related to the Ermakov invariant, which is an exact invariant that had been developed to tackle the time-dependent one-dimensional classical harmonic oscillator.[9,10] In this discussion, we generalize for $c\neq 0$. For systems obeying the Schrödinger equation, the Ermakov invariant is given by

$$I=[(\psi/\Lambda)^2+\hbar^2(\psi\partial\Lambda/\partial x-\Lambda\partial\psi/\partial x)^2]/(2\mu)^{1/2}\ , \qquad (5)$$

where $\Lambda$ satisfies the auxiliary equation of Ermakov and Lewis[10]

$$\partial^2\Lambda/\partial x^2+(2\mu/\hbar^2)(E-V)\Lambda=1/(\hbar^2\Lambda^3)\ . \qquad (6)$$

If we substitute $\Lambda=[2\mu(E-U)]^{-1/4}$ into Eq. (6), we generate our auxiliary equation given by Eq. (3) (Ref. 11). So our auxiliary equation is equivalent to the auxiliary equation of Ermakov and Lewis. Thus the Ermakov invariant could be specified with the solutions to the Schrödinger equation and Eq. (1). Had we substituted $\Lambda=P^{-1/2}=(\partial W/\partial x)^{-1/2}$ into Eq. (6), we would have generated the alternate Hamilton-Jacobi equation for continuous quantum motion[2]

$$\frac{(\partial W/\partial x)^2}{2\mu}+V-E=\frac{-\hbar^2}{4\mu}\frac{\partial^3W/\partial x^3}{\partial W/\partial x}+\frac{3\hbar^2}{8\mu}\left[\frac{\partial^2W/\partial x^2}{\partial W/\partial x}\right]^2 .$$

We may now evaluate $I$ for the closed-form representation

$$\psi=\alpha\phi+\beta\theta\ ,$$

where $\alpha$ and $\beta$ are coefficients that satisfy the boundary conditions imposed upon the Schrödinger equation, and

$$\Lambda=P^{-1/2}=(a\phi^2+b\theta^2+c\phi\theta)^{1/2}/(2\mu)^{1/4}\ .$$

After some more tedious but again straightforward algebra, Eq. (5) may be evaluated to render

$$I=\frac{a\beta^2+b\alpha^2-c\alpha\beta}{ab-c^2/4}\ .$$

Thus, the Ermakov invariant may be expressed in terms of the alternate set of hidden variables.


[1] E. R. Floyd, Phys. Rev. D 26, 1339 (1982).

[2] E. R. Floyd, Phys. Rev. D 29, 1842 (1984).

[3] E. R. Floyd, Phys. Rev. D 25, 1547 (1982).

[4] See AIP document No. PAPS PRVDA-25-1547-20 (the supplement to Ref. 3) which numerically describes the modified potential and effective action variable for the bound state.

[5] E. R. Floyd, J. Math. Phys. 20, 83 (1979).

[6] The product set $[\phi^2,\phi\theta,\theta^2]$ are the independent solutions for

$$\partial^3\sigma/\partial x^3+[(2\hbar^2/\mu)(E-V)]\partial\sigma/\partial x-[(\hbar^2/\mu)dV/dx]\sigma=0\ ,$$

where $\mathscr{W}(\phi^2,\phi\theta,\theta^2)=2\mathscr{W}^3(\phi,\theta)$. Here, $\phi^2$ represents the quantum probability density because the wave function in one dimension may be expressed by a real function with an unimportant phase factor [cf. L. D. Landau and E. M. Lifshitz, *Quantum Mechanics* (Addison-Wesley, Reading, MA, 1958), pp. 52 and 160].

[7]For symmetric $V$ and for $\phi$ and $\theta$ individually being either symmetric or antisymmetric, a nonsymmetric $U$ manifests $c \neq 0$ (cf. Refs. 3 and 4).

[8]A. Messiah, *Quantum Mechanics* (Wiley, New York, 1961), Vol. I, p. 88.

[9]V. P. Ermakov, Univ. Izv. Kiev **20**, 1 (1880).

[10]H. R. Lewis, Jr., Phys. Rev. Lett. **18**, 510 (1967).

[11]From this transformation, I recognized that the closed-form solution for Eq. (1) may be derived from the closed-form solution for Eq. (6) [cf. C. J. Eliezer and A. Gray, SIAM J. Appl. Math. **30**, 463 (1976)].