Vortices and electrically charged vortices in non-Abelian gauge theories

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(Received 10 June 1986)

Vortex solutions for a spontaneously broken SU(N) theory are explicitly constructed. N Higgs fields in the adjoint representation are needed in order to ensure topological stability. $(N-1)$ topologically different solutions exist with magnetic flux Φ quantized according to the relation $\Phi=(2\pi/e)n/\sqrt{N}$ with $n=1,2,\ldots,N-1$. When a Chern-Simons term is added, the model exhibits electrically charged vortex solutions. A novel feature of these solutions is that their electric charge q is quantized in units of the fundamental charge e, $q=mne/\sqrt{2N}$, with $m\in Z$. In addition, their angular momentum J is nonzero and also quantized, $J=nm/2N$.

I. INTRODUCTION AND RESULTS

Gauge theories exhibit a rich spectrum of stable regular classical solutions. Vortices, monopoles, and instantons[1] are elegant topological objects having relevant physical implications in quantum field theory and cosmology.

Vortices and monopoles arise in spontaneously broken gauge theories in two and three space dimensions. For both kinds of static solutions, the magnetic flux is quantized due to its topological properties. Monopoles admit an electrically charged generalization with finite energy: the dyon.[2] Its charge is classically continuous and only becomes quantized at the quantum level.[3]

Vortices, both in Abelian[4] and non-Abelian Higgs models[5–7] do not admit finite-energy electrically charged generalizations.[2–8]

The addition of a Chern-Simons (CS) term[9–11] radically changes this situation. In the presence of the CS term vortices acquire electric charge keeping a finite energy both in Abelian[12] and non-Abelian[13] gauge theories. We presented in Ref. 13 a charged vortex solution in SU(2) gauge theory with two Higgs field which spontaneously broke the symmetry down to Z_2. The topological character of the CS term leads to the quantization of the vortex electric charge already at the classical level. As a consequence the angular momentum takes also discrete values (compare this result with that arising in three space dimensions: the addition of a θ term (Pontryagin density) does not change the classical dyon charge but, at the quantum level, renders it noninteger.[14,15]

In the present article we construct vortex solutions for a SU(N) gauge theory with Higgs fields. We analyze both cases: With and without the CS term. This leads to electrically uncharged and charged vortices, respectively.

Our analysis shows that N Higgs fields in the adjoint representation of SU(N) are needed in order to produce topologically stable vortices. There are $(N-1)$ different solutions corresponding to $(N-1)$ nontrivial homotopy classes. This should be compared with the Abelian case where only one Higgs is necessary and where one finds an arbitrary number of topologically stable vortices.[4]

The Lagrangian we choose reads in $(2+1)$ dimensions

$$\mathscr{L}=-\tfrac{1}{2}\mathrm{Tr}F_{\mu\nu}F^{\mu\nu}+\mathrm{Tr}\sum_{A=1}^{N-1}D_\mu\Phi^A D^\mu\Phi^A+\mathrm{Tr}D_\mu\psi D^\mu\psi -V(\Phi,\psi)+\frac{\mu}{2}\epsilon_{\alpha\beta\gamma}\mathrm{Tr}(F^{\alpha\beta}A^\gamma-\tfrac{2}{3}A^\alpha A^\beta A^\gamma)\ , \qquad (1.1)$$

where

$$A_\mu=A_\mu^a t^a,\ \ \mathrm{Tr}(t^a t^b)=\tfrac{1}{2}\delta^{ab};\ \ a,b=1,2,\ldots,N^2-1\ ,$$

$$D_\mu=\partial_\mu+e[A_\mu,\ ]\ ,\ \ F_{\mu\nu}=\partial_\mu A_\nu-\partial_\nu A_\mu+[A_\mu,A_\nu]\ ,$$

and

$$\mu=\frac{e^2 m}{4\pi},\ \ m\in Z\ .$$

In order to have stable vortex solutions we find that the vacuum must have special properties: the ψ field has to be in the Cartan algebra while the Φ^A ones $(A=1,2,\ldots,N-1)$ in orthogonal directions (E_α's) to the Cartan algebra for the vacuum state.

Let us briefly describe the vortex solutions we construct in this work. They bear a Z_N topological charge which is connected to the (singular) gauge transformation $\Omega_m(\varphi)$,

$$\Omega_n(\varphi+2\pi)=e^{2\pi i n/N}\Omega_n(\varphi)\ . \qquad (1.2)$$

(The roots of unity, $\{e^{2\pi i n/N};\ n=0,1,\ldots,N-1\}$, provide a representation of the Abelian group Z_N.) Asymptotically the vortex fields coincide with the gauge transformation of the vacuum under $\Omega_n(\varphi)$ [Eq. (2.8)]. The field ψ remains in its (constant) vacuum value in all space while Φ^A and A_μ read

$$\Phi^A=F^A(\rho)\Omega_n{}^{-1}(\varphi)(E_{\alpha_A}+E_{-\alpha_A})\Omega_n(\varphi)\ , \qquad (1.3)$$

$$A_\varphi=\frac{na(\rho)}{e}M,\ \ A_\rho=0\ , \qquad (1.4)$$

$$M=\mathrm{diag}[1/N,1/N,\ldots,1/N,(1-N)/N]\ . \qquad (1.5)$$

In addition, when a CS term is added, one necessarily has a nonzero A_0 field which we take

$$A_0=\frac{n}{e}a_0(\rho)M\ . \qquad (1.6)$$

This ansatz separates the variables in the equations of motion. The radial functions $a(\rho)$, $a_0(\rho)$, and $F^A(\rho)$ obey the system of coupled ordinary differential equations given in Sec. II [Eqs. (2.52)–(2.54)].

An electromagnetic tensor can be introduced in order to characterize the vortices:

$$\mathscr{F}_{\mu\nu}=\frac{\mathrm{Tr}[MF_{\mu\nu}]}{\mathrm{Tr}[M^2]} . \quad (1.7)$$

Then, the flux associated to the magnetic field $\mathscr{F}_{12}$ reads for the n-vortex solution in SU(N):

$$\Phi=n\Phi_0 \quad (1.8)$$

with

$$\Phi_0=\frac{2\pi}{\sqrt{N}}\frac{1}{e} . \quad (1.9)$$

When the CS term is present in the Lagrangian, the vortex also bears an electric charge which turns out to be quantized:

$$Q=-mnQ_0 \quad (1.10)$$

with $Q_0=e/\sqrt{2N}$ and $m\in Z$ defined by the quantization of the CS coefficient, Eq. (1.2). In addition, the vortex has a nonzero angular momentum

$$J=-\frac{nm}{2N} . \quad (1.11)$$

The quantization of the angular momentum in units of $1/2N$ is characteristic of Z_N-symmetric field theories.[16] Concerning the charge quantization, being of topological origin, one should expect it remains valid at the quantum level.

The paper is organized as follows. In Sec. II we discuss the topological aspects involved in the construction of SU(N) vortices (Sec. II A and II B) and give an explicit example for the SU(3) case (Sec. II C). In Sec. III we discuss the principal features of the vortex solution both in the neutral and charged cases.

In particular, we prove Eqs. (1.8)–(1.11) for the flux, charge, and angular momentum of the vortex solution. For large N it follows the SU(N) vortices asymptotically become the U(1) Abelian vortices.

Finally, we discuss in an appendix the symmetry-breaking pattern for the SU (3) case.

II. TOPOLOGICAL ASPECTS

A. The Chern-Simons term

The Chern-Simons term $\mathscr{L}_{\mathrm{CS}}$ added to the Higgs Lagrangian,

$$\mathscr{L}_{\mathrm{CS}}=-\frac{\mu}{2}\epsilon^{\mu\nu\alpha}\mathrm{Tr}(F_{\mu\nu}A_\alpha-\tfrac{2}{3}A_\mu A_\nu A_\alpha) , \quad (2.1)$$

violates both parity and time-inversion invariance but not charge-conjugation invariance. Although it leads to gauge-covariant equations of motion, it is not itself gauge invariant; rather, it changes by total derivatives under a gauge transformation. It then follows that the response of the action S to a gauge transformation $U_m(x)$ is

$$S\underset{U_m}{\rightarrow}S+\mu\frac{8\pi^2}{e^2}\omega(U_m) , \quad (2.2)$$

where $\omega(U_m)$ is the winding number of the gauge transformation,

$$\omega(U_m)=\frac{1}{24\pi^2}\int d^3x\, \epsilon^{\alpha\beta\gamma} \times\mathrm{Tr}(U_m{}^{-1}\partial_\alpha U_m U_m^{-1}\partial_\beta U_m U_m{}^{-1}\partial_\gamma U_m) \quad (2.3)$$

Equation (2.3) can be converted to a surface integral which is not zero but takes an integer value m:

$$\omega(U_m)=m,\ \ m\in Z \quad (2.4)$$

which characterizes the homotopy equivalence class to which U_m belongs. Only for homotopically trivial U_0 does $\omega(U_0)$ vanish. Then, the requirement that the phase exponent of the action be gauge invariant enforces a quantization condition on the parameters μ and e (Ref. 10):

$$\frac{4\pi\mu}{e^2}=m . \quad (2.5)$$

B. Vortices in SU(N)

Vortex configurations exist whenever the gauge symmetry is spontaneously broken via Higgs fields, leaving the vacuum invariant under a subgroup H of the gauge group G. Then, in order to have topologically stable static solutions in two space dimensions, the relevant homotopy group $\Pi_1(G/H)$ must be nontrivial.

Let us consider $G=\mathrm{SU}(N)$ and the Higgs fields in the adjoint representation. It is convenient to have maximum symmetry breaking so that the vacuum is invariant only under the unit matrix in the adjoint representation. Since the matrices of the center of SU(N),

$$I_N e^{2\pi ik/N}\in Z_N , \quad (2.6)$$

with I_N the $N\times N$ unit matrix are mapped onto the unit matrix in the adjoint representation, $\Pi_1(G/H)=\Pi_1(\mathrm{SU}(N)/Z_N)=Z_N$ and one has $(N-1)$ topologically nontrivial classes besides the ordinary vacuum (trivial class). One can obtain a representative of each of these classes by a nontrivial gauge rotation $\Omega_n\in\mathrm{SU}(N)$, $n=1,2,\ldots,N-1$, of the trivial vacuum.

Since we are working in two space dimensions, infinity is characterized by an angle φ (the direction in which one goes to infinity) so that Ω_n is a mapping of the form $\Omega_n=\Omega_n(\varphi)$. This mapping must be in one of the homotopy classes referred to above and satisfies, when one makes a turn around a closed contour,

$$\Omega_n(2\pi)=e^{2\pi in/N}\Omega_n(0),\ \ n=1,2,\ldots,N-1 . \quad (2.7)$$

Condition (2.7) can be realized for an Abelian subgroup of gauge rotations:

$$\Omega_n(\varphi)=\mathrm{diag}[e^{in\varphi/N},e^{in\varphi/N},\ldots,e^{in\varphi/N}, e^{-i(1-1/N)n\varphi}] \quad (2.8)$$

the last element being adjusted so that $\det\Omega_n(\varphi)=1$ for all φ. With this choice, the ansatz for the gauge-field configuration corresponding to a vortex may be chosen to be in the Cartan algebra of SU(N):

$$A_\mu = A_\mu^a t^a = \frac{m}{e} a_\mu M \ , \tag{2.9}$$

where t_a are the SU(N) generators normalized as in (1.2) and M is given by:

$$M = -\frac{i}{n}\Omega_n{}^{-1}\partial_\rho \Omega_n = \text{diag}\left[\frac{1}{N},\frac{1}{N},\ldots,\frac{1}{N},\frac{1-N}{N}\right] . \tag{2.10}$$

Let us call H_i,$(N-1)$ diagonal matrices spanning the Cartan subalgebra of SU(N) and $E_{\pm\alpha}$ the remaining generators of the standard Cartan-Weyl basis for the SU(N) generators, satisfying

$$[H_i,E_{\pm\alpha}] = \pm\alpha_i E_{\pm\alpha} \ ,$$
$$[E_\alpha,E_{-\alpha}] = \sum_{i=1}^{N-1} \alpha_i H_i \ , \tag{2.11}$$

where α_i are the roots and we choose an orthonormal basis so that $\alpha^i=\alpha_i$.

We can then write M in the form

$$M = \sum_{i=1}^{N-1} m^i H_i \ , \tag{2.12}$$

where m_i are related to the "magnetic" weights introduced in Ref. 17. One can easily prove from (2.10) and (2.11) that they verify the condition

$$\mathbf{m}\cdot\boldsymbol{\alpha} = \sum_{i=1}^{N-1} m^i \alpha_i \in Z \ , \tag{2.13}$$

where we have written $\boldsymbol{\alpha}=(\alpha^i)$, $\mathbf{m}=(m^i)$.

Now, since finite action requires that

$$F_{\mu\nu} \to 0 \quad \text{for } \rho\to\infty \tag{2.14}$$

A_μ must be a pure gauge at infinity. Moreover, the Higgs fields ϕ have to take there their vacuum value and also

$$D_\mu\phi = \partial_\mu\phi + ie[A_\mu,\phi] \to 0 \quad \text{for } \rho\to\infty \ . \tag{2.15}$$

This condition can be achieved in two ways. Either ϕ does not depend on φ at infinity (and commutes with A_φ)

$$\phi = \psi \to \sum_{\gamma=1}^{N-1} C_j H_j \quad \text{for } \rho\to\infty \tag{2.16}$$

so that each term in the covariant derivative vanishes or

$$\phi = \Phi \to \Omega_n{}^{-1}(\varphi) F_0 \Omega_n(\varphi) \quad \text{for } \rho\to\infty \tag{2.17}$$

with

$$F_0 = \sum_{\pm\alpha} F_0^\alpha E_\alpha \tag{2.18}$$

and it is the sum of both terms in D_φ [Eq. (2.15)] that cancels. It is this last possibility which is topologically nontrivial.

Let us then describe the vacuum structure in the SU(N) theory in order to have vortex solutions. The scalar fields are obviously constant for an ordinary vacuum and they will be taken either in the Cartan algebra or in the orthogonal direction to it (defined by the E_α generators). The potential V will be then chosen so that the vacuum results in this way. We call $\psi^1,\psi^2,\ldots,\psi^S$ the Higgs fields in the Cartan algebra and $\Phi^1,\Phi^2,\ldots,\Phi^G$ those in the E_α directions.

The φ dependence of the fields for the vortex configuration follows from the action of the gauge transformation $\Omega_n(\varphi)$ on the ordinary vacuum (concerning the A_0 component, see below). In addition, we take ψ^B to be constant everywhere, thus leading to the ansatz

$$\Phi^A = \Omega_n{}^{-1}(\varphi) F^A(\rho) \Omega_n(\varphi), \quad A=1,2,\ldots,R \ ,$$
$$\Psi^B = \sum_{\partial=1}^{N-1} C_j^B H_j, \quad B=1,2,\ldots,S \ , \tag{2.19}$$
$$A_\varphi = \frac{n}{e} a(\rho) M, \quad A_\rho = 0 \ .$$

The A_0 potential will be chosen parallel to A_φ in internal space:

$$A_0 = \frac{n}{e} a_0(\rho) M \ . \tag{2.20}$$

Concerning $F^A(\rho)$,

$$F^A(\rho) = \sum_{\pm\alpha} F_\alpha^A(\rho) E_\alpha \tag{2.21}$$

it must take the vacuum value at infinity

$$F^A(\infty) = \eta^A, \quad A=1,2,\ldots,R \ . \tag{2.22}$$

Ansatz (2.19) yields to a covariant derivative for Φ^A of the form

$$D_\varphi\Phi^A = \partial_\varphi\Phi^A + ina(\rho)[M,\Phi^A] \tag{2.23}$$

but since

$$n[M,\Phi^A] = i\partial_\varphi\Phi^A \tag{2.24}$$

one then has

$$D_\varphi\Phi^A = [1-a(\rho)]\partial_\varphi\Phi^A \ , \tag{2.25}$$

$$D_0\Phi^A = -a_0(\rho)\partial_\varphi\Phi^A \ , \tag{2.26}$$

and hence, the finite energy condition leads to

$$a(\infty)=1, \quad a_0(\infty)=0 \ . \tag{2.27}$$

Let us now discuss the main problem in the search of vortex solutions, namely the separability of the equations of motion under the ansatz (2.19) and (2.20).

The equation of motion for A_μ reads

$$D_\mu F^{\mu\nu} = \frac{\mu}{2}\epsilon^{\nu\alpha\beta}F_{\alpha\beta} + J^\nu \ , \tag{2.28}$$

where the current J^ν is given by

$$J_\nu = ie \sum_{A=1}^{R} [D_\nu\Phi^A,\Phi^A] \ . \tag{2.29}$$

(Because of our ansatz, ψ-type fields do not contribute to J_ν.)

Concerning the scalar fields, their equations of motion read

$$D_\mu D^\mu \Phi^A = \frac{\delta V}{\delta \Phi^A} . \tag{2.30}$$

Let us first analyze Eq. (2.29). Inserting the ansatz for A_μ one gets

$$\frac{1}{\rho}\frac{\partial}{\partial \rho}\left[\rho \frac{\partial \Phi^A}{\partial \rho}\right] - n^2\left[\left[\frac{1-a}{\rho}\right]^2 + a_0{}^2\right][M,[M,\Phi^A]] = \frac{\delta V}{\delta \Phi^A} . \tag{2.31}$$

For simplicity, we shall consider potentials such that

$$\frac{\delta V}{\delta \Phi^A} \propto \Phi^A . \tag{2.32}$$

[An explicit example for SU(3) is presented in Sec. II C.]

Then, separability requires that

$$n^2[M,[M,\Phi^A]] = -\frac{\partial^2 \Phi^A}{\partial \varphi^2} = C_n^A(\rho)\Phi^A . \tag{2.33}$$

As we shall see, this puts constraints on the choice of F^A. The equations of motion reduce to

$$\frac{d}{d\rho}\left[\rho \frac{dF^A}{d\rho}\right] - C_n^A(\rho)\left[\left[\frac{1-a}{\rho}\right]^2 + a_0{}^2\right] = \Omega_n \frac{\delta V}{\delta \Phi^A}\Omega_n{}^{-1} . \tag{2.34}$$

Concerning the gauge-field equation, after use of the ansatz, one gets for A_φ,

$$\rho \frac{d}{d\rho}\left[\frac{1}{\rho}\frac{da}{d\rho}\right]M = -e^2[1-a(\rho)] \sum_{A=1}^{R} [\Phi^A,[\Phi^A,M]] + \mu\rho \frac{da_0}{dp} M . \tag{2.35}$$

We then see that separability now imposes the condition

$$\sum_{A=1}^{R} [\Phi^A,[\Phi^A,M]] = MB_n(\rho) . \tag{2.36}$$

Concerning the A_0 potential, the equation of motion it obeys can be written as

$$\frac{1}{\rho}\frac{d}{d\rho}\left[\rho \frac{da_0}{d\rho}\right] + e^2 B_n(\rho) a_0 = \frac{\mu}{\rho}\frac{da}{dp}(\rho) . \tag{2.37}$$

In summary, separability of the equations of motion into radial and angular parts imposes the two conditions

$$n^2[M,[M,\Phi^A]] = C_n^A(\rho)\Phi^A, \quad A = 1,2,\ldots,R , \tag{2.38a}$$

$$\sum_{A=1}^{R} [\Phi^A,[\Phi^A,M]] = MB_n(\rho) \tag{2.38b}$$

thus constraining the possible choices of $F^A(\rho)$ defined by (2.19). It is easy to see that the choice

$$F^A(\rho) = \frac{f^A(\rho)}{\sqrt{N}}(E_{\alpha_A} + E_{-\alpha_A}) , \tag{2.39}$$

where E_{α_A} is the step generator associated to the α_A root and $f^A(\rho)$ is a scalar function, gives for (2.38a):

$$C_n^A = n^2(\mathbf{m}\cdot\boldsymbol{\alpha}_A)^2 \tag{2.40}$$

or, according to (2.13)

$$C_n^A = (k_n^A)^2 n^2, \quad k_n^A \in Z . \tag{2.41}$$

Concerning condition (2.38b), it becomes

$$-2\sum_{A=1}^{R} k_n^A \frac{f^A(\rho)^2}{N}\boldsymbol{\alpha}_A = nB_n(\rho)\mathbf{m} \tag{2.42}$$

and hence, using Eq. (2.41),

$$B_n(\rho) = -\frac{2}{\mathbf{m}^2}\sum_{A=1}^{N-1}\frac{(k_n^A)^2}{N} f^A(\rho)^2 . \tag{2.43}$$

The number of Φ-type scalars that one has to add depends on the choice of the magnetic weights m, that is, on the topological properties of the vortex. For an $\Omega_n(\varphi)$ given by Eq. (2.8) and an M expressed as in (2.10), with the last diagonal element adjusted so that $\det\Omega_n = 1$ and $\mathrm{tr}M = 0$, one can see that $(N-1)$ Φ-type fields and one Ψ-type field are sufficient in order to have a topologically nontrivial configuration satisfying constraints (2.38) and hence leading to separable equations of motion. Then, the explicit ansatz for a SU(N) vortex configuration with topological charge n is, according to the above discussion,

$$\begin{gathered}\Phi^A(\rho) = \frac{f^A(\rho)}{\sqrt{N}}\Omega_n{}^{-1}(\varphi)(E_{\alpha_A})\Omega_n(\varphi) , \\ \psi = \sum_{j=1}^{N-1} C_j H_j, \quad A_\varphi = \frac{n}{e}a(\rho)M , \\ A_0 = \frac{n}{e}a_0(\rho)M, \quad A_\rho = 0 .\end{gathered} \tag{2.44}$$

Working in an orthogonal basis, the "last" Cartan generator can be chosen as

$$H_{N-1} = \frac{1}{\sqrt{2N(N-1)}}\mathrm{diag}(1,1,\ldots,1,N-1) \tag{2.45}$$

and hence the magnetic weight has just one component:

$$\mathbf{m} = \sqrt{(2/N)(N-1)}(0,0,\ldots,0,1) . \tag{2.46}$$

Then, choosing the step generators in the form

$$(E_{\alpha_A})_{ij} = \frac{1}{\sqrt{2}}\delta_{iA}\delta_{jN}, \quad A = 1,2,\ldots,N-1 , \tag{2.47}$$

the integers k_n^A can be explicitly seen to be independent of A since

$$k_n^A = \sqrt{2(1-1/N)}\alpha_{N-1}^A , \tag{2.48}$$

but

$$\alpha_{N-1}^A = \left[\frac{N}{2(N-1)}\right]^{1/2} , \tag{2.49}$$

and then

$$k_n^A = 1, \quad \forall A . \tag{2.50}$$

From this last result we get for C_n^A and B_n:

$$C_n^A=n^2\ ,$$
$$D\equiv B_n=-\frac{1}{N-1}\sum_{A=1}^{N-1}f^A(\rho)^2\ . \tag{2.51}$$

The equations of motion take then the form

$$\frac{1}{\rho}\frac{d}{d\rho}\left[\rho\frac{df^A}{d\rho}\right]-\left[\frac{n-A(\rho)}{\rho}\right]^2 f^A+e^2A_0{}^2f^A =v_A(\rho)f^A\ , \tag{2.52}$$

$$\rho\frac{d}{d\rho}\left[\frac{1}{\rho}\frac{dA}{d\rho}\right]+\frac{e}{\rho^2}D[n-A(\rho)]=\mu\rho\frac{dA_0}{d\rho}\ , \tag{2.53}$$

$$\frac{1}{\rho}\frac{d}{d\rho}\left[\rho\frac{dA_0}{d\rho}\right]+e^2DA_0=\frac{\mu}{\rho}\frac{dA}{d\rho}\ , \tag{2.54}$$

where we have rescaled the gauge field radial functions in the form

$$A(\rho)\equiv na(\rho),\quad A_0(\rho)\equiv na_0(\rho) \tag{2.55}$$

and defined $v_A(\rho)$ through the relation

$$f^Av_A=\Omega_n(\varphi)\frac{\delta V}{\delta\Phi^A}\Omega_n{}^{-1}(\varphi)\ , \tag{2.56}$$

this being always possible due to the choice (2.31).

C. An SU(3) example

In order to illustrate the discussion above in an explicit and simple example we describe in this subsection the construction of the SU(3) vortex [for the SU(2) case see Ref. 13]. For $G=\mathrm{SU}(3)$, two topologically nontrivial classes are possible. The associated $\Omega_n(\varphi)$ are

$$\Omega_1(\varphi)=\begin{bmatrix} e^{i\varphi/3} & 0 & 0\\ 0 & e^{i\varphi/3} & 0\\ 0 & 0 & e^{-2i\varphi/3}\end{bmatrix}\ ,$$
$$\Omega_2(\varphi)=[\Omega_1(\varphi)]^2\ . \tag{2.57}$$

One then has

$$M=\frac{1}{3}\begin{bmatrix}1&0&0\\0&1&0\\0&0&-2\end{bmatrix}\ . \tag{2.58}$$

An explicit realization of the Cartan algebra is

$$H_1=\frac{\lambda_3}{2},\quad H_2=\frac{\lambda_8}{2}\ , \tag{2.59}$$

where λ_3 and λ_8 are the usual diagonal Gell-Mann matrices. One then gets for the two-component magnetic weight

$$\mathbf{m}=\left[0,\frac{2}{\sqrt{3}}\right]\ . \tag{2.60}$$

Concerning the step generators E_α, they can be combined in the form

$$E_{\alpha_1}+E_{-\alpha_1}=\frac{1}{\sqrt{2}}\lambda_4\ ,$$
$$E_{\alpha_2}+E_{-\alpha_2}=\frac{1}{\sqrt{2}}\lambda_6\ , \tag{2.61}$$
$$E_{\alpha_3}+E_{-\alpha_3}=\frac{1}{\sqrt{2}}\lambda_1\ .$$

Roots $\boldsymbol{\alpha}_A$ are

$$\boldsymbol{\alpha}_1=\left[\frac{1}{2},\frac{\sqrt{3}}{2}\right],\quad \boldsymbol{\alpha}_2=\left[-\frac{1}{2},\frac{\sqrt{3}}{2}\right]\ ,$$
$$\boldsymbol{\alpha}_3=(1,0)\ . \tag{2.62}$$

The integers k_n are then

$$k_1=k_2=1 \tag{2.63}$$

in accord with (2.50). The Φ-type scalars take the form

$$\Phi^1=\frac{f^1(\rho)}{\sqrt{6}}\Omega_n{}^{-1}\lambda_4\Omega_n(\varphi)\ ,$$
$$\Phi^2=\frac{f^2(\rho)}{\sqrt{6}}\Omega_n{}^{-1}\lambda_6\Omega_n(\varphi)\ . \tag{2.64}$$

Concerning the ψ field, it can be written as

$$\psi=B\lambda_3+C\lambda_8 \tag{2.65}$$

with B and C two independent constants.

The vortex equations read

$$\frac{1}{\rho}\frac{d}{d\rho}\left[\rho\frac{df^A}{d\rho}\right]-\left[\left[\frac{n-A(\rho)}{\rho}\right]^2-A_0{}^2(\rho)\right]f^A =v_A(\rho)f^A(\rho),\quad A=1,2\ ,$$
$$\rho\frac{d}{d\rho}\left[\frac{1}{\rho}\frac{dA}{d\rho}\right]-\frac{e}{2}[(f^1)^2+(f^2)^2](n-A)=\mu\rho\frac{dA_0}{d\rho}\ , \tag{2.66}$$
$$\frac{1}{\rho}\frac{d}{d\rho}\left[\rho\frac{dA_0}{d\rho}\right]-\frac{e^2}{2}[(f^1)^2+(f^2)^2]A_0=\frac{\mu}{\rho}\frac{dA}{d\rho}\ .$$

In order to go further we have now to make explicit the choice of the potential V. Let us consider the following form for V:

$$V=\frac{\lambda}{4}[(\Phi_a^1\Phi_a^1-A_1{}^2)^2+(\Phi_a^2\Phi_a^2-A_2{}^2)^2] +\mu_1d_{abc}\Phi_1^a\Phi_1^b\psi^c+\mu_2d_{abc}\Phi_2^a\Phi_2^b\psi^c +\frac{\gamma}{2}(\Phi_1^a\Phi_2^a)^2+\frac{\tilde\lambda}{4}(\psi^a\psi^a-\tilde A^2)^2\ , \tag{2.67}$$

where we have written

$$\Phi^A=\sum_{a=1}^{8}\Phi_A^a\lambda^a$$

and d_{abc} for the SU(3) completely symmetric tensors. One can see that our choice of the same coupling constant λ in the two first terms of the right-hand side of (2.67) implies that

$$f_1(\rho)=f_2(\rho)=f(\rho) \tag{2.68}$$

thus making (2.66b) and (2.66c) identical to the Abelian case equations of motion. In the vacuum, the Φ fields take then the form

$$\Phi_1(\infty)=\frac{\eta}{\sqrt{6}}\Omega_n{}^{-1}(\varphi)\lambda_4\Omega_n(\varphi)\,,$$
$$\Phi_2(\infty)=\frac{\eta}{\sqrt{6}}\Omega_n{}^{-1}(\varphi)\lambda_6\Omega_n(\varphi) \tag{2.69}$$

with

$$\eta=f(\infty)\,. \tag{2.70}$$

Now, the condition that at infinity

$$\frac{\delta V}{\delta\Phi^1}=\frac{\delta V}{\delta\Phi^2}=\frac{\delta V}{\delta\Psi}=0 \tag{2.71}$$

determines η, B, and C in terms of the potential parameters

$$\eta^2=\frac{2\mu_1\mu_2(A_1{}^2-A_2{}^2)+\mu_1{}^2A_2{}^2-\mu_2{}^2A_1{}^2}{\mu_1{}^2-\mu_2{}^2}\,, \tag{2.72}$$

$$B=\frac{3\lambda}{2}\frac{A_1{}^2-A_2{}^2}{\mu_1-\mu_2},\quad C=\frac{\sqrt{3}}{2}\lambda\frac{A_1{}^2-A_2{}^2}{\mu_1+\mu_2}\,, \tag{2.73}$$

and imposes the constraint

$$\tilde{A}^2=\frac{1}{3\lambda\tilde{\lambda}(A_1{}^2-A_2{}^2)}\times[\mu_1(\mu_1-2\mu_2)A_2{}^2-\mu_2(\mu_2-2\mu_1)A_1{}^2]-3\lambda^2(A_1{}^2-A_2{}^2)\frac{(\mu_1{}^2+\mu_2{}^2-\mu_1\mu_2)}{(\mu_1{}^2-\mu_2{}^2)^2}\,. \tag{2.74}$$

Moreover, the condition that (2.65) and (2.69) correspond to a minimum of V, that is, the condition that V_{ab} defined through

$$V_{ab}=\left.\frac{\delta V}{\delta\chi^a\delta\chi^b}\right|_{\text{vacuum}}\,, \tag{2.75}$$

with $\chi=\Phi^1$, Φ^2, or ψ, is a positive-definite matrix implies certain inequalities to be satisfied by the potential parameters. We show in an appendix that the condition of positive definiteness of (V_{ab}) is satisfied in a certain domain of the space of parameters for the SU(3) case. No difficulty should appear in extending this analysis to the SU(N) model.

III. PROPERTIES OF THE VORTEX SOLUTIONS

A. The neutral vortex solution

Let us first consider the Lagrangian (1.1) with no CS term ($\mu=0$) in order to study neutral vortices. We just set $A_0=0$ for the SU(3) ansatz (thus corresponding to a purely magnetic vortex configuration); the equations of motion, according to (2.66) and (2.67) read

$$\rho\frac{d}{d\rho}\left[\frac{1}{\rho}\frac{dA}{d\rho}\right]+eF^2(n-A)=0\,, \tag{3.1a}$$

$$\frac{1}{\rho}\frac{d}{d\rho}\left[\rho\frac{df}{d\rho}\right]-\left[\frac{n-A}{\rho}\right]^2 f=\frac{\lambda}{6}(f^2-\eta^2)f \tag{3.1b}$$

with η given by (2.72). [Note that the two scalars Φ^1 and Φ^2 satisfy the same equation, (3.1b).]

These equations exactly coincide with those corresponding to the Abelian vortex solution.[4] Qualitatively one has then the following behavior: the magnetic field $\mathcal{H}\sim(1/\rho)(dA/d\rho)$ decreases monotonically from its value at the origin to zero at infinity with characteristic length $(e\eta)^{-1}$ while the scalars Φ^1 and Φ^2 increase with characteristic length $(\lambda\eta^2/3)^{-1/2}$ from zero at the origin to its vacuum value η at infinity. As it is well known vortices exist provided a Ginsburg-Landau-type parameter χ, which in field theory is related to the scalar- and vector-meson masses

$$\chi=\frac{m_{\text{scalar}}}{m_{\text{vector}}}\,, \tag{3.2}$$

satisfies the condition $\chi>1$ which corresponds to type-II superconductivity. In the present case

$$m_{\text{scalar}}=\left[\frac{\lambda\eta^2}{3}\right],\quad m_{\text{vector}}=e\eta\,, \tag{3.3}$$

and hence the type-II superconductivity conditions read

$$\frac{\lambda}{3e^2}>1\,. \tag{3.4}$$

As explained in Ref. 18, an exact solution can be found in the limiting case $\lambda=3e^2$. All the properties discussed there are then shared by the Z_3 vortex presented here. Namely, static vortices do not interact with each other and their energy simply adds. In order to obtain an appropriate magnetic flux, we start by defining an "electromagnetic" tensor $\mathcal{F}_{\mu\nu}$ from $F_{\mu\nu}$. Being last in the M (or λ_8) direction, it is natural to write

$$\mathcal{F}_{\mu\nu}=\tfrac{1}{2}\mathrm{Tr}(\lambda_8F_{\mu\nu})\,. \tag{3.5}$$

With this definition the magnetic field reads

$$B=\tfrac{1}{2}\mathrm{Tr}(\lambda_8F_{\rho\varphi})=\frac{1}{\sqrt{3}}\frac{A'(\rho)}{\rho}\,. \tag{3.6}$$

One can now compute the magnetic flux Φ which gives

$$\Phi=n\Phi_0,\quad n=1,2\,, \tag{3.7}$$

where Φ_0 is the quantum of the vortex flux:

$$\Phi_0=\frac{2\pi}{\sqrt{3}}\frac{1}{e}\,. \tag{3.8}$$

B. The SU(3) charged vortices and the SU(N) extension

When $A_0\neq0$, one has, instead of (3.1),

$$\frac{1}{\rho}\frac{d}{d\rho}\left[\rho\frac{df}{d\rho}\right]-\left[\left[\frac{n-A}{\rho}\right]^2-A_0{}^2\right]f=\frac{\lambda}{6}(f^2-n^2)f \tag{3.9a}$$

$$\rho\frac{d}{d\rho}\left[\frac{1}{\rho}\frac{dA}{d\rho}\right]+eF^2(1-A)=\mu\rho\frac{dA_0}{d\rho}\,, \tag{3.9b}$$

$$\frac{1}{\rho}\frac{d}{d\rho}\left[\frac{\rho dA_0}{d\rho}\right]-e^2F^2A_0=\frac{\mu}{\rho}\frac{dA}{d\rho} . \tag{3.9c}$$

From the definition (3.5) one has for the electric field

$$E=E_\rho=\mathscr{F}_{0\rho}=\tfrac{1}{2}\mathrm{Tr}(\lambda_8F_{0\rho})=-\frac{1}{\sqrt{3}}A_0' \tag{3.10}$$

Now, Eq. (3.9c) can be written in the form

$$\partial_\rho E_\rho+\sigma=\mu B , \tag{3.11}$$

where we have defined the charge density σ as

$$\sigma=-\tfrac{1}{2}\mathrm{Tr}(\lambda_8J_0)=e^2F^2A_0 . \tag{3.12}$$

Since $\lim_{\rho\to\infty}E_\rho=0$ according to the boundary conditions, one then gets from (3.11) a relation between the charge Q,

$$Q=\int d^2x\,\sigma , \tag{3.13}$$

and the flux

$$Q=\mu\Phi \tag{3.14}$$

(this relation is characteristic of gauge theories with CS term; it was first recognized in Ref. 11). Now, due to the quantization conditions satisfied by μ [Eq. (2.5)] and the form of the magnetic flux Φ, which is the same as in the neutral case [Eqs. (3.7) and (3.8)]. Eq. (3.14) becomes

$$Q=mnQ_0,\quad m=\pm1,\pm2,\ldots,\quad n=1,2 \tag{3.15}$$

and hence, as it was shown in Ref. 13 for the SU(2) case *the charge Q is quantized,* the smallest charge unit being

$$Q_0=\frac{e}{2\sqrt{3}} . \tag{3.16}$$

[The $1/\sqrt{3}$ factor is due to the fact the charged scalars are taken in the adjoint representation of SU(3).] Charge quantization can be connected with the angular momentum J of the vortex,

$$J=\int d^2x\,\epsilon^{ij}x_iT_{0j} , \tag{3.17}$$

where $T_{\mu\nu}$, the energy-momentum tensor is given by

$$T_{\mu\nu}=-\mathrm{Tr}\left[F_{\mu\alpha}F^\alpha_\nu-\sum_{A=1}^{R}D_\mu\Phi^AD_\nu\Phi^A\right]-g_{\mu\nu}\mathscr{L} \tag{3.18}$$

For the vortex solution, the only nontrivial contribution to J comes from

$$T_{0\varphi}=-\frac{4}{3en^2}[e^2(n-A)A_0F+A_0'A'] . \tag{3.19}$$

Inserting (3.19) in (3.17) and using the equations of motion one finally finds

$$J=-\frac{2Q}{\sqrt{3}e}-\frac{4\pi}{3}\frac{\mu m}{e^2}\frac{A^2(\infty)}{n^2}=-\frac{nm}{6} . \tag{3.20}$$

The generalization of these results to the SU(N) case is straightforward. Instead of (3.15) and (3.16) one has for the charge Q

$$Q=-mnQ_0,\quad m=\pm1,\pm2,\ldots,\quad n=1,2,\ldots,N-1 , \tag{3.21}$$

$$Q_0=\frac{e}{2\sqrt{N}} , \tag{3.22}$$

while the angular momentum now takes the form

$$J=-nm\frac{1}{2N} . \tag{3.23}$$

An angular momentum quantized in units of $1/2N$ has been found for particles bearing Z_N symmetry in 1 + 1 space-time. In that case there are no space rotations and the spin $\frac{1}{2}(1-1/N)$ is a "Lorentz spin" associated to hyperbolic rotations in two-dimensional space-time.[16]

It is also interesting to note the resemblences of these results and those obtained in Ref. 19 for a related model. Indeed, in their study of the vacuum-polarization effects of fermions interacting with Abelian gauge in (2+1) dimensions, these authors show that a Chern-Simons term is induced in the effective action when a vortex is taken as a background and also charge and angular momentum are induced for the vacuum, satisfying relations similar to (3.21)–(3.23).

Concerning the behavior of the scalar and gauge fields corresponding to the SU(3) vortex, one can easily analyze their asymptotic behavior from Eqs. (3.9). One finds two possible solutions for large ρ:

$$\begin{aligned}A(\rho)&=n+Z_\pm\frac{mp}{e}K_1(m_\pm\rho)[1+O(e^{-m_\pm\rho})] ,\\A_0(\rho)&=\pm nZ_\pm\frac{m}{e}K_o(m_\pm\rho)[1+O(e^{-m_\pm\rho})] ,\\f(\rho)&=\eta[1-Y_\pm K_0(m_\pm p)+O(e^{-2m_\pm\rho})] ,\end{aligned} \tag{3.24}$$

where $Z_\pm,Y_\pm$ are dimensionless constants, m is the scalar field mass,

$$m=\left[\frac{\lambda\eta^2}{3}\right]^{1/2} , \tag{3.25}$$

while $m_\pm$ are two distinct vector-meson masses,

$$m_\pm=\left[\frac{\mu^2}{4}+e^2\eta^2\right]^{1/2}\pm\frac{\mu}{2} . \tag{3.26}$$

(The fact that in the presence of CS term, the symmetry-breaking pattern implies two distinct masses for the two different polarizations of the vector field was first described in Ref. 20.) The type-II superconductivity condition reads now

$$\mu=\frac{me^2}{4\pi}>\eta\left[\frac{\lambda}{3}\right]^{1/2}\left[\frac{3e^2}{\lambda}-1\right] \tag{3.27}$$

for $m_{\text{vector}}=m_+$ and

$$0<\frac{me^2}{4\pi}<\eta\left[\frac{\lambda}{3}\right]^{1/2}\left[\frac{3e^2}{\lambda}-1\right] \tag{3.28}$$

for the m_- solution. Concerning the energy of the vortex solution, one easily obtains for the SU(N) case

$$\epsilon=T_{00}=\frac{N-1}{N}\left\{\frac{1}{e^2}\left[A_0{}^2+\left[\frac{1}{\rho}\frac{dA}{d\rho}\right]^2\right]+A_0{}^2F^2+\left[\frac{n-A}{\rho}\right]^2F^2\right\}. \quad (3.29)$$

From the asymptotic (3.24) it then follows that

$$E_N\sim\frac{N-1}{N}m_\pm{}^2\ln\frac{e^2}{m_\pm{}^2},$$

and hence the m_- solution has lower energy. The qualitative behavior of the magnetic and the Higgs fields is similar to that described in Sec. III A for the neutral case. Concerning the electric field, it vanishes at zero and at infinity reaching its maximum at some finite ρ.

Note that for $N\to\infty$ one essentially recovers the U(1) result due to the factor $(N-1)/N$ in T_{00}. Indeed, one finds any number of stable vortices and the energy of each one [Eq. (3.29) with $N=\infty$] coincides with the Abelian expression. Also charge and angular momentum become continuous in the $N=\infty$ limit, in accord with the results in Ref. 12.

In the Abelian case, one finds that only the first vortex solution $(n=1)$ is stable for $\chi>1$ (see Eq. (3.2)]. Higher-n configurations split into n vortices with one unit of flux that repel each other. Only when $\chi=1$ stable multivortices exist.[1,18] An analogous situation is to be expected for the SU(N) model except that n is in this case restricted, $1\le n\le N-1$ since an $n=N$ vortex can decay into the vacuum. This is a characteristic feature of Z_N symmetric states: each elementary state $(n=1)$ is the antiparticle of the bound state of $(N-1)$ elementary states.

ACKNOWLEDGMENTS

One of us (F.A.S.) was supported in part by Facultad de Ciencias Exactas, UNLP and CIC, Buenos Aires, Argentina. LPTHE is a Laboratoire associé au CNRS UA 280.

APPENDIX

In order to have a locally stable vacuum the matrix defined in Eq. (2.75),

$$V_{ab}=\frac{\delta^2V}{\delta X_a\delta X_b}, \quad (A1)$$

must be positive at the minimum

$$\Phi_1=\frac{\eta}{\sqrt{6}}\lambda_y, \qquad \Phi_2=\frac{\eta}{\sqrt{6}}\lambda_6, \Psi=B\lambda_3+C\lambda_8 \quad (A2)$$

with η^2, B, C given by Eqs. (2.72) and (2.73). Since $\eta^2>0$, we get from (2.72) the constraint

$$\frac{A_1{}^2}{A_2{}^2}>\frac{\mu_1(2\mu_2-\mu_1)}{\mu_2(2\mu_1-\mu_2)}. \quad (A3)$$

The positivity requirements on the characteristic polynomial coefficients of matrix (A1) yields to

$$\tilde\lambda>0, \quad (A4)$$

$$\lambda[2\tilde\lambda(B^2+C^2)+\lambda\eta^2]-\frac{\mu_1{}^2+\mu_2{}^2}{3}-\frac{\gamma^2\eta^2}{4}>0, \quad (A5)$$

$$3\lambda^2\tilde\lambda\frac{(A_1{}^2-A_2{}^2)}{\mu_1{}^2-\mu_2{}^2}[(4\lambda^2-\gamma^2)\eta^2(\mu_3{}^2+\mu_2{}^2-\mu_1\mu_2)-\mu_1{}^4-\mu_2{}^4]-\tfrac{2}{3}\eta^2[\gamma\mu_1\mu_2+2\lambda(\mu_1{}^2+\mu_2{}^2)]>0, \quad (A6)$$

$$9(A_1{}^2-A_2{}^2)(4\lambda^2-\gamma^2)-12\lambda(\mu_1{}^4+\mu_2{}^4)-8\mu_1{}^2\mu_2{}^2(\lambda+\gamma)>0. \quad (A7)$$

Conditions (A5)—(A7) trivially hold when $\lambda,\tilde\lambda\gg1$, the other parameters being fixed.

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