U(1) problem on a lattice. II. Strong-coupling expansion

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We calculate the mass difference between the π and η mesons in lattice QCD with Wilson fermions using the strong-coupling expansion. We obtain the result that the mass difference first appears at the order of $(1/g^2N)^6$ in the ordinary phase where the parity and flavor symmetry are conserved and that m_π(π meson mass) $< m_\eta$(η meson mass) and $\lim_{m_\pi\to 0} m_\eta \neq 0$. Furthermore, we point out that the dynamics which makes the η meson heavier than the π meson leads to the spontaneous parity and flavor-symmetry breaking ($\langle \bar\psi i\gamma_5\tau^3\psi\rangle \neq 0$) for $M_0{}^2 < M_c{}^2 = 4$ [or $K^2 > K_c{}^2 = \frac{1}{4}$, where the hopping parameter $K = 1/(2M_0)$].

I. INTRODUCTION

The U(1) problem is why the η meson (flavor singlet) is much heavier than the π meson (flavor nonsinglet). Recently we have calculated the mass difference between the singlet and the nonsinglet mesons, using the mesonic effective potential derived by lattice QCD with the Wilson fermion in the strong-coupling limit.[1] The present article is a companion of Ref. 1, hereafter called I. In I no mass difference was obtained ($m_\pi = m_\eta$) in the ordinary phase (where the parity and the flavor symmetry are conserved). The U(1) problem cannot be solved in the strong-coupling limit. We consider that there are two main reasons for this result. First, the calculation of the meson mass from the effective potential was made at the meson tree level in I. Presumably the effect of the meson loops is important for the mass difference. Second, the calculation in the strong-coupling limit is not enough to produce the mass difference. The high-order terms in strong-coupling expansion are needed to produce the mass difference.

Since it is very difficult to calculate the effect of the meson loops on a lattice we do not consider the first point in this paper. Instead we will calculate the mass difference at the tree level of the effective potential obtained by the strong-coupling expansion. An effort is made to make the present article as self-contained as possible, although some reliance on I is unavoidable. For an introduction to the subject containing more references, see I.

The formulation of both strong and $1/N$ expansions are needed to calculate the mass difference since it occurs at higher order in $1/N$ (Ref. 2). We formulate a method to calculate the partition function with the source in the double expansions in Sec. II and give a systematic diagrammatic procedure to calculate the effective potential in Sec. III. In Sec. IV, as an exercise, we calculate the $1/g^2N$ correction to the large-N limit and show that the pseudoscalar meson is the massless particle associated with the parity violation for one flavor case. In Sec. V we calculate the mass difference between the π and the η in the ordinary phase and obtain the main result of this paper. That is

$$m_\eta > m_\pi$$

and

$$\lim_{m_\pi\to 0} m_\eta \neq 0$$

if the β^6/N^5 correction is considered. ($\beta = 1/g^2N$.) In addition to the above result it is shown that the "vector" mesons have no mass difference between singlet and nonsinglet up to all orders in the strong-coupling expansion in Sec. VI. In Sec. VII we discuss physical implications of our results. We pointed out that the dynamics, which makes the η meson heavier than the π meson, leads to the spontaneous parity- and flavor-symmetry breaking ($\langle \bar\psi\gamma_5\tau^3\psi\rangle \neq 0$) for $M_0^2 < M_c{}^2 = 4$(or $K^2 > K_c{}^2 = \frac{1}{4}$) where the hopping parameter is related to the mass parameter M_0 such that $K = 1/(2M_0)$.

II. FORMULATION OF THE STRONG-COUPLING EXPANSION

In this section we formulate the strong-coupling expansion for a mesonic effective potential. The $1/g^2N$ correction to the effective potential in the large-N limit was calculated by Ichinose[3] and the $1/N$ expansion to the effective potential in the strong-coupling limit was calculated by the present author.[1] Combining two methods we generalize the strong-coupling expansion to an arbitrary order of the $1/N$ expansion.

The partition function with the source $J_n^{\hat\alpha\hat\beta}$ is given by

$$Z(J) = \int D\bar\psi D\psi D(U_{n,\mu}) \exp\left[S_F + S_G + \sum_n \mathrm{tr}(J_n \bar\psi_n \psi_n) \right], \tag{2.1}$$

where

$$S_F=\sum_{n,\mu}(\bar\psi_{n+\hat\mu}U^\dagger_{n,\mu}P_\mu\psi_n+\bar\psi_n U_{n,\mu}P_{-\mu}\psi_{n+\hat\mu})+\sum_n \bar\psi_n\hat M\psi_n\ , \tag{2.2}$$

$$S_G=\left[\frac{1}{g^2}\sum_{n,\mu>\nu}\mathrm{Tr}(U_{n,\mu}U_{n+\hat\mu,\nu}U^\dagger_{n+\hat\nu,\mu}U^\dagger_{n,\mu})+\mathrm{H.c.}\right]. \tag{2.3}$$

Here $(\bar\psi_n)^a_{\hat\alpha}$ is the quark field, a is the color index, $\hat\alpha=(\alpha,f)$ is the spinor-flavor index,

$$\hat M^{\hat\alpha\hat\beta}=(M_f a+4r)\delta_{ff'}\delta_{\alpha\beta}\ ,$$

M_f is a bare-quark mass of the flavor f, and

$$(P_{\pm\mu})^{\hat\alpha\hat\beta}=\tfrac{1}{2}(1\pm\gamma_\mu)_{\alpha\beta}\delta_{ff'}\ .$$

In order to formulate a strong-coupling expansion Eq. (2.1) is rewritten as

$$Z(J)=\int D\psi D\bar\psi\exp\left[\sum_n \mathrm{tr}[(J_n+\hat M)\bar\psi_n\psi_n]\right]\times\exp\left[\frac{1}{g^2}\sum_{n,\mu>\nu}\mathrm{Tr}\frac{\delta}{\delta\bar c_{n,\mu,}}\frac{\delta}{\delta\bar c_{n+\hat\mu,\nu}}\frac{\delta}{\delta c_{n+\hat\nu,\mu}}\frac{\delta}{\delta c_{n,\nu}}+\mathrm{H.c.}\right]\prod_{n,\mu}z(c_{n,\mu})\Big|_{c_{n,\mu}=0}\ , \tag{2.4}$$

where

$$\begin{aligned}
z(c_{n,\mu})&=\int dU_{n,\mu}\exp(\mathrm{Tr}U_{n,\mu}\bar D_{n,\mu}+\mathrm{Tr}D_{n,\mu}U^\dagger_{n,\mu})\ ,\\
(D_{n,\mu})_{ab}&=(c_{n,\mu})_{ab}-(\bar\psi_n)_aP^t_\mu(\psi_{n+\hat\mu})_b\\
&\equiv(c_{n,\mu}-A_{n,\mu})_{ab}\ ,\\
(\bar D_{n,\mu})_{ab}&=(\bar c_{n,\mu})_{ab}-(\bar\psi_{n+\hat\mu})_aP^t_{-\mu}(\psi_n)_b\\
&\equiv(\bar c_{n,\mu}-\bar A_{n,\mu})_{ab}\ ,
\end{aligned} \tag{2.5}$$

$dU_{n,\mu}$ is the Haar measure on U(N), N is the number of the color, tr means the trace over the spinor-flavor index, and Tr means the trace over the color index.

The integral (2.5) has been calculated in the $1/N$ expansion in I and the result is

$$z(c_{n,\mu})=\exp[NW(\Lambda_{n,\mu})]\ , \tag{2.6}$$

where

$$\begin{aligned}
W(\Lambda_{n,\mu})&=\sum_{k=0}^{\infty}(1/N)^k w_k(\Lambda_{n,\mu})\ ,\\
\Lambda_{n,\mu}&=(1/N^2)\bar D_{n,\mu}D_{n,\mu}\\
&=(1/N^2)(\bar c_{n,\mu}+\bar A_{n,\mu})(c_{n,\mu}+A_{n,\mu})\ ,\\
w_0(\Lambda)&=\mathrm{Tr}((1+4\Lambda)^{1/2}-1-\ln\{[1+(1+4\Lambda)^{1/2}]/2\}),\\
w_{2k}(\Lambda)&=\sum_{l=0}^{k}C^{(2k)}_{q_1\cdots q_{2l+1}}\lambda_{q_1}\cdots\lambda_{q_{2l+1}},\quad k\ge 1\ ,\\
w_{2k+1}(\Lambda)&=\sum_{l=0}^{k}C^{(2k+1)}_{q_1\cdots q_{2l+2}}\lambda_{q_1}\cdots\lambda_{q_{2l+2}},\quad k\ge 0\ ,
\end{aligned}$$

and

$$\lambda_q=\mathrm{Tr}\Lambda^q\ .$$

We will use only the value of $C^{(1)}_{11}=\frac{1}{2}$ and $C^{(2)}_{111}=\frac{2}{3}$, hereafter.

We can calculate the vacuum expectation value of any operators which include arbitrary number of $U_{n,\mu}$, by using (2.4) and (2.6). But hereafter we treat only the vacuum expectation value of operators without $U_{n,\mu}$ (a local mesonic operator). We define

$$\exp\{NS_1[1/g^2,M(n)]\}\equiv\exp\left[\frac{1}{g^2}\frac{\delta^4}{\delta c^4}+\mathrm{H.c.}\right]\times\exp\left[N\sum_{n,\mu}W(\Lambda_{n,\mu})\right]\ , \tag{2.7}$$

where

$$\frac{\delta^4}{\delta c^4}=\sum_{n,\mu>\nu}\mathrm{Tr}\frac{\delta}{\delta\bar c_{n,\mu,}}\frac{\delta}{\delta\bar c_{n+\hat\mu,\nu}}\frac{\delta}{\delta c_{n+\hat\nu,\mu}}\frac{\delta}{\delta c_{n,\nu}}$$

and evaluate it by expanding

$$\exp\left[\frac{1}{g^2}\frac{\delta^4}{\delta c^4}+\mathrm{H.c.}\right]=1+\left[\frac{1}{g^2}\frac{\delta^4}{\delta c^4}+\mathrm{H.c.}\right]+\cdots\ ,$$

which is the strong-coupling expansion. Finally we obtain

$$Z(J)=\int DM\exp[NS_{\mathrm{eff}}(1/g^2,M(n),J_n)]\ , \tag{2.8}$$

where

$$S_{\mathrm{eff}}(1/g^2,M(n),J_n)=\sum_n\mathrm{tr}[(J_n+\hat M)M(n)-\ln M(n)]+S_1(1/g^2,M(n))$$

and $M(n)^{\hat\alpha\hat\beta}=1/N(\bar\psi^{\hat\alpha}_n\psi^{\hat\beta}_n)$ is the meson field.[4]

Hereafter the calculation of $S_1(1/g^2,M(n))$ is considered. In Sec. III we show that $S_1(1/g^2,M(n))$ is made of all connected diagrams with respect to $M(n)$; therefore, we will calculate connected diagrams only. Furthermore in Sec. III we give a systematic diagrammatic procedure to calculate $S_1(1/g^2,M(n))$.

Finally we mention the meaning of the strong and $1/N$ expansions used in this section. Usually we expand Eq. (2.1) directly by considering both $g^2N=\lambda$ (fixed) and N

as large numbers. But our expansion is different from the usual one. We use the $1/N$ expansion to evaluate $W(\Lambda_{n,\mu})$ [see Eq. (2.6)], use the strong-coupling expansion to evaluate $\exp S_G$ and combine them. After combining two expansions, however, we get the consistent $1/g^2N$ and $1/N$ expansions (they have positive integer power) and can obtain the S_{eff} coincided with the result from the usual expansion.

III. THE CALCULATION PROCEDURE IN THE STRONG-COUPLING EXPANSION

In this section we give a diagrammatic procedure to calculate the effective potential. First we show that $S_1(1/g^2,M(n))$ is made of all connected diagrams with respect to $M(n)$. We define

$$Z(1/g^2,M)=\exp[NS_1(1/g^2,M)]$$
$$\equiv\sum_{n=0}^{\infty}\sum_{x_1,\ldots,x_n} M(x_1)\cdots M(x_n)f_n(x_1,\ldots,x_n,1/g^2)/n!\,. \tag{3.1}$$

The g_{n+1}, the connected part of f_{n+1}, is defined by induction such that

$$f_{n+1}(x_0,\ldots,x_n,1/g^2)=g_{n+1}(x_0,\ldots,x_n,1/g^2)+\sum_{0\le m<n}\sum_{i_1,\ldots,i_n} g_{m+1}(x_0,x_{i_1},\ldots,x_{i_n},1/g^2)f_{n-m}(x_{i_{m+1}},\ldots,x_{i_n},1/g^2)\quad\text{for } n\ge 0 \tag{3.2}$$

and the generating function of all connected diagrams is given by

$$R(1/g^2,M)=\sum_{n=0}^{\infty}\sum_{x_1,\ldots,x_n} M(x_1)\cdots M(x_n)g_n(x_1,\ldots,x_n,1/g^2)/n!\,. \tag{3.3}$$

By using (3.1)–(3.3) for any n it is easy to show that

$$\left.\frac{\delta^n}{\delta M^n}\frac{\delta Z(1/g,M)}{\delta M(x)}\right|_{M=0}=\left.\frac{\delta^n}{\delta M^n}\frac{\delta R(1/g^2,M)}{\delta M(x)}Z(1/g^2,M)\right|_{M=0} \tag{3.4}$$

then we obtain

$$\frac{\delta}{\delta M}(\ln Z-R)=0\,. \tag{3.5}$$

Therefore

$$Z(M)=\exp[R(M)-R(0)]$$

with $Z(0)=1$, means

$$NS_1(1/g^2,M)=R(1/g^2,M)-R(1/g^2,0)\,. \tag{3.6}$$

This completes the proof. It is noticed that the "connected" in our expansion (2.7) means that two loops are connected to each other if they have at least one "common" link, which belongs to one $C^{(n)}_{q_1\cdots q_t}\lambda_{q_1}\cdots\lambda_{q_t}$ $(t\ge 2)$.

Next we summarize the procedure to calculate $NS_1(1/g^2,M)$.

(1) We expand $W(\Lambda_{n,\mu})$ with respect to $c_{n,\mu}$ and $\bar{c}_{n,\mu}$, where

$$\Lambda_{n,\mu}=(\bar{c}_{n,\mu}+\bar{A}_{n,\mu})(c_{n,\mu}+A_{n,\mu})/N^2\,.$$

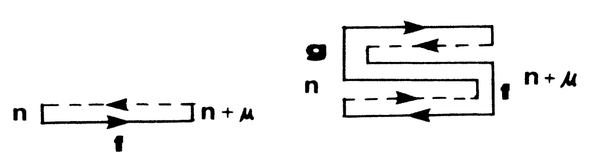
If we want to calculate NS_1 up to the order of $(1/g^2)^t$ we have to expand W up to the order of $\bar{c}^{\,r}_{n,\mu}c^{\,s}_{n,\mu}$ with $r+s=t$.

(2) We use diagrams to represent terms obtained in the above expansion. For example Fig. 1(a) represents $\mathrm{Tr}\bar{c}_{n,\mu}f(\bar{A}_{n,\mu},A_{n,\mu})$ and Fig. 1(b) represents $\mathrm{Tr}\bar{c}_{n,\mu}fc_{n,\mu}g$, where f and g are some functions of $\bar{A}_{n,\mu}$ and $A_{n,\mu}$ obtained in (1) and a closed loop means the trace over color index.

(3) We combine four diagrams into one by the $(1/g^2)\mathrm{Tr}\delta^4/\delta c^4$ operation which appeared in (2.7). For example, Fig. $2=(1/g^2)\mathrm{Tr}(f_1f_2f_3f_4)$. This process is repeated t times.

(4) We generate all necessary diagrams by the above procedure (3). In order to calculate NS_1, we consider only "connected" diagrams.

(5) When we expand $\exp[(1/g^2)\mathrm{Tr}\delta^4/\delta c^4]$, terms such as $(1/n!)[(1/g^2)\mathrm{Tr}\delta^4/\delta c^4]^n$ appear. We can neglect the factor $1/n!$ in the same way as ordinary Feynman rules since the operation of the derivative $(\delta^4/\delta c^4)^n$ generates each term $n!$ times.



(a) (b)

FIG. 1. (a) The diagram which represents $\mathrm{tr}(\bar{c}_{n,\mu}f)$ and (b) the diagram which represents $\mathrm{tr}(\bar{c}_{n,\mu}fc_{n,\mu}g)$, where $\bar{c}_{n,\mu}=n--\!\leftarrow\!--n+\mu$ and $c_{n,\mu}=n--\!\rightarrow\!--n+\mu$.

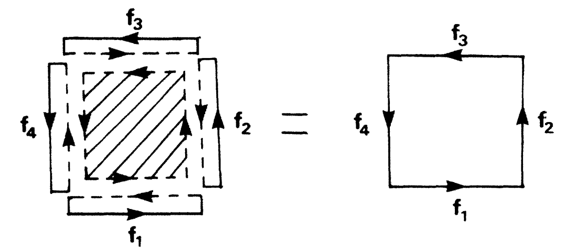


FIG. 2. A diagram made of four links by the strong-coupling expansion.

(6) By using the formula such as

$$\mathrm{Tr}[(\bar{A}_{n,\mu}A_{n,\mu})/N^2]^k = -\mathrm{tr}[M(n)P^t_\mu M(n+\hat{\mu})P^t_{-\mu}]^k ,$$

we rewrite terms obtained by the above procedures, in terms of $M(n)$ and tr (tr means the trace over the flavor-spinor index). We must be careful for the sign factor of the trace. From the procedure (1)—(6) we obtain the final expression of $NS_1(1/g^2,M)$ in the form of the strong-coupling expansion.

IV. THE $1/g^2N$ CORRECTION TO THE LARGE-N LIMIT

Although we are interested in the mass difference between singlet and nonsinglet mesons we calculate the $1/g^2N$ corrections to the large-N limit as an exercise, before calculating such quantities. Such a correction has been already calculated in Ref. 3. We pointed out that the pseudoscalar meson is the massless particle associated with the parity violation for one flavor case in the strong-coupling limit.[1,5] In order to confirm that this property is unchanged by higher-order terms in the strong-coupling expansion we analyze the $1/g^2N$ correction to the large-N limit.

Up to the first order of $\beta = 1/g^2N$ in the large-N limit we obtain

$$\begin{aligned} NS_{\text{eff}}(\beta,M) = N\sum_n \mathrm{tr}\Bigg[& \hat{M}M(n) - \ln M(n) + \sum_\mu (\ln\{[1+(1+4\Lambda_{n,\mu})^{1/2}]/2\} + 1 - (1+4\Lambda_{n,\mu})^{1/2}) \\ & -\beta\sum_{\mu\neq\nu} M(n)P^t_\mu f(\bar{\Lambda}_{n,\mu})M(n+\hat{\mu})P^t_\nu f(\bar{\Lambda}_{n+\hat{\mu},\nu})M(n+\hat{\mu}+\hat{\nu})P^t_{-\mu}f(\Lambda_{n+\hat{\nu},\mu}) \\ & \times M(n+\hat{\nu})P^t_{-\nu}f(\Lambda_{n,\nu})\Bigg] + O(\beta^2) , \end{aligned} \tag{4.1}$$

where

$$\Lambda_{n,\mu} = M(n)P^t_\mu M(n+\hat{\mu})P^t_{-\mu}, \quad \bar{\Lambda}_{n,\mu} = M(n+\hat{\mu})P^t_{-\mu}M(n)P^t_\mu ,$$

and

$$f(x) = 2/[1+(1+4x)^{1/2}] .$$

In order to investigate a vacuum structure we assume $M(n) = \sigma e^{i\theta\gamma_5}$ and solve the gap equation for S_{eff} (Refs. 1 and 5). The solution is given by

$$\sigma = \begin{cases} (1-6\beta/M_0^2)/M_0 & \text{for } M_0^2 \geq 4(1+9\beta/16) , \\ [3/(16-M_0^2)]^{1/2}[1+\beta(5M_0^2+16)(M_0^2-16)/576] & \text{for } M_0^2 \leq 4(1+9\beta/16) , \end{cases} \tag{4.2}$$

$$\sin^2\theta = \begin{cases} 0 & \text{for } M_0^2 \geq 4(1+9\beta/16) , \\ 4[4-M_0^2+\beta M_0^2(-37M_0^4+704M_0^2-1792)/768]/(16-M_0^2) & \text{for } M_0^2 \leq 4(1+9\beta/16) , \end{cases} \tag{4.3}$$

where M_0 is the mass parameter. The value of M_0 where the phase transition occurs satisfies $\sin\theta = 0$ in (4.3) and it gives $M_0^2 = 4(1+9\beta/16)$.

Next we must calculate the π meson mass and show that it vanishes at the value $M_0^2 = 4(1+9\beta/16)$. To see this it is enough to calculate the pseudoscalar meson mass in the case of $\theta = 0$. After a few calculations we obtain

$$\begin{aligned} V^{(2)}_{\text{eff}} &\equiv \sum_{m,n} \frac{1}{2}\frac{\delta^2 S_{\text{eff}}}{\delta M(m)\delta M(n)}\Bigg|_{M=\sigma}[M(m)-\sigma][M(n)-\sigma] \\ &= \int_p \sum_{A,B} \Pi_A(-p)D_{AB}(p)\Pi_B(p) , \end{aligned} \tag{4.4}$$

where

$$M(n)-\sigma=\int_p e^{ip_\mu}\Pi_A(p)\Gamma^A,\quad \int_p=\int_{-\pi/a}^{\pi/a}\frac{d^4p}{(2\pi)^4}\ .$$

$\{\Gamma^A\}$ is a basis of 4×4 matrix that is given by $\Gamma^S=\frac{1}{2}$, $\Gamma^P=\gamma_5/2$, $\Gamma^{A(\rho)}=i\gamma_\rho^t\gamma_5/2$, $\Gamma^{V(\rho)}=\gamma_\rho^t/2$, and $\Gamma^{T(\rho\sigma)}=[\gamma_\rho^t,\gamma_\sigma^t]/(2i\times 2^{1/2})$. In order to obtain the pseudoscalar-meson mass the $D_{P-A}(p)$, the pseudoscalar and axial-vector part of $D_{AB}(p)$, is only used and it is given as

$$D_{PP}(p)=1/(2\sigma^2)-\tfrac{1}{2}-\beta\left[\tfrac{3}{2}(\sigma^2+\sigma^4)\sum_\mu \cos p_\mu a+\tfrac{1}{4}\sigma^2\sum_{\mu\neq\nu}\cos p_\mu a\,\cos p_\nu a\right],$$

$$D_{PA_\alpha}(p)=-D_{A_\alpha P}(p)=\sin p_\alpha a\left[1+\beta\left[3\sigma^2+3\sigma^4+\sigma^2\sum_{\nu\neq\alpha}\cos p_\nu a\right]\right]\Big/2\ ,$$

$$D_{A_\alpha A_\gamma}(p)=\delta_{\alpha\gamma}\left\{1/(2\sigma^2)-\tfrac{1}{2}\cos p_\alpha a+\beta\left[\tfrac{3}{2}\sigma^2\sum_\mu \cos p_\mu a-\cos p_\alpha a\left[3\sigma^4+6\sigma^2+\tfrac{3}{2}\sigma^2\sum_{\mu\neq\alpha}\cos p_\mu a\right]\Big/2\right.\right.$$
$$\left.\left.+\tfrac{1}{4}\sigma^4\sum_{\nu\neq\mu\neq\alpha}\cos p_\mu a\,\cos p_\nu a\right]\right\}+\beta\sigma^2\sin p_\alpha a\,\sin p_\gamma a(1-\delta_{\alpha\gamma})/2\ . \tag{4.5}$$

We calculate the pseudoscalar meson from the equation

$$\det D_{P-A}(p_0=im_\pi,p_k=0)=0$$

then we obtain

$$\cosh m_\pi a=1+2D/[(1+3\beta\sigma^4+6\beta\sigma^2)\times(1/\sigma^2-\tfrac{3}{2}-9\beta\sigma^4/2)]\ , \tag{4.6}$$

where

$$D=(\tfrac{1}{2}+3\beta\sigma^4/2+3\beta\sigma^2)(2-1/\sigma^2+6\beta\sigma^4+3\beta\sigma^2)+[1/(2\sigma^2)+6\beta\sigma^2][1/(2\sigma^2)-\tfrac{3}{2}-9\beta\sigma^4/2-6\beta\sigma^2]$$

and

$$\sigma=(1-6\beta/M_0{}^4)/M_0\ .$$

Since $D=0$ for $M_0{}^2=4(1+9\beta/16)+O(\beta^2)$ we obtain $m_\pi=0$ at $M_0{}^2=4(1+9\beta/16)$ (Ref. 6). Therefore we conclude that the pseudoscalar meson is the massless particle associated with the parity violation for one flavor case in the first order of $1/g^2N$ in the large-N limit.

V. THE MASS DIFFERENCE BETWEEN THE SINGLET AND THE NONSINGLET PSEUDOSCALAR MESONS

In this section we calculate the mass difference between π (nonsinglet) and η (singlet) in the strong-coupling expansion. Hereafter, we consider the ordinary phase ($\theta=0$). It is easy to see that the lowest-order terms which contribute to the mass difference have the order β^6. Since it is too difficult to calculate $S_1(1/g^2,M)$ up to the β^6, we will calculate only the mass difference ($m_\eta-m_\pi$), not the absolute value of the meson mass (m_η or m_π).

In the lowest order the mass terms which contribute only to the singlet sector are made of the product of two loops and each loop has the order of β^3. There are two different types: Fig. 3(a) and 3(b). Figure 3(a) is given by

$$-\mathrm{tr}[M(n)P_\mu M(n+\mu)P_\nu M(n+\mu+\nu)P_\alpha M(n+\mu+\nu+\alpha)P_\beta M(n+\mu+\nu+\alpha+\beta)P_{-\nu}$$
$$\times M(n+\mu+\alpha+\beta)P_{-\beta}M(n+\mu+\alpha)P_{-\mu}M(n+\alpha)P_{-\alpha}] \tag{5.1}$$

and Fig. 3(b) is given by

$$-\mathrm{tr}[M(n)P_\mu M(n+\mu)P_\nu M(n+\mu+\nu)P_{-\mu}M(n+\nu)P_{-\nu}M(n)P_\alpha M(n+\alpha)P_\mu M(n+\mu+\alpha)P_{-\alpha}$$
$$\times M(n+\mu)P_\beta M(n+\mu+\beta)P_{-\mu}M(n+\beta)P_{-\beta}]\ . \tag{5.2}$$

The product of two loops must have at least one "common" link since they must be connected in $S_{\rm eff}$. In this case a "common" link means a link joined in $C_{11}^{(1)}$ or $C_{111}^{(0)}$. But after little calculation it is shown that (i) in the case that one link is common there is no contribution to the pseudoscalar sector, and (ii) in the case that two links are common, contributions to the pseudoscalar sector cancel each other.

Therefore two loops must have three common links for the calculation of mass difference. After little calculations we find that contributions from Fig. 3(b) again cancel each other.

First we consider each loop in the product of two loops. The condition that two loops must have three common

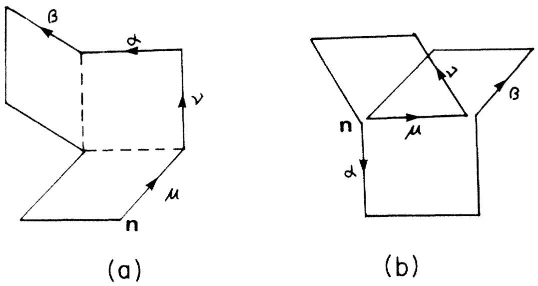


FIG. 3. The lowest-order diagrams in the product of two loops which contribute to the singlet sector.

links is satisfied by fixing three links in each loop. Under this condition we classify such one-loop diagrams, which have nonzero contributions, as follows (see Fig. 4):

(A-1) (n,μ), $(n+\mu,\nu)$,and $(n+\mu+\nu+\alpha,\beta)$ are fixed ,

(A-2) (n,μ), $(n+\mu+\nu,\alpha)$, and $(n+\alpha+\beta,\mu)$ are fixed ,

(A-3) (n,μ), $(n+\mu+\nu,\alpha)$, and $(n+\nu+\alpha,\beta)$ are fixed ,

(A-4) (n,μ), $(n+\nu,\mu)$, and $(n+\nu+\alpha,\beta)$ are fixed ,

(A-5) (n,μ),$(n+\nu,\mu)$, and $(n+\nu,\beta)$ are fixed ,

(A-6) (n,μ), $(n+\nu,\mu)$, and $(n+\nu+\beta,\alpha)$ are fixed ,

(A-7) (n,μ), $(n+\mu,\nu)$, and $(n+\nu,\beta)$ are fixed ,

(A-8) (n,μ), $(n+\mu,\nu)$, and $(n+\nu+\beta,\alpha)$ are fixed ,

(A-9) (n,μ), $(n+\mu,\nu)$, and $(n+\nu+\alpha,\beta)$ are fixed ,

(A-10) (n,μ), $(n+\mu+\alpha,\nu)$, and $(n+\alpha+\beta,\mu)$ are fixed ,

(A-11) (n,μ), $(n+\mu+\nu,\alpha)$, and $(n+\nu,\beta)$ are fixed ,

(A-12) (n,μ), $(n+\mu,\nu)$, and $(n+\mu+\alpha,\beta)$ are fixed .

Second we insert $M(n)=\sigma+\int \Pi(p)\exp(ip\cdot n)$ into Eq. (5.1), consider the linear terms of $\Pi(p)$, and sum up such terms within the same class. They are denoted

$$\int_p D^{(A\text{-}i)}_{\mu\nu\alpha\beta}(p)\exp(ip\cdot n),\quad i=1-12\ . \tag{5.3}$$

For example,

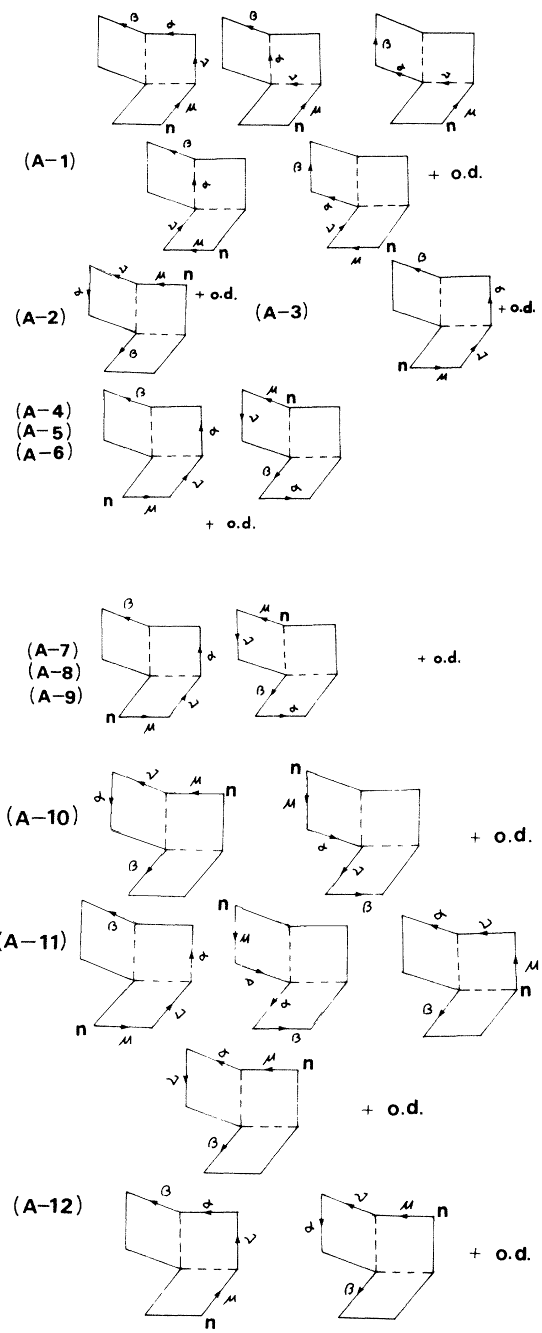


FIG. 4. A classification of the lowest-order loops with three links fixed. o.d. means the same diagrams with the opposite direction.

$$\begin{aligned}D^{(A\text{-}1)}_{\mu\nu\alpha\beta}(p)=\frac{\sigma^7}{32}e^{\mu\nu\alpha\beta}\mathrm{tr}\Pi(p)[&\gamma_5(1+e^{ip_\mu}+e^{ip_{\mu+\nu}}+e^{ip_{\mu+\nu+\alpha}}+e^{ip_{\mu+\nu+\alpha+\beta}}+e^{ip_{\mu+\alpha+\beta}}+e^{ip_{\mu+\alpha}}+e^{ip_\alpha})\\
&+\gamma_5\gamma_\beta(-1-e^{ip_\mu}-e^{ip_{\mu+\nu}}-e^{ip_{\mu+\nu+\alpha}}+e^{ip_{\mu+\nu+\alpha+\beta}}+e^{ip_{\mu+\alpha+\beta}}-e^{ip_{\mu+\alpha}}-e^{ip_\alpha})\\
&+\gamma_5\gamma_\alpha(-1-e^{ip_\mu}-e^{ip_{\mu+\nu}}+e^{ip_{\mu+\nu+\alpha}}+e^{ip_{\mu+\nu+\alpha+\beta}}+e^{ip_{\mu+\alpha+\beta}}+e^{ip_{\mu+\alpha}}+e^{ip_\alpha})\\
&+\gamma_5\gamma_\nu(-1-e^{ip_\mu}+e^{ip_{\mu+\nu}}+e^{ip_{\mu+\nu+\alpha}}+e^{ip_{\mu+\nu+\alpha+\beta}}-e^{ip_{\mu+\alpha+\beta}}-e^{ip_{\mu+\alpha}}-e^{ip_\alpha})\\
&+\gamma_5\gamma_\mu(-1+e^{ip_\mu}+e^{ip_{\mu+\nu}}+e^{ip_{\mu+\nu+\alpha}}+e^{ip_{\mu+\nu+\alpha+\beta}}+e^{ip_{\mu+\alpha+\beta}}+e^{ip_{\mu+\alpha}}-e^{ip_\alpha})]\ .\end{aligned}\tag{5.4}$$

Third, we produce the term of (5.3) with i fixed and sum up the different n, μ, ν, α, and β. Since there exists a double counting for diagrams we multiply the factor $\frac{1}{2}$ for $i=7-12$. We denote the results as

$$\int_p D^i(p)=\int_p \int_q \sum_{n,\mu,\nu,\alpha,\beta} D^{(A-i)}_{\mu\nu\alpha\beta}(p)D^{(A-i)}_{\mu\nu\alpha\beta}(q)\exp[i(p+q)n]\times C^i$$
$$=\int_p \sum_{\mu,\nu,\alpha,\beta} D^{(A-i)}_{\mu\nu\alpha\beta}(p)D^{(A-i)}_{\mu\nu\alpha\beta}(-p)\times C^i\,, \tag{5.5}$$

where

$$C^i=1 \text{ for } i=1-6 \text{ or } =\tfrac{1}{2} \text{ for } i=7-12\,.$$

In order to obtain the singlet mass we put $p_0=im_\eta a$ and $p_k=0$ in (5.5). The results are

$$D^1(p)=D^2(p)=D^3(p)=\tfrac{3}{2}\sigma^{14}\Big[\Pi_p(p)\Pi_p(-p)(37+27X)-\Pi_{A_0}(p)\Pi_{A_0}(-p)(37-27X)$$
$$+[\Pi_P(p)\Pi_{A_0}(-p)-\Pi_{A_0}(p)\Pi_P(-p)]\times 27iY\Big]\,, \tag{5.6}$$

$$D^4(p)/2=D^5(p)/2=D^6(p)/2=D^7(p)=D^8(p)=D^9(p)=D^{10}(p)=D^{11}(p)/3=D^{12}(p)$$
$$=\tfrac{3}{4}\sigma^{14}\Big[\Pi_P(p)\Pi_P(-p)(1-X)-\Pi_{A_0}(p)\Pi_{A_0}(-p)(1+X)-\Big[\Pi_P(p)\Pi_{A_0}(-p)-\Pi_{A_0}(p)\Pi_P(-p)\Big]\times iY\Big]\,, \tag{5.7}$$

where

$$X=\cosh(m_\eta a)$$

and

$$Y=\sinh(m_\eta a)\,.$$

Summing up $D^i(p)$ ($p_0=im_\eta a$, $p_k=0$) for i and multiply the factor $\beta^6\times(C^{(1)}_{11})^3\times(C^{(0)}_1)^7/N^5$ we obtain the final result

$$D^{\text{singlet}}(p)=[\beta^6\times(C^{(1)}_{11})^3\times(C^{(0)}_1)^7/N^5]3\sigma^{14}\Big[\Pi_P(p)\Pi_P(-p)(59+37X)-\Pi_{A_0}(p)\Pi_{A_0}(-p)(59-37X)$$
$$+\Big[\Pi_P(p)\Pi_{A_0}(-p)-\Pi_{A_0}(p)\Pi_P(-p)\Big]\times 37iY\Big]\,. \tag{5.8}$$

Combining (5.8) with the result in the strong-coupling limit we obtain

$$2D^{\text{nonsinglet}}_{P-A_0}(p_0=im_\pi a,\ p_k=0)=\begin{array}{c|cc} & P & A_0\\ \hline P & 1/\sigma^2-3-X & iY\\ A_0 & -iY & 1/\sigma^2-X\end{array}\,,$$

$$2D^{\text{singlet}}_{P-A_0}(p_0=im_\eta a,\ p_k=0)=\begin{array}{c|cc} & P & A_0\\ \hline P & 1/\sigma^2-3-X+t(59+37X) & iY(1-37t)\\ A_0 & -iY(1-37t) & 1/\sigma^2-X-t(59-37X)\end{array}\,, \tag{5.9}$$

where

$$t=2n_f[\beta^6\times(C^{(1)}_{11})^3\times(C^{(0)}_1)^7/N^5]3\sigma^{14}$$
$$=n_f\times\beta^6\times 6\sigma^{14}/8N^5$$

and $\sigma=1/M_0+O(\beta)$.

From $\det D^{\text{nonsinglet}}_{P-A_0}(p)=0$ and $\det D^{\text{singlet}}_{P-A_0}(p)=0$ we obtain

$$\cosh(m_\eta a)=1+\frac{(M_0{}^2-4+96t)(M_0{}^2-1-22t)}{(1-37t)(2M_0{}^2-3)}$$
$$\cosh(m_\pi a)=1+\frac{(M_0{}^2-4)(M_0{}^2-1)}{(2M_0{}^2-3)}\,. \tag{5.10}$$

(Notice that we neglect the common corrections to $m_\eta a$ and $m_\pi a$ in the strong-coupling expansion.) From (5.10) we obtain

$$\Delta ma=\cosh(m_\eta a)-\cosh(m_\pi a)$$
$$=t(37M_0{}^4-111M_0{}^2+140-2112t)/[(2M_0{}^2-3)(1-37t)]$$
$$=6n_f\beta^6(37M_0{}^4-111M_0{}^2+140)/[8N^5M_0{}^{14}(2M_0{}^2-3)]+O(\beta^7)>0\,. \tag{5.11}$$

Furthermore

$$\lim_{m_\pi \to 0} \Delta ma = \lim_{M_0{}^2 \to 4} \Delta ma$$
$$= n_f \beta^6 \times 3^3 / (N^5 \times 2^{11} \times 5) \neq 0 . \quad (5.12)$$

From (5.11) and (5.12) we conclude that

$$m_\eta > m_\pi$$

and

$$\lim_{m_\pi \to 0} m_\eta \neq 0 .$$

This is a desired property for solving the U(1) problem. The dynamical mechanism which realizes the above property is discussed in Sec. VII.

VI. NO MASS DIFFERENCE BETWEEN THE "VECTOR" MESONS

Here we will show that there exists no mass difference between singlet and nonsinglet "vector" (V-T) mesons in all orders in the strong-coupling expansion. $V_{\rm eff}^{(2)}$ terms which contribute to the mass difference have general forms such as (remember Sec. V)

$$\text{const} \times \text{tr}\pi(n)(P_c + P_{\bar{c}}) \times \text{tr}\pi(n)(P_{c'} + P_{\bar{c}'}) . \quad (6.1)$$

where

$$P_c = P_{\mu_1} P_{\mu_2} \cdots P_{\mu_k}, \quad \sum_{i=1}^{k} \hat{\mu}_i = 0 ,$$
$$P_{\bar{c}} = P_{-\mu_k} P_{-\mu_{k-1}} \cdots P_{-\mu_1} ,$$

and $\pi(n) = M(n) - \sigma$. In other words, c is the oriented closed loop starting from n and $\bar{c}$ is the loop c with opposite orientation. The vector-tensor component in (6.1) must have γ_μ or $\gamma_\mu \gamma_\nu (\mu \neq \nu)$ in P_c. For example, we consider

$$\text{tr}\pi(n)\gamma_\mu$$

or

$$\text{tr}\pi(n)\gamma_\mu\gamma_\nu .$$

But in $P_{\bar{c}}$ such a term considered appears as

$$\text{tr}\pi(n)(-\gamma_\mu)$$

or

$$\text{tr}\pi(n)(-\gamma_\nu)(-\gamma_\mu) .$$

Therefore the vector-tensor component in $\text{tr}\pi(n)(P_c + P_{\bar{c}})$ is always zero by the above cancellation. Since the term (6.1) is the general form in the $1/g^2N$ expansion we conclude that the vector tensor sector has no mass-difference between the singlet and the nonsinglet channel in the effective potential. This result may explain the fact that "there is no U(1) problem for the vector mesons in nature."

VII. DISCUSSION

In Sec. V we obtained the main result of this paper. Up to the order of β^6

$$m_\eta > m_\pi$$

and

$$\lim_{m_\pi \to 0} m_\eta \neq 0 .$$

Qualitatively this result is a desired solution to the U(1) problem on a lattice. But quantitatively the mass difference is much smaller than the experimental data. When we put $M_0{}^2 = 4 (m_\pi = 0)$, $a^{-1} = 900$ MeV, $N = 3$, $n_f = 2$, and $\beta = 4.0$ (not strong coupling) into (5.11) we obtain

$$m_\eta \approx 380 \text{ MeV} .$$

Inversely when we put $M_0{}^2 = 4 (m_\pi = 0)$, $a^{-1} = 900$ MeV, $N = 3$, and $M_\eta = 550$ MeV we obtain

$$\beta \approx 4.6 .$$

This result shows that the higher order in the strong-coupling expansion is necessary for the large singlet meson mass. But the calculation of higher and higher orders in the strong-coupling expansion becomes more and more difficult. Therefore we will not go to this direction.

Since Monte Carlo (MC) simulation is a powerful method to calculate the hadron mass, it is desired that the mass difference is calculated by the MC method. But until now it seems very difficult to measure it by the quenched approximation.[7] Including the dynamical quark is necessary to calculate the mass difference. Further progress[8] for this field is desired.

In paper I we pointed out that the pseudoscalar meson is the massless particle associated with the parity-violating phase transition on a lattice. But this is not correct if we consider the many flavors case. To explain this we define

$$\Pi^a(n) = i\bar{\psi}\gamma_5\tau^a\psi(n)$$

and

$$\Pi^0(n) = i\bar{\psi}\gamma_5\psi(n) .$$

If we assume that the flavor symmetry is conserved, it is easy to see

$$\langle \Pi^a(0)\Pi^b(n) \rangle = 0 \quad \text{for } a \neq b ,$$
$$\langle \Pi^a(0)\Pi^a(n) \rangle = \langle \Pi^b(0)\Pi^b(n) \rangle \quad \text{for any } a \text{ and } b ,$$
$$\langle \Pi^a \rangle = 0 \quad \text{for any } a .$$

Furthermore if the parity is conserved

$$\langle \Pi^0 \rangle = 0 .$$

If we regard $\langle \Pi^0 \rangle$ as the order parameter of the phase transition, then there exists a phase where $\langle \Pi^0 \rangle \neq 0$. On such a phase transition some particles may become massless. Ordinarily we think the particle corresponding to the order parameter become massless, then $\Pi^0(n)$ (the flavor-singlet meson) should become massless. But the result in Sec. V shows that $\Pi^a(n)$ (the flavor-nonsinglet mesons) rather than $\Pi^0(n)$ become massless. Physically this is desired.

In the above argument the assumption that the flavor symmetry is conserved is not true. To see this we investigate Eq. (5.10) first. From Eq. (5.10) the π meson becomes massless at $M_0^2=4$ while the η meson does at $M_0^2=4-96t$. If we change M_0^2 from a larger value, the π meson becomes massless first. For $M_0^2<4$ since the existence of the tachyon is forbidden by the positivity of the action "the flavor symmetry" must be broken; for $^{\exists}a$,

$$\langle \Pi^a \rangle \neq 0 ,$$

contrary to our naive assumption. This is our new scenario. To show that this is true we investigate the effective potential

$$S_{\text{eff}}(\beta,M)=\sum_n \mathrm{Tr}\left[\hat{M}M(n)-\ln M(n)+\sum_\mu (\ln\{[1+(1+4\Lambda_{n,\mu})^{1/2}]/2\}+1-(1+4\Lambda_{n,\mu})^{1/2})\right]+b\sum_n[\mathrm{Tr}\gamma_5 M(n)]^2/4 \quad (7.1)$$

with $n_f=2$ and $b>0$. This effective potential is ($\beta\to 0$ and $N\to\infty$) + (the bilinear term for the singlet sector). At first we take the flavor-dependent quark mass term such that

$$\hat{M}=\begin{bmatrix} m_u a+4r & 0 \\ 0 & m_d a+4r \end{bmatrix}=\begin{bmatrix} M_1 & 0 \\ 0 & M_2 \end{bmatrix} .$$

Therefore we assume

$$M(n)=\begin{bmatrix} \sigma_1\exp(i\theta_1) & 0 \\ 0 & \sigma_2\exp(i\theta_2) \end{bmatrix} \quad (7.2)$$

in the vacuum. The gap equations become

$$M_1\sigma_1=\cos\theta_1 ,$$
$$M_2\sigma_2=\cos\theta_2 , \quad (7.3)$$

$$\begin{bmatrix} -1+\dfrac{8\sigma_1^2}{1+(1-4\sigma_1^2\sin^2\theta_1)^{1/2}}-2b\sigma_1^2 & -2b\sigma_1\sigma_2 \\ -2b\sigma_1\sigma_2 & -1+\dfrac{8\sigma_2^2}{1+(1-4\sigma_2^2\sin^2\theta_2)^{1/2}}-2b\sigma_2^2 \end{bmatrix}\begin{bmatrix} \sin\theta_1 \\ \sin\theta_2 \end{bmatrix}=0 . \quad (7.4)$$

In the case of $M_1=M_2=M$ the gap equation (7.4) has two solutions;

(I) $\sigma_1=\sigma_2$, $\sin\theta_1=-\sin\theta_2$ for $M^2<4$,

(II) $\sigma_1=\sigma_2$, $\sin\theta_1=\sin\theta_2$ for $M^2<4-4b$;

then we must calculate the difference of the effective potential between two solutions:

$$V_{\text{eff}}(\sin\theta_1=-\sin\theta_2)-V_{\text{eff}}(\sin\theta_1=\sin\theta_2)=4b(\sigma_1\sin\theta_1)^2>0 .$$

Therefore the solution (I) is realized. This solution means that

$$\langle \bar{\psi}i\gamma_5\psi\rangle=0 ,$$
$$\langle \bar{\psi}i\gamma_5\tau^3\psi\rangle=2\sigma_1\sin\theta_1 =4\frac{[3(4-M^2)]^{1/2}}{16-M^2} .$$

The parity and flavor symmetry is broken. For $M^2\geq 4$, where the parity and flavor symmetry is conserved, the masses of π and η are given as

$$\cosh(m_\eta a)=1+\frac{(M^2-4+4b)(M^2-1)}{2M^2-3+4b} ,$$
$$\cosh(m_\pi a)=1+\frac{(M^2-4)(M^2-1)}{2M^2-3} \quad (7.5)$$

then

$$\cosh(m_\eta a)-\cosh(m_\pi a)=\frac{4b(M^2+1)}{(2M^2-3)(2M^2-3+4b)}>0 .$$

Therefore the U(1) problem is also solved for $M^2>4$.

The above scenario is the hidden dynamical mechanism for the solution to the U(1) problem on a lattice. The analysis of the effective potential derived from a real lattice QCD in the strong-coupling expansion, including the flavor dependence of the quark mass, will be published elsewhere.[9]

In the strong-coupling limit both solutions (I) and (II) have the same energy. This degeneracy is accidental, not the result from some symmetries. If we consider the effect of the meson loops, the solution (I) becomes a true vacuum. In the ordinary phase the η meson becomes heavier than the π meson by the effect of the meson loops even in the strong-coupling limit as pointed out by Wilson.[10] Quantitatively such an effect of the meson loops to the mass difference may be bigger than the effect of the strong-coupling expansion. Therefore it is necessary to calculate the loop effect. Since it is very difficult to calculate it using the complicated meson propagator on a lattice, the random walk technique[11] is suitable. In the future we will investigate this problem.

Before ending this section we mention other uses of the strong-coupling expansion obtained in Sec. II. In principle using (2.4) and (2.8) we can calculate the vacuum expectation of any operators in lattice QCD. Furthermore the Wess-Zumino term or the chiral anomaly may be also calculated in this formulation.

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[1]S. Aoki, Phys. Rev. D **33**, 2399 (1986).

[2]E. Witten, Nucl. Phys. **B160**, 57 (1979).

[3]I. Ichinose, Nucl. Phys. **B249**, 715 (1985).

[4]N. Kawamoto and J. Smit, Nucl. Phys. **B192**, 100 (1981).

[5]S. Aoki, Phys. Rev. D **30**, 2653 (1984).

[6]It is noticed that the value of $M_0{}^2$ is different from the result in Ref. 11, where the random walk technique has been used. We believe that the value of $M_0{}^2=4(1+9\beta/16)$ is correct. Probably the sign of the term $\beta\sigma^4$ in Ref. 11 may be wrong. If we change $\beta\sigma^4 \to -\beta\sigma^4$ in (4.18), we obtain $M_0{}^2=4(1+3\beta/16)$ by solving $\tilde{D}=0$. This value coincides with the value in Ref. 11.

[7]M. Fukugita, T. Kaneko, and A. Ukawa, Phys. Lett. **130B**, 199 (1983).

[8]A. Ukawa and M. Fukugita, Phys. Rev. Lett. **55**, 1854 (1985); G. G. Batrouni *et al.*, Phys. Rev. D **32**, 2736 (1985).

[9]S. Aoki, University of Tokyo Report No. UT-488 (unpublished).

[10]K. Wilson, in *New Phenomena in Subnuclear Physics,* proceedings of the 14th Course of the International School of Subnuclear Physics, Erice, 1975, edited by A. Zichichi (Plenum, New York, 1977).

[11]N. Kawamoto, Nucl. Phys. **B190** [FS3], 617 (1981).