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Fermions and the Gaussian effective potential

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The effect of fermions on the Gaussian effective potential is studied in a variety of fermion-scalar models in 2, 3, and 4 dimensions. Both $g\phi\bar{\psi}\psi$ and $g\phi^2\bar{\psi}\psi$ couplings are considered. Stability requires the bare g to be infinitesimal; $g_B{}^2=G^2/I_0$ with I_0 a divergent integral. This contrasts with large-N studies in which g_B remains finite. The presence of fermions encourages spontaneous symmetry breaking, and in 3 + 1 dimensions the fermions destabilize the already "precarious" ϕ^4 theory.

I. INTRODUCTION

The Gaussian-effective-potential (GEP) approach has now been used to study a variety of scalar field theories.[1–6] It aims to give a general picture of the physics, particularly with regard to the properties and stability of possible vacuum states. In essence, the GEP is a variational approximation to the effective potential using Gaussian trial wave functionals. One writes the scalar field as $\phi_0+\hat{\phi}$, where ϕ_0 is a constant classical field, and $\hat{\phi}$ is a free field of a variable mass Ω, and computes $\langle\mathscr{H}\rangle$ in the free field's vacuum state. Minimizing the result with respect to Ω gives $\bar{V}_G(\phi_0)$, the GEP. The method is nonperturbative, and has several important advantages over the traditional loop-expansion approach.[4]

In the present paper we extend the method to theories containing both scalars and fermions. We shall limit our ambition to computing the effects of the fermions on the effective potential of the bosons. Ideally, perhaps, one would like to calculate an effective potential as a function of both ϕ_0 *and* "$\langle\bar{\psi}\psi\rangle$" (in some sense), but it is not clear to us how to achieve this goal. (The suggestion in Ref. 2 to include a shift in the fermion field, $\psi=\psi_0+\hat{\psi}$, leaves the spinor ψ with a nonzero vacuum expectation value, which violates Lorentz invariance.)

Consequently, we proceed to write the fermion field simply as a free field of a variable mass M (Ref. 7):

$$\psi=\int(dk)_M\sum_\lambda[u_M^\lambda(\mathbf{k})b_M(\mathbf{k},\lambda)e^{-ik\cdot x}+v_M^\lambda(\mathbf{k})d_M^\dagger(\mathbf{k},\lambda)e^{ik\cdot x}]\,, \qquad (1)$$

where, in $\nu+1$ dimensions,

$$(dk)_M=\frac{d^\nu k}{(2\pi)^\nu 2\omega_k(M)},\quad \omega_k(M)\equiv(\mathbf{k}^2+M^2)^{1/2}\,. \qquad (2)$$

The spinors are normalized to $2M$, and the $b,b^\dagger$ and $d,d^\dagger$ operators obey the usual anticommutation relations. The summation index λ is the helicity label. Our trial vacuum state $|0\rangle$ is the state annihilated by the b_M and d_M operators, as well as by the boson annihilation operator a_Ω. Although not explicitly indicated by the notation, the wave functional $|0\rangle$ depends on M, Ω, and the boson field shift ϕ_0.

The GEP is obtained from the Hamiltonian density $\mathscr{H}$ as $\langle 0|\mathscr{H}|0\rangle$, minimized with respect to Ω and M. Consequently, we shall need the matrix elements of the fermion kinetic and mass terms. A straightforward calculation gives

$$\begin{aligned}\langle 0|\bar{\psi}(-i\gamma\cdot\nabla)\psi|0\rangle&=-2\left[\sum_\lambda\right](I_1'-M^2I_0')\,,\\ \langle 0|\bar{\psi}\psi|0\rangle&=-2\left[\sum_\lambda\right]I_0'M\,,\end{aligned} \qquad (3)$$

where $(\sum_\lambda)$ is the number of helicity states, $=\frac{1}{2}\mathrm{Tr}(\mathbb{1})$, $=2^{N-1}$ in $\nu+1=2N$ or $2N+1$ spacetime dimensions. The corresponding bosonic results[1–4] are

$$\begin{aligned}\langle 0|\tfrac{1}{2}[\dot{\phi}^2+(\nabla\phi)^2]|0\rangle&=I_1-\tfrac{1}{2}\Omega^2I_0\,,\\ \langle 0|\phi^2|0\rangle&=\phi_0{}^2+I_0\,,\\ \langle 0|\phi^4|0\rangle&=\phi_0{}^4+6\phi_0{}^2I_0+3I_0{}^2\,,\\ \langle 0|\phi^6|0\rangle&=\phi_0{}^6+15\phi_0{}^4+45\phi_0{}^2I_0{}^2+15I_0{}^3\,.\end{aligned} \qquad (4)$$

Matrix elements of $\phi\bar{\psi}\psi$ and $\phi^2\bar{\psi}\psi$ follow immediately, since they factorize.

In the above formulas I_n', I_n are shorthand for $I_n(M)$, $I_n(\Omega)$, respectively, where

$$I_n(\Omega)\equiv\int(dk)_\Omega[\omega_k(\Omega)]^n\,. \qquad (5)$$

These divergent integrals can be manipulated using the formulas in Tables I and II, which are reproduced from paper II of Ref. 4. Another important property is

$$\frac{dI_n}{d\Omega}=(2n-1)\Omega I_{n-1}\,. \qquad (6)$$

Note that I_{-1} is finite in 1 + 1 and 2 + 1 dimensions, but logarithmically divergent in 3 + 1 dimensions.

It will be implicit throughout that any divergent I_n integrals are to be regulated with an ultraviolet cutoff which

TABLE I. Useful formulas for the differences of I_N integrals in $\nu+1$ dimensions. ($x=\Omega^2/m^2$.)

$\nu=1$ or 2
$I_1(\Omega)-I_1(m)=\frac{1}{2}(\Omega^2-m^2)I_0(m)-m^{\nu+1}L_2(x)/(8\pi)$
$I_0(\Omega)-I_0(m)=-m^{\nu-1}L_1(x)/(4\pi)$
$I_{-1}(\Omega)=\begin{cases}1/(2\pi\Omega^2), & \nu=1\ ,\\ 1/(4\pi\Omega), & \nu=2\end{cases}$
$\nu=3$ or 4
$I_1(\Omega)-I_1(m)=\frac{1}{2}(\Omega^2-m^2)I_0(m)-\frac{1}{8}(\Omega^2-m^2)^2I_{-1}(m)+m^{\nu+1}L_3(x)/(32\pi^2)$
$I_0(\Omega)-I_0(m)=-\frac{1}{2}(\Omega^2-m^2)I_{-1}(m)+m^{\nu-1}L_2(x)/(16\pi^2)$
$I_{-1}(\Omega)-I_{-1}(m)=-m^{\nu-3}L_1(x)/(8\pi^2)$

is taken to infinity at the end. However, in the Appendix we shall have some remarks about the curious differences that occur if dimensional regularization is used instead.

The plan of the paper is as follows. In Sec. II we examine the simplest Yukawa model, without boson self-interactions, and explain the necessity for a rather unconventional renormalization of the Yukawa coupling constant. Models including boson self-couplings are considered in Sec. III. In particular we demonstrate that ϕ^4 coupled to fermions in 3 + 1 dimensions is unstable. We return to lower dimensions in Sec. IV to consider theories with a $g\phi^2\bar{\psi}\psi$ interaction, which preserves a $\phi\to-\phi$ symmetry. Again we find that stability requires us to make g_B infinitesimal. This leads to results very different from $1/N$ expansion analyses of similar models,[8-10] and to problems with trying to impose supersymmetry. We discuss these difficulties in Sec. V, before summarizing our conclusions in Sec. VI.

Our formalism and notation follows the series of papers in Ref. 4 (hereafter referred to as I, II, III), and some familiarity with these works will undoubtedly aid the reader. In particular, we rely heavily on the results of III. However, we have tried to make the main points of the present article self-contained.

II. THE SIMPLEST YUKAWA MODEL

We begin by studying the simplest model of a Dirac fermion field[11] coupled, through a Yukawa term, to a non-self-interacting boson field in 1 + 1 or 2 + 1 dimensions. The Hamiltonian density is

$$\mathscr{H}=\bar{\psi}(-i\gamma\cdot\nabla+\mu_B)\psi+g_B\phi\bar{\psi}\psi+\tfrac{1}{2}\dot{\phi}^2+\tfrac{1}{2}(\nabla\phi)^2+a_B\phi+\tfrac{1}{2}m_B{}^2\phi^2\ . \qquad (7)$$

Note that we include a linear term for the ϕ field. This could, of course, be eliminated by use of the freedom to redefine the field as $\phi'=\phi+c$, with a consequent redefinition of μ_B. However, in the absence of a $\phi\to-\phi$ symmetry, quantum effects induce a shift in the vacuum, so that it is more convenient to use the above-mentioned freedom to arrange that the *quantum* vacuum, rather than the classical vacuum, is at $\phi_0=0$; i.e., we shall impose the condition

$$a_R\equiv d\bar{V}_G/d\phi_0\,|_{\phi_0=0}=0\ , \qquad (8)$$

and fix a_B accordingly.

From the Hamiltonian density and Eqs. (3) and (4) one obtains [N.B. $(\sum_\lambda)=1$ in 1 + 1 or 2 + 1 dimensions]

TABLE II. The $L_i(x)$ functions, and their expansions in $y\equiv(x-1)$ or $z\equiv\sqrt{x}-1$.

$\nu=$odd	
$L_1(x)=\ln x$	$=y(1-\frac{1}{2}y+\frac{1}{3}y^2-\cdots)$
$L_2(x)=x\ln x-(x-1)$	$=\frac{1}{2}y^2(1-\frac{1}{3}y+\cdots)$
$L_3(x)=\frac{1}{4}[2x^2\ln x-2(x-1)-3(x-1)^2]$	$=\frac{1}{6}y^3(1+\cdots)$
$\nu=$even	
$L_1(x)$	$=(\sqrt{x}-1)=z$
$L_2(x)=\frac{1}{3}(\sqrt{x}-1)^2(2\sqrt{x}+1)$	$=z^2(1+\frac{2}{3}z)$
$L_3(x)=\frac{1}{30}(\sqrt{x}-1)^3(8x+9\sqrt{x}+3)$	$=\frac{2}{3}z^3(1+\frac{5}{4}z+\frac{2}{5}z^2)$

$$V_G(\phi_0;\Omega,M)=I_1+\tfrac{1}{2}(m_B{}^2-\Omega^2)I_0+a_B\phi_0+\tfrac{1}{2}m_B{}^2\phi_0{}^2 -2[I'_1-M(M-\mu_B)I'_0+g_BM\phi_0I'_0] \ . \quad (9)$$

The GEP itself, $\overline{V}_G(\phi_0)$, is obtained by minimizing this expression with respect to M and Ω. Differentiation of (9) yields the optimum values of M and Ω:

$$\overline{M}=\mu_B+g_B\phi_0 \ , \quad \overline{\Omega}^2=m_B{}^2 \ , \quad (10)$$

and it is easily checked that, in this case, these conditions do give the global minimum of V_G. When the $\overline{M}$ equation is used in Eq. (9), it reduces the fermionic and Yukawa terms to just $-2I_1(\overline{M})$. This feature persists in the other models we shall study later.

Also general is the result that $\overline{M}$ and $\overline{\Omega}$, evaluated at a minimum of the GEP, correspond to the particle masses in that vacuum. One shows this by computing the energy of a one-fermion state $b_M^\dagger|0\rangle$ or a one-boson state $a_\Omega^\dagger|0\rangle$ over and above the vacuum energy, just as in Refs. 1–4. In the present case this means that the bare parameters μ_B and m_B are in fact the physical masses in the $\phi_0=0$ vacuum, and hence must be finite. We may therefore drop the B subscript.

Equation (9) can be simplified by imposing condition (8), using the $\overline{M}$ and $\overline{\Omega}$ equations, and employing the formulas of Table I:

$$\overline{V}_G(\phi_0)=D+\tfrac{1}{2}[m^2-2g_B{}^2I_0(\mu)]\phi_0{}^2 +2\mu^{\nu+1}L_2(x')/(8\pi) \ , \quad (11)$$

where D is the divergent vacuum-energy constant, and

$$x'\equiv\overline{M}^2/\mu^2=(1+g_B\phi_0/\mu)^2 \ . \quad (12)$$

The presence of the divergent integral $I_0(\mu)$ indicates that some renormalization is necessary. One cannot simply appeal to a boson-mass renormalization $m_R{}^2=m^2-2g_B{}^2I_0(\mu)$ because, as we mentioned above, m is itself the physical mass, as determined by a first-principles calculation, and must be finite. We are forced to conclude that g_B must be infinitesimal, with

$$g_B{}^2=G^2/I_0(\mu) \ , \quad (13)$$

where G is finite. This makes x' infinitesimally close to unity, so that the $L_2(x')$ term vanishes like $O(1/I_0)$, leaving the GEP proportional to $\phi_0{}^2$.

A possible objection to this conclusion is that we are ignoring wave-function renormalization. Could we not render the second term in Eq. (11) finite by a suitable rescaling of ϕ_0? This does not work as we can easily show. Suppose we define $\phi_0=Z^{1/2}\Phi_0$, with

$$Z^{-1}=f\left[1-\frac{2g_B{}^2}{m^2}I_0(\mu)\right] \ , \quad (14)$$

where f is some finite number, which will have to be negative if $Z^{1/2}$ is to be real. Since this makes $\phi_0\sim I_0{}^{-1/2}$, one would still find that x' is infinitesimally close to unity, so that the $L_2(x')$ term again vanishes like $O(1/I_0)$. This leaves $\overline{V}_G$ proportional to $\Phi_0{}^2$, but now with a *negative* coefficient. This pinpoints the real difficulty with assuming that g_B is finite—it would cause the GEP to be unbounded below. Since the GEP is a variational approximation, it should be an upper bound on the true effective potential. Therefore, there seems to be no escape from the conclusion that g_B cannot be finite, and must be renormalized as in Eq. (13).

It is instructive to contrast this with the situation in the loop expansion. As noted in I and II, the unrenormalized one-loop effective potential (1LEP) can be read off from the GEP result. For the present simple model, the two are in fact identical. This can be seen by substituting (10) into (9) to obtain

$$\overline{V}_G(\phi_0)=(a_B\phi_0+\tfrac{1}{2}m^2\phi_0{}^2) +\hbar[I_1(m)-2I_1(\mu+g_B\phi_0)] \ , \quad (15)$$

where $\hbar$ has been reinstated. Separating out the vacuum-energy constant leads again to Eq. (11). However, in the loop expansion one would now proceed to invoke a wave-function renormalization

$$\phi_0=\left[1-\hbar\frac{2g_B{}^2}{m^2}I_0(\mu)\right]^{-1/2}\Phi_0 =\left[1+\hbar\frac{g_B{}^2}{m^2}I_0(\mu)+O(\hbar^2)\right]\Phi_0 \ . \quad (16)$$

This converts the second term in Eq. (11) into $\frac{1}{2}m^2\Phi_0{}^2$, while the $L_2(x')$ term, which already carries an explicit factor of $\hbar$ may be evaluated with just $\phi_0=\Phi_0+O(\hbar)$, so that

$$x'=(1+g_B\Phi_0/\mu)^2+O(\hbar) \ .$$

This gives an x' which differs finitely from unity, so that the $L_2(x')$ term gives a nonvanishing contribution. Indeed, this term would be interpreted as the one-loop correction to the classical potential, $\frac{1}{2}m^2\Phi_0{}^2$.

The huge difference between this result and our previous conclusion comes about because in the loop expansion one allows oneself to treat the factor in (16) as an expansion in $\hbar$. That is, one is implicitly regulating the divergent integral I_0 and taking the limit $\hbar\to0$ *before* allowing I_0 to go to infinity. (Note that the same effect is achieved by expanding in powers of g, so that the one-loop approximation is essentially a perturbative approximation.) It is this interchange of limits that completely alters the picture. In the true theory I_0 has to be taken to infinity *first*—indeed $\hbar$ and g (the renormalized g) should remain finite, and not be taken to zero at all.

Returning to the GEP approach, then, we are forced to renormalize g as $g_B{}^2=G^2/I_0(\mu)$, which leaves the GEP as

$$\overline{V}_G(\phi_0)-D=\tfrac{1}{2}(m^2-2G^2)\phi_0{}^2 \ . \quad (17)$$

Thus, the fermions tend to destabilize the $\phi_0=0$ vacuum, and one requires $G^2<\frac{1}{2}m^2$ for the theory to be stable. Although the GEP is purely parabolic, we do not conclude that the theory is entirely noninteracting. The coefficient of $\frac{1}{2}\phi_0{}^2$, which can be viewed as the inverse propagator at zero momentum, does not match with the physical mass squared, implying that the propagator is nontrivial.

III. YUKAWA MODELS

A. 1 + 1 and 2 + 1 dimensions

The results of Sec. II may be generalized to models which include boson self-interaction terms, such as ϕ^4 and ϕ^6. Results for the scalar sector can be read off from II and III, and the only effect of the fermions is to add a term $-G^2\phi_0^2$ to the GEP. This tends to destabilize the $\phi_0=0$ vacuum, but it does not affect the ultimate stability of the theory, which is now governed by the boson self-interactions.

Actually, in the absence of a $\phi\to-\phi$ symmetry there would be no reason to exclude ϕ^3 and ϕ^5 terms from the potential, necessitating a generalization of the results in II and III. We choose instead to focus on a subclass of theories in which odd terms are forbidden by a discrete chiral invariance

$$\phi\to-\phi,\quad \psi\to\gamma_5\psi\ . \tag{18}$$

The most general such Hamiltonian density, consistent with renormalizability in 2 + 1 dimensions, is

$$\mathcal{H}=\bar\psi(-i\gamma\cdot\nabla)\psi+g_B\phi\bar\psi\psi+\tfrac{1}{2}\dot\phi^2+\tfrac{1}{2}(\nabla\phi)^2+\tfrac{1}{2}m_B{}^2\phi^2+\lambda_B\phi^4+\xi\phi^6\ . \tag{19}$$

The symmetry forbids a bare fermion mass term, and this requires a slight modification to our previous analysis.

The fermions give the same contribution to the GEP as in Eq. (9), but now with $\mu_B=0$. The $\overline{M}$ equation becomes

$$\overline{M}=g_B\phi_0\ , \tag{20}$$

so that, as before, the fermionic contribution simplifies to just $-2I_1(\overline{M})$. From this we must subtract out the contribution to the vacuum-energy constant, viz., $-2I_1(0)$, which is not infrared singular, and can be expressed as

$$-2I_1(0)=-2[I_1(\mu)-\tfrac{1}{2}\mu^2I_0(\mu)-\mu^{\nu+1}L_2(0)/(8\pi)] \tag{21}$$

for arbitrary μ. Subtracting this constant leaves a fermionic contribution

$$-2\left\{\frac{1}{2}g_B{}^2\phi_0{}^2I_0(\mu)-\frac{\mu^{\nu+1}}{8\pi}\left[L_2\left[\frac{g_B{}^2\phi_0{}^2}{\mu^2}\right]-L_2(0)\right]\right\}. \tag{22}$$

To avoid the theory being unstable, one must renormalize g as

$$g_B{}^2=G^2/I_0(\mu)\ . \tag{23}$$

The L_2 terms then give a vanishing contribution, and one is left with just $-G^2\phi_0{}^2$, as usual. [Note that μ is arbitrary in the above, and does not appear in the final result. Changing μ only affects G by an infinitesimal amount: $O(1/I_0)$.]

Analysis of the scalar sector proceeds exactly as in III, to which we refer the reader for full details. It is convenient to use the α,β parameters, introduced in III, which are finite, dimensionless measures of the ϕ^4 and ϕ^6 coupling strengths, respectively. They are defined as

$$\alpha\equiv\frac{3\lambda_r}{2\pi}m_R^{\nu-3},\quad \beta\equiv\frac{45}{8\pi^2}\xi m_R^{2\nu-4} \tag{24}$$

where $\nu=1,2$ is the spatial dimension, and

$$\lambda_r=\lambda_B+15\xi I_0(m_R)\ , \tag{25}$$

with m_R being the physical mass of the boson. Although the fermions add only the single term $-G^2\phi_0{}^2$ to the GEP, this leads to surprisingly rich behavior. For clarity, we first discuss the results taken at face value, and add some caveats at the end.

Figures 1 and 2 show the α,β parameter space for the (1 + 1)- and (2 + 1)-dimensional theories, respectively. For reasons explained in III the parameter space is restricted in certain ways, indicated by the boundary lines in the figures. We may divide the remaining parameter space into regions of "unbroken symmetry" and "broken symmetry" according to whether the $\phi_0=0$ vacuum is or is not the lowest minimum of the GEP. (Strictly speaking, in 1 + 1 dimensions this terminology, and some of that used below in discussing "phase transitions," is not correct, because of the possibility of intervacuum mixing. We return to this point later.)

The fermionic term $-G^2\phi_0{}^2$ tends to destabilize the $\phi_0=0$ vacuum, causing the unbroken-symmetry region to shrink as G^2 increases. Finally, when G^2 exceeds $\frac{1}{2}m_R{}^2$ the second derivative of $\overline{V}_G$ at $\phi_0=0$ becomes negative, so that the unbroken-symmetry region then disappears altogether. At the critical $G^2=\frac{1}{2}m_R{}^2$, the symmetric phase

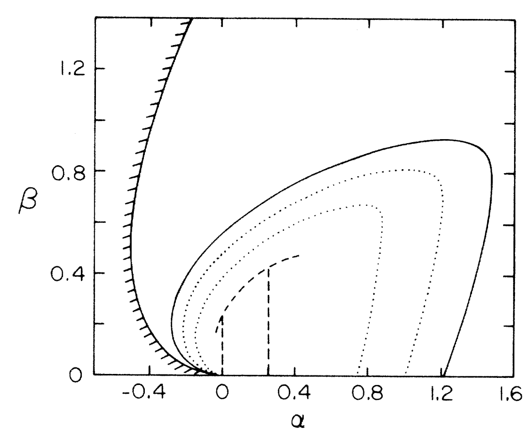

FIG. 1. The α,β parameter space for the (1 + 1)-dimensional version of the Yukawa model [Eq. (19)], showing the shrinkage of the "unbroken symmetry" region as the Yukawa coupling is increased. For $G^2=0$ the region extends out to the solid curve (cf. Fig. 1 of III). As G^2 increases, the region shrinks as illustrated by the dotted curves corresponding to $G^2/m_R{}^2=\frac{1}{6}$ and $\frac{1}{3}$. At $G^2/m_R{}^2=\frac{1}{2}$ the region is delimited by the dashed lines. (See also Fig. 3.) For larger G^2's the symmetric vacuum is always unstable, because $\phi_0=0$ becomes a local maximum of the GEP. (The left-hand boundary on the overall parameter space is explained in III.)

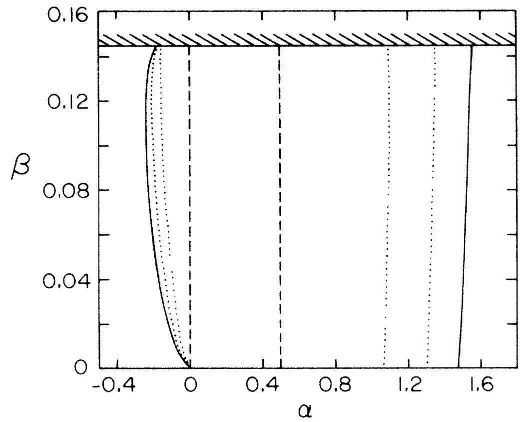

FIG. 2. Same as Fig. 1, but for the (2 + 1)-dimensional version of the Yukawa model (19). (cf. Fig. 2 of III.) (The overall parameter space is bounded by $\alpha > -1$ and $0 < \beta < 0.145$. See III.)

is bounded in α by $0<\alpha<\frac{1}{4}(\nu=1)$ or $0<\alpha<\frac{1}{2}(\nu=2)$, corresponding to the condition for the fourth derivative of $\overline{V}_G$ to be positive at $\phi_0=0$.

In the (1 + 1)-dimensional case the situation can be quite complex because there may be two distinct (pairs of) nonzero minima, at $\phi_0=\pm c_F$ and $\phi_0=\pm c_N$, which we call the "far" and "near" vacua, respectively, according to their proximity to the origin (i.e., $c_F > c_N$). For example, at the critical $G^2=\frac{1}{2}m_R{}^2$ one finds the phase diagram shown in Fig. 3. There are two "triple points" where three phase coexist. Going around a triple point we can pass from the symmetric phase (S) to the near-vacuum phase (N) through a second-order transition, and then from the N phase to the far-vacuum phase (F) by a first-order transition, and then, through another first-order transition, back to the S phase. However, as with the liquid-vapor phases in a pressure-temperature diagram, it is possible to pass from the N to the F phase without going through a phase transition at all. This is because as one goes away from the triple point along the N-F transition line, the minima at c_F and c_N approach each other and coalesce, leaving only a single minimum. At this point the N-F line stops.

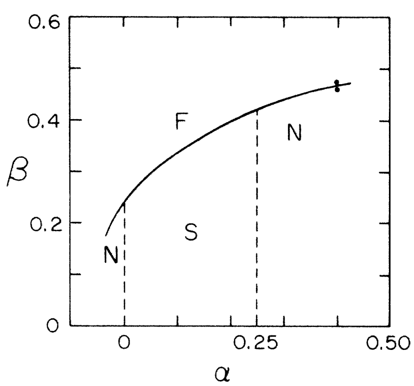

FIG. 3. A more detailed view of the situation at $G^2/m_R{}^2=\frac{1}{2}$ in the (1 + 1)-dimensional Yukawa model. The three "phases" are labeled S, N, F according to whether the symmetric, the near, or the far vacuum has the lowest energy (see text). In the absence of intervacuum mixing, the transitions $S \Longleftrightarrow F$ and $N \Longleftrightarrow F$ would be first order, while the $S \Longleftrightarrow N$ transition (at this critical $G^2/m_R{}^2=\frac{1}{2}$) would be second order. The lines separating the F and N phases end when the far and near minima come together and coalesce. The dots near the right-hand end of the N-F line indicate roughly the location of the parameter values used in Fig. 4: the spacing of the dots has been exaggerated for greater visibility.

As an illustration, we show in Fig. 4 the shape of the GEP for parameter values lying on either side of the N-F line, as indicated by the two dots in Fig. 3. One can see clearly that raising β, and hence crossing the N-F line, causes the far vacuum to become deeper than the near vacuum. Note also that at these parameter values the origin is unstable, even locally: the second derivative vanishes because $G^2=\frac{1}{2}m_R{}^2$, and the fourth derivative is negative because $\alpha > \frac{1}{4}$.

The phase picture of Fig. 3 persists for values of G^2 slightly less than $\frac{1}{2}m_R{}^2$. The S region in then somewhat larger, and the S-N transition weakly first order. Also, the N-F lines become shorter as G^2 is decreased, and soon disappear: the left-hand line has gone by $G^2 \simeq 0.48m_R{}^2$, and the right-hand line shrinks away by $G^2 \simeq 0.42m_R{}^2$. For smaller G^2's one then has a simple two-phase picture, as in the scalar theory. If instead we go to values of G^2 greater than $\frac{1}{2}m_R{}^2$, the S region is absent—it has merged with the N region—but the N-F line remains as an isolated feature, like a mouth, in the middle of the phase diagram. The line shrinks as G^2 increases, and vanishes altogether for $G^2 \gtrsim 0.56m_R{}^2$, leaving only a single, broken-symmetry, phase.

The foregoing description in terms of "phases" and "phase transitions" is actually too naive, because we have ignored the possibility of mixing between degenerate, or almost-degenerate minima. One knows that in 0 + 1 di-

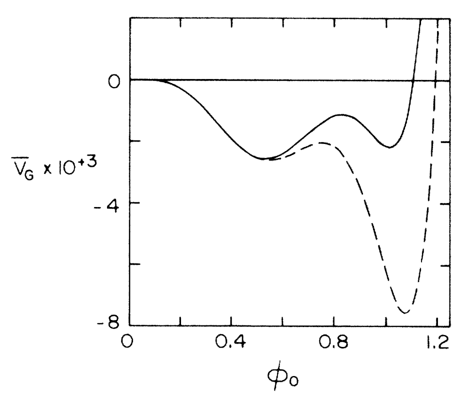

FIG. 4. The GEP for the (1 + 1)-dimensional Yukawa model, for the parameter values $G^2/m_R{}^2=\frac{1}{2}$, $\alpha=0.4$, and $\beta=0.465$ (solid line) or $\beta=0.472$ (dashed line). These values correspond to points on either side of the N-F transition line, as indicated by the two dots in Fig. 3. (The units are such that $m_R=1$.)

mensions (quantum mechanics) there is such mixing, so that even when the GEP has a marked double-well shape, the $\phi \to -\phi$ symmetry is never truly broken. However, when the mixing is weak, the symmetry would *appear* to be broken, if the system were observed only over a short timescale. In 1 + 1 dimensions the situation is analogous, in that there are finite-energy configurations—solitons—which interpolate between the naive vacua, enabling them to mix. Because of this mixing, the "phase transitions" we have been discussing above probably correspond to smooth, but somewhat abrupt, changes.

This viewpoint was discussed previously in II in the context of ϕ^4 theories (see also Ref. 12). It is a natural extrapolation from the experience with (0 + 1)-dimensional models in I, where the physics is well understood. We stress, however, that the GEP itself does not tell us whether there can be mixing between its various minima. To investigate this point, other methods must be brought to bear. For instance, one could attempt to calculate the intervacuum mixing effects by semiclassical methods.[13,14] In 2 + 1 (and higher) dimensions one would expect intervacuum mixing to be suppressed by factors involving the spatial volume, so that it is presumably correct to speak of distinct phases in that case.

Finally, we must not neglect to warn that the GEP is an *approximate* method. Experience with quantum mechanics suggests that, while it can be remarkably accurate in both weak- and strong-coupling regimes, it is not at its best in describing the transitional regions. Consequently, quite apart from the question of intervacuum mixing, if the GEP shows, say, two minima of nearly equal depth, it is not easy to be certain which is truly the lower-energy state. Relatively small changes in the shape of GEP can of course have a considerable effect on where the phase boundaries occur in parameter space. Thus, for instance, the triple-point structures in Fig. 3 have to be viewed cautiously. (Also, mixing effects are likely to be strongest in this region, since the intervacuum barriers are low.) The picture in Fig. 3 should be regarded mainly as a general indication of some rather rich behavior in this parameter range.

B. 3 + 1 dimensions

We now proceed to 3 + 1 dimensions. The most general, renormalizable fermion-scalar model is

$$\mathscr{H}=\bar{\psi}(-i\gamma\cdot\nabla+\mu_B)\psi+g_B\phi\bar{\psi}\psi+\tfrac{1}{2}[\dot{\phi}^2+(\nabla\phi)^2+m_B{}^2\phi^2]+a_B\phi+b_B\phi^3+\lambda_B\phi^4\ . \tag{26}$$

We recall that the scalar sector alone is a rather peculiar theory.[3-5] The renormalization requires a negative, infinitesimal λ_B, $\sim -1/(6I_{-1})$, making the theory "precarious." It is stable only when the ultraviolet cutoff is removed,[3,4] and then only if the potential is symmetric.[5] As we shall see, the addition of fermions also destabilizes the theory.

Calculating $\langle 0|\mathscr{H}|0\rangle$ one obtains [N.B. $(\sum_\lambda)=2$ in 3 + 1 dimensions]

$$\begin{aligned}V_G(\phi_0;\Omega,M)=&I_1+\tfrac{1}{2}(m_B{}^2-\Omega^2)I_0+a_B\phi_0+\tfrac{1}{2}m_B{}^2\phi_0{}^2\\&+b_B(\phi_0{}^3+3\phi_0I_0)\\&+\lambda_B(\phi_0{}^4+6\phi_0{}^2I_0+3I_0{}^2)\\&-4[I_1'-M(M-\mu_B)I_0'+g_BM\phi_0I_0']\ .\end{aligned} \tag{27}$$

Minimizing with respect to M and Ω leads to

$$\bar{M}=\mu_B+g_B\phi_0\ , \tag{28}$$

$$\bar{\Omega}^2=m_B{}^2+6b_B\phi_0+12\lambda_B[\phi_0{}^2+I_0(\bar{\Omega})]\ . \tag{29}$$

[However, as in ϕ^4, at large values of ϕ_0 (29) gives only a local minimum of V_G, with the global minimum being at $\Omega=0$.] As before, consideration of the one-particle states' energies shows that the fermion and scalar masses are given by $\bar{M}\,|_{\phi_0=0}=\mu_B$ (so that we may discard the B subscript hereafter) and $\bar{\Omega}\,|_{\phi_0=0}\equiv m_R$, where

$$m_B{}^2=m_R{}^2-12\lambda_BI_0(m_R)\ . \tag{30}$$

Imposing our requirement (8) gives

$$\left.\frac{d\bar{V}_G}{d\phi_0}\right|_{\phi_0=0}=a_B+3b_BI_0(m_R)-4g_B\mu I_0(\mu)=0\ , \tag{31}$$

placing the vacuum, if there is one, at $\phi_0=0$.

By virtue of the $\bar{M}$ equation, the fermionic terms again reduce to $-4I_1(\bar{M})$. Using the formula of Table I we can express this fermionic contribution as

$$-2g_B{}^2\phi_0{}^2(I_0'-\mu^2I_{-1}')+2g_B{}^3\phi_0{}^3\mu I_{-1}'+\tfrac{1}{2}g_B{}^4\phi_0{}^4I_{-1}'-\frac{\mu^4L_3(x')}{8\pi^2}\ , \tag{32}$$

where $I_n'\equiv I_n(\mu)$ here, and $x'=\bar{M}^2/\mu^2=(1+g_B\phi_0/\mu)^2$. In the above expression we have discarded the constant and linear terms, which merely contribute to the vacuum-energy constant and to the cancellation noted in Eq. (31), respectively.

The most singular term in (32) is the first, containing the quadratically divergent integral $I_0(\mu)$. This already suggests that we must renormalize g_B as

$$g_B{}^2=\frac{G^2}{I_0(\mu)}\ , \tag{33}$$

and we may demonstrate that this is the only possibility by considering the second derivative of $\bar{V}_G$ at the origin:

$$\left.\frac{d^2\bar{V}_G}{d\phi_0{}^2}\right|_{\phi_0=0}=m_R{}^2-\frac{9b_B{}^2I_{-1}(m_R)}{1+6\lambda_BI_{-1}(m_R)}-4g_B{}^2[I_0(\mu)-\mu^2I_{-1}(\mu)]\ . \tag{34}$$

The first term is finite, and a cancellation between the second and third terms is impossible, because both have the same sign. [Note that $I_{-1}(1+6\lambda_BI_{-1})>0$ is the condition for $d^2V_G/d\Omega^2|_{\bar{\Omega}}$ to be positive, which must be true, otherwise $\bar{V}_G(\phi_0)=-\infty$.] Thus each term in (34) must be finite, dictating the renormalization specified by Eq. (33).

Consequently, all terms in Eq. (32) except the first will vanish in the limit of infinite cutoff, leaving only $-2G^2\phi_0^2$ as the fermion contribution. The analysis of the scalar sector proceeds as in Ref. 5 (see also II). One may quickly eliminate all possible renormalizations of b and λ other than

$$\lambda_B=\frac{-1}{6I_{-1}(m_R)}+\frac{C_1}{[I_{-1}(m_R)]^2}\ ,\qquad b_B=\frac{C_2}{I_{-1}(m_R)}\ , \tag{35}$$

where C_1,C_2 are finite, and $C_1>0$. [In the notation of II, $C_1=-1/(12\lambda_R)=\kappa/(48\pi^2)$.] From Eq. (34) one sees that the $\phi_0=0$ vacuum is locally stable for $G^2<\frac{1}{4}(m_R^2-3C_2^2/2C_1)$.

However, one also finds that at sufficiently large $|\phi_0|$ the GEP is governed by the $\Omega=0$ end point, which gives

$$V_G(\phi_0,\Omega=0)=\text{const}+\tfrac{3}{2}C_2m_R^2\phi_0-2G^2\phi_0^2\ . \tag{36}$$

Thus the fermionic contribution destabilizes the theory. Only if $G=0$ and $C_2=0$, can the GEP be bounded below.

Since (3 + 1)-dimensional models seem doomed to instability, we return to lower dimensions and consider a different kind of fermion-scalar interaction term.

IV. THE $g\phi^2\bar{\psi}\psi$ MODEL

In 2 + 1 dimensions the interaction term $\phi^2\bar{\psi}\psi$ is renormalizable, according to power counting. This fact motivates us to consider a model defined by the Hamiltonian density

$$\mathcal{H}=\bar{\psi}(-i\gamma\cdot\nabla+\mu_B)\psi+g_B\phi^2\bar{\psi}\psi+\tfrac{1}{2}\dot{\phi}^2+\tfrac{1}{2}(\nabla\phi)^2+\tfrac{1}{2}m_B^2\phi^2+\lambda_B\phi^4+\xi\phi^6 \tag{37}$$

which is the most general renormalizable form possessing a $\phi\to-\phi$ symmetry. We shall also consider the model in 1 + 1 dimensions, where the analysis is very similar.

Evaluating $\langle 0|\mathcal{H}|0\rangle$ one obtains

$$V_G(\phi_0;\Omega,M)=I_1+\tfrac{1}{2}(m_B^2-\Omega^2)I_0+\tfrac{1}{2}m_B^2\phi_0^2+\lambda_B\phi_0^4+\xi\phi_0^6+6\lambda_BI_0\phi_0^2+3\lambda_BI_0^2+15\xi(I_0\phi_0^4+3I_0^2\phi_0^2+I_0^3)-2[I_1'-M(M-\mu_B)I_0'+g_BMI_0'(\phi_0^2+I_0)]\ . \tag{38}$$

Minimizing V_G with respect to Ω and M leads to equations for $\bar{\Omega}$ and $\bar{M}$ which are now *coupled:*

$$\bar{\Omega}^2=m_B^2+12\lambda_B[I_0(\bar{\Omega})+\phi_0^2]+30\xi\{\phi_0^4+6\phi_0^2I_0(\bar{\Omega})+3[I_0(\bar{\Omega})]^2\}-4g_B\bar{M}I_0(\bar{M})\ , \tag{39}$$

$$\bar{M}=\mu_B+g_B[\phi_0^2+I_0(\bar{\Omega})]\ . \tag{40}$$

(Later on, we shall have to enquire whether these conditions give us the true global minimum of V_G.) Note that the $\bar{M}$ equation reduces the fermionic contribution in Eq. (38) to just $-2I_1(\bar{M})$. As before, consideration of the one-particle states shows that $\bar{\Omega}|_{\phi_0=0}$ and $\bar{M}|_{\phi_0=0}$ are the physical particle masses in the $\phi_0=0$ vacuum. Thus, we define the renormalized mass parameters by

$$m_R^2=m_B^2+12\lambda_BI_0(m_R)+90\xi[I_0(m_R)]^2-4g_B\mu_RI_0(\mu_R)\ ,\qquad \mu_R=\mu_B+g_BI_0(m_R)\ . \tag{41}$$

To discover how the coupling constants renormalize we may examine the derivatives of $\bar{V}_G$ at the origin. The second derivative is found to coincide with m_R^2 (as in the scalar case, but unlike the situation in Yukawa models). The fourth derivative is calculated to be

$$\left.\frac{1}{4!}\frac{d^4\bar{V}_G}{d\phi_0^4}\right|_{\phi_0=0}\equiv\lambda_R=\lambda_R'-\tfrac{2}{3}g_B^2[I_0(\mu_R)-\mu_R^2I_{-1}(\mu_R)]\ , \tag{42a}$$

where

$$\lambda_R'=\lambda_r'\frac{1-12\lambda_r'I_{-1}(m_R)}{1+6\lambda_r'I_{-1}(m_R)}\ , \tag{42b}$$

$$\lambda_r'=\lambda_r-\tfrac{1}{3}g_B^2[I_0(\mu_R)-\mu_R^2I_{-1}(\mu_R)]\ , \tag{42c}$$

$$\lambda_r=\lambda_B+15\xi I_0(m_R)\ . \tag{42d}$$

From this chain of relations we can see that the finiteness of λ_R implies the finiteness of λ_r. This is true irrespective of the behavior of g_B, since either $g_B^2I_0$ is finite, in which case Eqs. (42a)–(42c) are entirely finite, or else $g_B^2I_0$ is infinite, in which case λ_R' in (42a) is infinite and (42b) reduces to

$$\lambda_r'=-\tfrac{1}{2}\lambda_R'+\frac{1}{4I_{-1}(m_R)}+O\left(\frac{1}{\lambda_R'}\right)\ ,$$

so that the infinite terms cancel out in (42c).

Consideration of the large-ϕ_0 behavior of the GEP shows that, as in the scalar case, ξ remains finite. At this stage, then, it remains only to ascertain the proper renormalization of g_B. We proceed to rewrite V_G in terms of g_B and the finite parameters $m_R^2,\mu_R,\lambda_r,\xi$, using (41) and (42d). The result is

$$V_G(\phi_0;\Omega,M)-D=V_G^{(\phi)}(\phi_0,\Omega)+2\mu_R^{\nu+1}\frac{L_2(x')}{8\pi}+2M(\Delta I_0')[M-\mu_R-g_B(\phi_0^2+\Delta I_0)]+I_0(\mu_R)(M-\mu_R)[M-\mu_R-2g_B(\phi_0^2+\Delta I_0)]\ , \tag{43}$$

where

$$(\Delta I_0) = -m_R^{\nu-1}\frac{L_1(x)}{4\pi}\ , \quad x=\frac{\Omega^2}{m_R{}^2}\ ,$$
$$(\Delta I_0') = -\mu_R^{\nu-1}\frac{L_1(x')}{4\pi}, \quad x'=\frac{M^2}{\mu_R{}^2}\ . \tag{44}$$

The vacuum-energy constant D is

$$D = I_1(m_R) - 3\lambda_r[I_0(m_R)]^2 + 15\xi[I_0(m_R)]^3 - 2[I_1(\mu_R) - g_B\mu_R I_0(\mu_R) I_0(m_R)] \tag{45}$$

and

$$V_G^{(\phi)}(\phi_0,\Omega) = \tfrac{1}{2}m_R{}^2\phi_0{}^2 + \lambda_r\phi_0{}^4 + \xi\phi_0{}^6 - m_R^{\nu+1}\frac{L_2(x)}{8\pi} + \tfrac{1}{2}(m_R{}^2-\Omega^2)(\Delta I_0) + 3\lambda_r(\Delta I_0)^2 + 6\lambda_r(\Delta I_0)\phi_0{}^2 + 15\xi(\Delta I_0)[\phi_0{}^4 + 3\phi_0{}^2(\Delta I_0) + (\Delta I_0)^2]\ , \tag{46}$$

which is identical to the renormalized result in the scalar case [cf. Eq. (2.17) of III].

The right-hand side of Eq. (43) is free of divergences, except for the last term, involving $I_0(\mu_R)$. To obtain the GEP we need to minimize (43) with respect to Ω and M. [Note that in deriving (43) we did *not* make use of the $\overline{\Omega}$ and $\overline{M}$ equations.] However, if g_B is assumed to be finite it is easy to convince oneself that no minimum exists, since the coefficient of $I_0(\mu_R)$ can be made *negative,* and arbitrarily large, by suitable choices of Ω and M. This means that a theory with finite g_B is totally unstable. This conclusion can be extended to any g_B which does not vanish at least as fast as

$$g_B{}^2 = \frac{G^2}{I_0(\mu_R)}\ , \tag{47}$$

where G is finite.

One may also understand this conclusion in terms of the conditions that the $\overline{\Omega}$ and $\overline{M}$ equations correspond to a minimum of V_G, not a maximum or a saddle point. Evaluating the second derivatives of V_G from (38) one obtains

$$(V_G)_{MM} = 2(I_0' - M^2 I'_{-1})\ ,$$
$$(V_G)_{M\Omega} = 2g_B\Omega I_{-1}(I_0' - M^2 I'_{-1})\ , \tag{48}$$
$$(V_G)_{\Omega\Omega} = \Omega^2 I_{-1}\{1 + 6I_{-1}[\lambda_B + 15\xi(I_0+\phi_0{}^2)]\}\ ,$$

where

$$(V_G)_{MM} \equiv d^2V_G/dM^2\,|_{\Omega=\overline{\Omega},\ M=\overline{M}}\ ,$$

etc. Clearly $(V_G)_{MM}$ is positive, so one will have a minimum provided that

$$(V_G)_{MM}(V_G)_{\Omega\Omega} - [(V_G)_{M\Omega}]^2 > 0\ ,$$

i.e.,

$$1 + 6I_{-1}[\lambda_r + 15\xi(\Delta I_0 + \phi_0{}^2) - \tfrac{1}{3}g_B{}^2(I_0' - M^2 I'_{-1})] > 0\ , \tag{49}$$

where (42d) has been used to eliminate λ_B in favor of the finite parameter λ_r. Obviously, this condition cannot be satisfied unless g_B is renormalized as in Eq. (47).

With this matter settled, we can now simplify V_G. Eliminating μ_B in favor of μ_R, we may write the $\overline{M}$ equation as

$$\overline{M} - \mu_R = g_B(\Delta I_0 + \phi_0{}^2)\ . \tag{50}$$

Using this result, and noting that it makes $x'=M^2/\mu_R{}^2$ infinitesimally close to unity, we may simplify (43) to

$$\overline{V}_G(\phi_0) - D = V_G^{(\phi)}(\phi_0,\Omega) - G^2(\Delta I_0 + \phi_0{}^2)^2\ . \tag{51}$$

This can be rewritten as

$$\overline{V}_G(\phi_0) - D = V_G^{(\phi)}(\phi_0,\Omega;\lambda_r\to\widetilde{\lambda}_r) - \tfrac{2}{3}G^2\phi_0{}^4\ , \tag{52}$$

where

$$\widetilde{\lambda}_r \equiv \lambda_r - \tfrac{1}{3}G^2\ . \tag{53}$$

This means that *the GEP has exactly the same form as for the scalar sector alone, except for an additional term* $-\frac{2}{3}G^2\phi_0{}^4$. All the other fermionic effects are absorbed into the renormalized mass [Eq. (41)], and into a finite shift of the quartic coupling constant [Eq. (53)].

Confirmation of this result may be obtained by using Eqs. (41), (42d), (50), and (53) to reexpress the $\overline{\Omega}$ equation as

$$\overline{\Omega}{}^2 = m_R{}^2 + 12\widetilde{\lambda}_r(\Delta I_0 + \phi_0{}^2) + 30\xi[\phi_0{}^4 + 6\phi_0{}^2(\Delta I_0) + 3(\Delta I_0)^2]\ , \tag{54}$$

which coincides with the scalar-model result [III, Eq. (2.18)] with $\lambda_r\to\widetilde{\lambda}_r$. We may therefore make use of the results of III. It is convenient, then, to introduce dimensionless parameters

$$\widetilde{\alpha} = \frac{3\widetilde{\lambda}_r}{2\pi}m_R^{\nu-3},\quad \beta = \frac{45}{8\pi^2}\xi m_R^{2\nu-4},\quad \gamma = \frac{G^2}{3\pi}m_R^{\nu-3} \tag{55}$$

so that, in the style of III, Eq. (2.23)

$$\mathscr{V}_G = \mathscr{V}_G^{(\phi)}(\alpha\to\widetilde{\alpha}) - \frac{\gamma F^2}{(8\pi)}\ , \tag{56}$$

where $F \equiv 4\pi\phi_0{}^2/m_R{}^{\nu-1}$.

Afficionados of III will know that the analysis involves various complications. We have carefully checked that all aspects generalize to be consistent with the statement in italics above. This is briefly discussed in the two paragraphs below, which the general reader may omit.

The behavior at large ϕ_0 is unaffected by the extra $-\gamma F^2$ term, so that the stability considerations of III, Sec. III B are unchanged. Thus, in particular, the bound $\beta < \beta_c = 0.145$, in the (2 + 1)-dimensional case, still applies. (However, in the special cases $\beta=\beta_c$ or $\beta=0$ the criterion for stability *will* be affected by the $-\gamma F^2$ term.) The other parameter-space restrictions obtained in Sec. III C and III D of III are derived from the condition for the $\overline{\Omega}$ equation to give a proper *minimum* of V_G. In our case this condition is Eq. (49), which reduces to the form in III with $\lambda_r \to \tilde{\lambda}_r$. Consequently, the whole analysis goes through with $\tilde{\alpha}$ replacing α. The transformations mentioned in (3.10) and (3.23) should obviously be supplemented by $\gamma' = (m_R{}^2/m_R'{}^2)\gamma = \gamma e^{-z_0}$ and $\gamma' = (m_R/m_R')\gamma = \gamma/(1+z_0)$, respectively, but we have not been able to derive analogs of (3.17)–(3.19) and (3.26) in which bare and renormalized parameters are explicitly separated.

In 2 + 1 dimensions there are certain instances where the $\overline{\Omega}$ equation is inoperative, and the $\Omega = 0$ end point provides the global minimum of V_G. The discussion of this complication in III, Sec. IV B, also goes through with $\alpha \to \tilde{\alpha}$. The treatment of the $m_R{}^2 = 0$ and $m_R{}^2 = -1$ special cases also generalizes, with the latter requiring $\tilde{\alpha} = -\beta$. In both these cases the symmetry is already spontaneously broken, so that the additional $-\frac{2}{3}G^2\phi_0{}^4$ term does not produce any dramatic change.

We can now describe our numerical results. As we did for the Yukawa models earlier, we may summarize them in terms of regions of "unbroken" and "broken" symmetry (according to whether or not $\phi_0 = 0$ is the lowest minimum of the GEP) in the $\tilde{\alpha}, \beta$ parameter space. This terminology, although not strictly correct in the (1 + 1)-dimensional case, saves us from cumbersome circumlocutions. However, we ask the reader to bear in mind the discussion in Sec. III A of this paper of the possibility of intervacuum mixing by solitons in 1 + 1 dimensions. In Fig. 5 and 6 we show, for the (1 + 1)- and (2 + 1)-dimensional cases, respectively, how the unbroken-symmetry region shrinks as the fermion coupling γ is increased. Unlike the Yukawa case, the region here disappears by shrinking down to a point, which lies on the $\tilde{\alpha}$ axis. The coordinates of this critical point are $\beta = 0$ and

$$\begin{aligned} &\gamma = 0.119\,598, \quad \tilde{\alpha} = 0.5698 \quad (\nu = 1)\,, \\ &\gamma = 0.124\,442, \quad \tilde{\alpha} = 0.6791 \quad (\nu = 2)\,. \end{aligned} \tag{57}$$

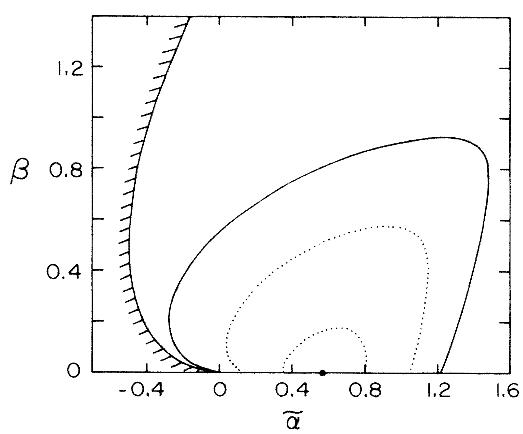

FIG. 5. The $\tilde{\alpha}, \beta$ parameter space for the (1 + 1)-dimensional version of the $g\phi^2\overline{\psi}\psi$ model [Eq. (37)]. For $\gamma = 0$ the "unbroken symmetry" region extends out to the solid curve. As γ increases, the region shrinks, as illustrated by the dotted curves corresponding to $\gamma = 0.05$ and $\gamma = 0.1$. At $\gamma = 0.1196$ the region has shrunk to a single point, and thereafter it disappears entirely.

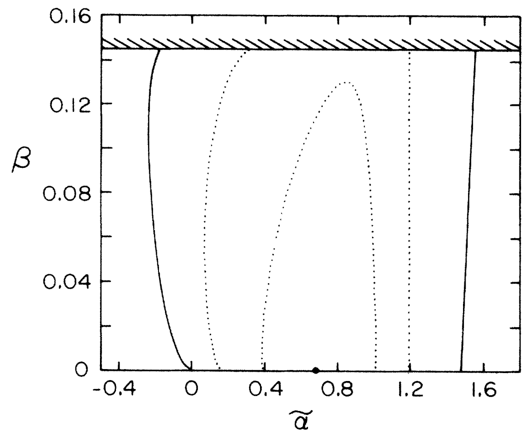

FIG. 6. Same as Fig. 5, but for the (2 + 1)-dimensional version of the $g\phi^2\overline{\psi}\psi$ model (37). The unbroken-symmetry region shrinks down to a point when γ reaches 0.1244.

There are no complications analogous to Figs. 3 and 4 because only a single pair of nonzero minima, $\phi_0 = \pm c$, ever develops.

Finally, following Refs. 1, 2, 4, and 6, we may calculate the mass of a two-boson state in the Gaussian approximation. It turns out that the $g_B\phi^2\overline{\psi}\psi$ term only produces a contribution which, together with other terms from the scalar potential, cancels by virtue of the $\overline{\Omega}$ equation: it serves, in fact, to produce the additional fermionic term present in the new $\overline{\Omega}$ equation, (39). Thus, the equation for the bound-state mass M_2 remains, as in III (see also Ref. 6),

$$1 + 6\lambda_{\text{eff}} \int (dp) \frac{1}{\omega_p(2\omega_p - M_2)} = 0\,, \tag{58}$$

where

$$\lambda_{\text{eff}}(\phi_0) = \lambda_B + 15\xi[I_0(\Omega) + \phi_0{}^2]\,, \tag{59}$$

which reduces to λ_r (not $\tilde{\lambda}_r$, note) at $\phi_0 = 0$. Thus when λ_r is negative, there can be bound states in the $\phi_0 = 0$ vacuum. However, one may show, by a straightforward generalization of the argument in III, that λ_{eff} is positive in any $\phi_0 \neq 0$ vacuum. Hence, the boson-boson interaction is repulsive, and bound states do not form in those vacua, according to the GEP approach. Fermion-antifermion and fermion-boson states can be investigated in the same fashion, but these show no nontrivial effects. Presumably, such interactions are only seen when one goes beyond the Gaussian approximation.[15]

V. COMPARISON WITH 1/N ANALYSES

Our results for the $g\phi^2\overline{\psi}\psi$ model are markedly different from those of other recent studies of similar models.[8–10] A direct comparison is not easy since these works study the $O(N)$-symmetric version of the model in the limit of large N. Nevertheless, it is evident that the crucial difference lies in the renormalization, since in these references g_B remains finite.

In our analysis, an $I_0(\mu_R)$ term—linearly divergent in

$2+1$ dimensions—remains after mass and λ renormalizations [see Eq. (43)]. It can only be tamed by taking $g_B{}^2=O(1/I_0)$. The crucial difference in Ref. 10 is that at the corresponding stage, the coefficient of the $I_0(\mu_R)$ term is just the square of the $\overline{M}$ equation. Thus once V_G is minimized with respect to M, the divergence disappears; everything is fine, and g_B can remain finite.

Now, λ renormalization in Ref. 10 differs from ours by including a $g_B{}^2 I_0$ term. Recall that we found earlier [Eq. (42) and the following ones] that, in order for the fourth derivative of V_G to be finite, we need $\lambda_r \equiv \lambda_B + 15\xi I_0(m_R)$ to be finite. If instead we try to follow Ref. 10 we would want to suppose that

$$\lambda_n \equiv \lambda_r - g_B{}^2 I_0(\mu_R) \tag{60}$$

is finite. Equation (43) would then become

$$\begin{aligned} V_G - D = V_G^{(\phi)}(\lambda_r \to \lambda_n) &+ 2\mu_R^{\nu+1}\frac{L_2(x')}{8\pi} \\ &+ 2M(\Delta I_0')[M-\mu_R - g_B(\phi_0{}^2+\Delta I_0)] \\ &+ I_0(\mu_R)[M-\mu_R-g_B(\phi_0{}^2+\Delta I_0)]^2 \\ &+ I_0(\mu_R) g_B{}^2 2\Delta I_0(\Delta I_0+2\phi_0{}^2)\ . \end{aligned} \tag{61}$$

Were it not for the presence of the last term, we would now be in the same situation as Ref. 10, and could take λ_n and g_B as finite. However, the last term, after minimization with respect to Ω (which gives $\Delta I_0 = -\phi_0{}^2$, corresponding to $\Omega = m_R + 4\pi\phi_0{}^2$), becomes $-2g_B{}^2 I_0(\mu_R)\phi_0{}^4$. Not only is this an uncanceled divergence, it also renders the potential unbounded below. One can experiment with variants of Eq. (60) without affecting the basic conclusion: finite g_B leads to a disastrous instability.

Although an analysis of the O(N)-symmetric generalization of the $\phi^2\bar{\psi}\psi$ model is beyond the scope of this paper, we have investigated this question. We find that in the generalization of Eq. (61) *the last term is suppressed by one power of* $1/N$ *relative to the other terms.* Thus, if we take the formal $N\to\infty$ limit, this term drops out and we recover the results of Ref. 10. However, at any finite N the instability remains—and can only be tamed by taking $g_B{}^2=O(1/I_0)$.

Thus it seems to us that the $1/N$ expansion is misleading here: it does not see the troublesome divergent term because it improperly takes the $N\to\infty$ limit before taking the cutoff $\to\infty$ limit. An alternative viewpoint is that the fault lies with our Gaussian approximation. However, since the GEP, as a variational approximation, is an upper bound on the true effective potential, we see no escape from the conclusion that the instability problem is a real effect.

The same problem leads to a puzzle concerning supersymmetry. If we convert from Dirac to Majorana fermions[11] and set

$$m_B=\mu_B,\quad 3\lambda_B = g_B\mu_B,\quad 18\xi = g_B{}^2\ , \tag{62}$$

the theory becomes supersymmetric. [See, eg., Ref. 8, Eq. (2.6) with $N=1$.]

In the $1/N$ expansion[8–10] everything works as one would expect: both ξ and g_B remain finite; the renormalized masses come out equal; and the vacuum-energy constant vanishes. For us, none of these desirable things happen. Imposing the relations (62) appears to be incompatible with the mass renormalizations, for instance: one cannot make m_R, μ_R both finite, let alone equal, unless $g_B{}^2=O(1/I_0{}^2)$, which is even smaller than usual.

Curiously, though, if we impose a modified version of (62) with $30\xi=g_B{}^2$, we find equal renormalized masses and a vanishing vacuum-energy constant. However, as usual, there is a negative $g_B{}^2 I_0(\mu_R)\phi_0{}^4$ term which ruins everything. Even if we now renormalize g_B as $g_B{}^2=G^2/I_0$, the (quasi)supersymmetric relations will make ξ and λ_r both vanish, leaving $V_G = \frac{1}{2}m_R{}^2\phi_0{}^2 - \frac{1}{3}G^2\phi_0{}^4$, which is unbounded below.

We do not know what to make of these results. It is not out of the question that the supersymmetry is anomalous (i.e., illusory) in these models, but that seems a very drastic conclusion to draw. It is slightly surprising that the Gaussian ansatz, although it seems to treat the bosons and fermions evenhandedly, does not preserve supersymmetry. Perhaps it is possible to formulate the Gaussian ansatz at the superfield level, thereby respecting the supersymmetry at all stages.

VI. CONCLUSIONS

We have found that the addition of fermions to a scalar theory tends to destabilize it. To avoid the theory becoming completely unstable, the bare coupling constant must be infinitesimal of the form

$$g_B{}^2 = G^2/I_0\ , \tag{63}$$

with G^2 finite. This conclusion applies to both $g_B\phi\bar{\psi}\psi$ and $g_B\phi^2\bar{\psi}\psi$ interaction terms.

We should stress the strangeness of this result: it does not resemble any renormalization we have encountered before. For instance, there is no sense in which $g_B = G[1+O(G)]$. Indeed, g_B and G have different mass dimensions, in general, and this upsets expectations based upon power counting.

Although we are somewhat uncomfortable with Eq. (63), we have been at pains to show that, in the context of the GEP approach, there is no possible alternative: anything "bigger" (vanishing less quickly as the cutoff tends to infinity) would make the theory unstable, while anything "smaller" effectively decouples the fermions altogether. Furthermore, if the GEP—a variational approximation—indicates that a theory is unstable, it seems inescapable that the instability is present in the exact theory. Thus $g_B{}^2$ must be at least as "small" as in Eq. (63). (These conclusions apply in the context of an ultraviolet cutoff, or a lattice regularization. The situation in dimensional regularization is somewhat different, and is discussed briefly in the Appendix.) This point is at the heart of our differences with the $1/N$-expansion results,[8–10] and causes difficulties with supersymmetry, as we discussed in the last section. The implications are disturbing, but we can see no escape from our conclusions.

Once the renormalization (63) is accepted, the net effect of the fermions on the GEP is easily summarized: for a Yukawa interaction the fermions produce an extra $-(\sum_\lambda)G^2\phi_0{}^2$ term, while for a $\phi^2\bar{\psi}\psi$ interaction they produce an extra $-\frac{2}{3}G^2\phi_0{}^4$ term, as well as inducing a finite change in the renormalized quartic coupling constant. There does not seem to be any explicit dependence upon the fermion mass, μ_R. We also observe that the results ultimately involve only the *square* of g_B, so that the sign of the interaction term in the Lagrangian is immaterial (just as with fermion mass terms).

Although the fermions essentially add only a single term to the GEP, this term has interesting effects upon the physics. Most strikingly, in the $(3+1)$-dimensional ϕ^4 Yukawa model it destabilizes the theory entirely. In the lower-dimensional models it, roughly speaking, encourages spontaneous symmetry breakdown. This effect is illustrated in the figures. We have warned that in the delicate transitional regions one cannot expect the GEP to give more than a qualitative description, and also that in $1+1$ dimensions the interpretation must take into account the possibility of intervacuum mixing by solitons. That is, our results should not be viewed as a definitive description of the behavior of these models, but rather as a general overview which can serve as a guide to further exploration. Furthermore, the problems raised in Sec. V deserve further study.

Note added. Since this paper was written it has been discovered [P. M. Stevenson and R. Tarrach, Phys. Lett. **176B**, 436 (1986)] that $(\lambda\phi^4)_{3+1}$ can be renormalized in a completely different way from that of Refs. 3–5. The resulting form of the theory is not "precarious" (it has positive λ_B) and it seems likely that it can be consistently coupled to fermions. This question is currently under investigation.

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APPENDIX: DIMENSIONAL REGULARIZATION

In the text we have treated the divergent integrals as though they were implicitly regulated by an ultraviolet cutoff. One would normally expect to obtain the same results when dimensional regularization is used instead, but such is not the case here. The point is that the nature of the results depends critically upon the *sign* of the divergent integrals.

In all the models studied, the condition for the $\bar{M}$ equation to give a minimum, not a maximum of V_G is

$$\left.\frac{d^2\bar{V}_G}{dM^2}\right|_{M=\bar{M}}=2\left[\sum_\lambda\right][I_0(M)-M^2I_{-1}(M)]>0\ . \tag{A1}$$

This is obviously satisfied with an ultraviolet cutoff, but in dimensional regularization it becomes problematic. In ν-ϵ spatial dimensions one has

$$\begin{aligned} I_0(M)&=1/(2\pi\epsilon)+O(1),\quad \nu=1\\ &=-M/(4\pi)+O(\epsilon),\quad \nu=2\\ &=-M^2/(8\pi\epsilon)+O(1),\quad \nu=3\ , \end{aligned} \tag{A2}$$

while $I_{-1}(M)$ is convergent for $\nu=1,2$, and is $1/(4\pi^2\epsilon)+O(1)$ for $\nu=3$.

The results can depend upon whether we approach the physical dimension from above or below. For example, in $1+1$ dimensions we would recover our previous results in the limit $\epsilon\to 0+$, but we could get nothing sensible out of $\epsilon\to 0-$. In $3+1$ dimensions the situation is the other way around. Dimensional regularization endows I_0 with a *finite* and negative value in $2+1$ dimensions, and there is no way to recover, our previous results. Note that if (A1) is not satisfied, then the energy is minimized by $M\to\infty$, so that the fermion degrees of freedom "freeze-out" and effectively decouple from the theory altogether.

[1]The ideas date back to L. I. Schiff, Phys. Rev. **130**, 458 (1963). References to other closely related approaches can be found in the papers of Ref. 4.

[2]T. Barnes and G. I. Ghandour, Phys. Rev. D **22**, 924 (1980).

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[8]W. A. Bardeen, K. Higashijima, and M. Moshe, Nucl. Phys. **B250**, 437 (1985).

[9]R. Gudmundsdottir and G. Rydnell, Nucl. Phys. **B254**, 593 (1985).

[10]T. Suzuki, Phys. Rev. D **32**, 1017 (1985).

[11]Throughout the paper we consider Dirac fermions. However, it is trivial to modify the results in the Majorana case: (i) in Eq. (1) the $d^\dagger$ operator is replaced by $b^\dagger$, so that the field is self-conjugate; (ii) the free Hamiltonian and the Yukawa term should be multiplied by $\frac{1}{2}$. Hence, the factor of $2(\sum_\lambda)$ multiplying the fermionic terms in the GEP [e.g., Eq. (9)] becomes a factor of $(\sum_\lambda)$ only. (These factors correspond to the number of degrees of freedom of a Dirac fermion and a Majorana fermion, respectively.) The conversion factor of $\frac{1}{2}$ feeds through systematically so as to multiply all terms of fermionic origin in the subsequent equations.

[12]S. J. Chang, Phys. Rev. D **13**, 2778 (1976).

[13]For reviews, see S. Coleman, in *The Whys of Subnuclear Physics*, proceedings of the International School of Subnuclear Physics, Erice, 1977, edited by A. Zichichi (Plenum, New York, 1979); R. Rajaraman, Phys. Rep. **21C**, 227 (1975).

[14]A possible approach is to use the methods of Ref. 13 starting with the GEP, pretending that it is a classical potential. Of course, this will lead to a double counting of the zero-point-fluctuation effects, which are already included in the GEP, but one may drop such terms, retaining only the instanton/soliton terms, which are distinguished by being exponentially small in $\hbar$.

[15]A possible way to calculate corrections to the Gaussian approximation is discussed in I.