Debye scalar potentials for the electromagnetic fields

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The electromagnetic fields away from sources are known to be expressible as linear combinations of $\mathbf{L}\partial_0$ and $\mathbf{M}$ operators on two Debye scalar potentials. It is shown that these $\mathbf{L},\mathbf{M}$ operators (or more precisely, a suitably rescaled version thereof) satisfy the O(3,1) dynamic symmetry. For the static case, an explicit example of a singular Debye potential is given to accommodate the magnetic monopoles.

It is well known that the electromagnetic fields may be expressed in terms of a pair of electric and magnetic Hertz vector potentials.[1–6] In the source-free region, the $\mathbf{E},\mathbf{B}$ fields may be expressed as a linear combination of vector operators $\mathbf{L}\partial_0$ and $\mathbf{M}$ (defined below) operating on a pair of scalar Debye potentials.[6,7] (The Debye potentials[2,5,6] are the purely radial Hertz potentials.)

The purpose of this paper is to make the following two observations.

(a) The $\mathbf{L},\mathbf{M}$ operators[6,7] ($\mathbf{L}=-\mathbf{r}\times\nabla$, $\mathbf{M}=\nabla\times\mathbf{L}$) [or more precisely, a suitably rescaled $\mathbf{M}$ operator (see below)] satisfy the usual O(3,1) algebra. Loosely speaking, the $\mathbf{M}$ operator resembles the role of the Runge-Lenz vector in the familiar Kepler problem in classical mechanics and the hydrogen atom problem in quantum mechanics.[8] [There, it is the O(4) algebra.] This O(3,1) result may be interpreted as the most transparent and effortless manifestation of Lorentz covariance of electrodynamics formulated purely from the three-vector notations.

(b) Inasmuch as the $\mathbf{E},\mathbf{B}$ fields formulated in the usual Debye potential language are source-free and the multipole expansion is usually *without* the monopole term,[5] can singular and/or non-single-valued potentials produce monopole sources? The answer is affirmative. We give here an explicit example of a singular magnetic Debye potential which corresponds to the familiar vector potential $\mathbf{A}$ for a Dirac monopole.[9]

(1) For the sake of readability, we briefly summarize the formalism of the Debye scalar potentials.[4–6] Let

$$\mathbf{L}=-\mathbf{r}\times\nabla\ , \qquad (1)$$

$$\mathbf{M}=\nabla\times\mathbf{L}\ . \qquad (2)$$

Then the $\mathbf{E},\mathbf{B}$ fields expressed as the $\mathbf{L}\partial_0,\mathbf{M}$ vector operators on ψ_E,ψ_M ($c=1$, $\partial_0=\partial/\partial t$) given by

$$\begin{bmatrix}\mathbf{E}\\ \mathbf{B}\end{bmatrix}=\begin{bmatrix}\mathbf{M} & \mathbf{L}\partial_0\\ \mathbf{L}\partial_0 & -\mathbf{M}\end{bmatrix}\begin{bmatrix}\psi_E\\ \psi_M\end{bmatrix} \qquad (3)$$

satisfy the Maxwell equations away from the source

$$\begin{bmatrix}\nabla\times & \partial_0\\ -\partial_0 & \nabla\times\end{bmatrix}\begin{bmatrix}\mathbf{E}\\ \mathbf{B}\end{bmatrix}=0\ , \qquad (4a)$$

$$\begin{bmatrix}\nabla\cdot & 0\\ 0 & \nabla\cdot\end{bmatrix}\begin{bmatrix}\mathbf{E}\\ \mathbf{B}\end{bmatrix}=0 \qquad (4b)$$

provided that ψ_E,ψ_M satisfy the wave equations

$$(\partial_0{}^2-\nabla^2)\begin{bmatrix}\psi_E\\ \psi_M\end{bmatrix}=0\ . \qquad (5)$$

We note in passing that the usual scalar potential ϕ and the vector potential $\mathbf{A}$ can be expressed in terms of ψ_E,ψ_M as

$$\begin{bmatrix}\phi\\ \mathbf{A}\end{bmatrix}=\begin{bmatrix}-(1+\mathbf{r}\cdot\nabla) & 0\\ \mathbf{r}\partial_0 & -\mathbf{L}\end{bmatrix}\begin{bmatrix}\psi_E\\ \psi_M\end{bmatrix} \qquad (6a)$$

with the usual relations to the fields

$$\begin{bmatrix}\mathbf{E}\\ \mathbf{B}\end{bmatrix}=\begin{bmatrix}-\nabla & -\partial_0\\ 0 & \nabla\times\end{bmatrix}\begin{bmatrix}\phi\\ \mathbf{A}\end{bmatrix}\ . \qquad (6b)$$

The duality is manifest here in the following sense:

$$\begin{bmatrix}\mathbf{E}\\ \mathbf{B}\end{bmatrix}\rightarrow\begin{bmatrix}\mathbf{B}\\ -\mathbf{E}\end{bmatrix}$$

is equivalent to

$$\begin{bmatrix}\psi_E\\ \psi_M\end{bmatrix}\rightarrow\begin{bmatrix}-\psi_M\\ \psi_E\end{bmatrix}\ ,$$

and

$$\begin{bmatrix}\phi\\ \mathbf{A}\end{bmatrix}\rightarrow\begin{bmatrix}\tilde{\phi}\\ \tilde{\mathbf{A}}\end{bmatrix}$$

with

$$\begin{bmatrix}\mathbf{B}\\ -\mathbf{E}\end{bmatrix}=\begin{bmatrix}-\nabla & -\partial_0\\ 0 & \nabla\times\end{bmatrix}\begin{bmatrix}\tilde{\phi}\\ \tilde{\mathbf{A}}\end{bmatrix}\ , \qquad (7a)$$

where

$$\begin{bmatrix}\tilde{\phi}\\ \tilde{\mathbf{A}}\end{bmatrix}=\begin{bmatrix}-(1+\mathbf{r}\cdot\nabla) & 0\\ \mathbf{r}\partial_0 & -\mathbf{L}\end{bmatrix}\begin{bmatrix}-\psi_M\\ \psi_E\end{bmatrix}\ . \qquad (7b)$$

(2) The O(3,1) algebra. The vector operators $\mathbf{L},\mathbf{M}$, as defined in Eqs. (1) and (2), satisfy the following commutation relations:

$$[L_i,L_j]=\epsilon_{ijk}L_k\ , \qquad (8a)$$

$$[L_i,M_j]=\epsilon_{ijk}M_k\ , \qquad (8b)$$

$$[M_i,M_j]=-\epsilon_{ijk}L_k\nabla^2 \ . \tag{8c}$$

Since ∇^2 commutes with both $\mathbf{L}$ and $\mathbf{M}$ (Ref. 6) and in fact

$$\mathbf{M}^2=\mathbf{L}^2\nabla^2+2\nabla^2 \ , \tag{9}$$

we can formally rescale the $\mathbf{M}$ operator. Define

$$\hat{\mathbf{L}}=i\mathbf{L}=\mathbf{r}\times\mathbf{p} \ , \tag{10a}$$

$$\hat{\mathbf{M}}=\frac{1}{\omega}\mathbf{M} \ , \tag{10b}$$

where

$$\omega=(-\nabla^2)^{1/2} \ . \tag{11}$$

Then we have

$$[\hat{\mathbf{L}}_i,\hat{\mathbf{L}}_j]=i\epsilon_{ijk}\hat{L}_k \ , \tag{12a}$$

$$[\hat{L}_i,\hat{M}_j]=i\epsilon_{ijk}\hat{M}_k \ , \tag{12b}$$

$$[\hat{M}_i,\hat{M}_j]=-i\epsilon_{ijk}\hat{L}_k \ . \tag{12c}$$

Equations (12) are seen to be a basis of the familiar O(3,1) Lie algebra.[10]

(3) The static monopole solution. From Eq. (3), we have, for the static case,

$$\mathbf{B}=-\mathbf{M}\psi_M \ . \tag{13}$$

Equivalently, the vector potential $\mathbf{A}$ is given by Eq. (6a) as

$$\mathbf{A}=-\mathbf{L}\psi_M \ . \tag{14}$$

For the Dirac monopole, the vector potential has a purely azimuthal component

$$A_\phi=\frac{g(1-\cos\theta)}{r\sin\theta} \ . \tag{15}$$

Since

$$L_\phi=-\frac{\partial}{\partial\theta} \ , \tag{16}$$

we see that (14) admits a solution of the form

$$\psi_M=-\frac{2g}{r}\ln\cos\frac{\theta}{2} \ . \tag{17}$$

It is interesting to note that the Dirac string feature associated with the vector potential (15) is herewith transferred to a singular Debye scalar potential. The potential ψ_M of (17) when substituted into (13) yields directly the monopole field $\mathbf{B}=g\hat{\mathbf{r}}/r^2$ by noting that

$$\mathbf{M}=-\mathbf{r}\nabla^2+\nabla(\mathbf{r}\cdot\nabla)+\nabla \ . \tag{18}$$

While the $\nabla(\mathbf{r}\cdot\nabla)+\nabla$ combination would annihilate the $1/r$ potential, the polar angle part of the Laplacian yields the desired $1/r^2$ field and the angular dependence miraculously disappears.

We note in passing that the duality discussed in Eq. (7) above suggests that an electric Debye scalar potential of the form

$$\psi_E=\frac{2e}{r}\ln\cos\frac{\theta}{2} \tag{19}$$

would yield the electrostatic Coulomb field from Eq. (3):

$$\mathbf{E}=\mathbf{M}\psi_E=\frac{e}{r^2}\hat{\mathbf{r}} \ . \tag{20}$$

On the other hand, the conventional scalar potential ϕ of Eq. (6a) seems ill defined for this case. Despite the singularity in the Debye potentials (17) or (19), the $\mathbf{E},\mathbf{B}$ fields themselves are well defined except at the charge.

We conclude that *singular* Debye potentials can provide an interesting escape clause from the usual predicament of the lack of accommodation for the monopole sources.

Note added

(a) The alternative to dealing with Dirac stringlike singularities is to introduce the coordinate-patch language of the fiber bundles.[11] There one can find two copies of the potentials (one in each sector) which are related by a gauge transformation in the overlap region.

(b) One of the advantages of invoking the Debye scalar potentials is that the electromagnetic fields involve only *two* degrees of freedom (ψ_E,ψ_M) (Ref. 6). The duality discussed above reflects this simplicity. The vector potential $\mathbf{A}$ and its dual $\tilde{\mathbf{A}}$ are related to the *same* set of Debye potentials.

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