Hamiltonian dynamics of gauge systems

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(Received 7 April 1986)

Gauge systems are approached by deemphasizing the role of the gauge group and replacing it by a subgroup of point transformations on the phase space $\mathscr{S}=T^*\mathscr{M}$ which is a cotangent bundle over a "big" configuration manifold $\mathscr{M}$. These transformations are generated by a proper subalgebra $\mathscr{V}$ of the Poisson algebra of dynamical variables linear in the momenta. The orbits of $\mathscr{V}$ which lie on the constraint surface $\mathscr{C}$, on which all v from $\mathscr{V}$ vanish, form the physical phase space $\mathfrak{s}$. Observables are identified with dynamical variables on $\mathscr{S}$ which are constant along the orbits in $\mathscr{C}$, and physical variables are identified with equivalence classes of observables. Special observables (including the Hamiltonian of the system) are at most quadratic in the momenta. The kinetic part of the Hamiltonian endows $\mathscr{M}$ with a metric which, together with the gauge algebra $\mathscr{V}$, leads to a unique splitting of all special observables into standard physical parts and gauge parts. The splitting also leads to observables which represent conjugate canonical variables in the physical space. The Poisson brackets of all special observables can be explicitly evaluated, and the gauge theory can be explicitly reduced to a physical theory. The canonical formalism is manifestly covariant under point transformations in $\mathscr{S}$ and in $\mathfrak{s}$, and under changes of the basis in $\mathscr{V}$. It enables us to construct a covariant factor ordering leading to a consistent canonical quantization of gauge systems.

I. INTRODUCTION

A. Motivation

In physics, one often deals with systems whose action is invariant under an infinitely dimensional group. It is common, though unfortunate, to label all such systems as gauge theories. Once their dynamical aspects are made explicit through the canonical formalism, important distinctions start to emerge. The invariance leads to constraints which restrict the canonical variables and, on the constraint surface, string them into orbits. Sometimes, the points on an orbit can be considered as equivalent descriptions of the same physical state, sometimes as different stages in the dynamical evolution of the system. Only in the first case it is physically appropriate to talk about gauge. The second case is associated with the arbitrariness in the choice of time, and its inclusion among canonical variables. This process is often called parametrization.

In all theories of physical interest, the constraints associated with gauge are linear homogeneous functions of the canonical momenta, while those associated with parametrization are quadratic functions (though not necessarily quadratic forms) of the momenta. This distinction is preserved by point transformations, but not by arbitrary canonical transformations. This underscores the importance of the cotangent bundle structure of the "big" phase space for drawing the line between gauge and parametrization.

Examples of finite-dimensional gauge systems will be discussed in the last section of this paper. The best known example of a finite-dimensional parametrized system is a single relativistic particle moving in a given spacetime. The four-momentum of such a particle is restricted, by a quadratic constraint, to lie on the mass shell. Among fields systems, electrodynamics and Yang-Mills theories are typical examples of gauge theories in the proper sense of the word. The constraints in these theories are linear and homogeneous in the canonical momenta.[1,2] The prototypes of parametrized theories are general relativity and the nonlinear σ model. The constraints associated with the arbitrariness in the choice of time (super-Hamiltonian constraints) in those theories are quadratic in the canonical momenta. Besides these constraints, there are also linear homogeneous constraints (supermomenta constraints) associated with invariance under the spatial diffeomorphisms which play the role of a gauge group.[3,4]

The last two theories illustrate an important feature of the canonical decomposition, namely, a possible breakdown of the group structure. In proper gauge theories, the Lie algebra of the underlying group is represented by the Poisson algebra of the linear homogeneous constraints. In general relativity, and for the nonlinear σ model, this is true only for supermomenta constraints (which are associated with gauge) but not for the super-Hamiltonians (which are associated with parametrization). The Poisson brackets between super-Hamiltonians lead to structure functions rather than structure constants, and the total set of Poisson brackets among all constraints does not represent any Lie algebra.[5] To represent the original invariance of the action under spacetime diffeomorphisms requires a further extension of the phase space by the embedding variables.[6] Indeed, one can trace the origin of the structure functions to the introduction of an anholonomic basis in the space of embeddings.[7] Unfortunately, to recover the physical theory, one must still impose in the end the original constraints which do not represent a Lie algebra.

The breakdown of the Lie algebra and the appearance of structure functions presents a serious obstacle to canonical quantization of parametrized theories. It is difficult to factor order the constraints in such a way that their commutators remain consistent[8] and that the quantization of the parametrized theory is equivalent to the quantization of the corresponding physical theory. The classical formalism introduced in this paper (referred to as I) and the canonical quantization scheme proposed in its sequel[8] (referred to as II) aim at a single goal: to resolve such problems in a simpler context of gauge theories and initiate a program on how to face them in parametrized theories.

Viewed superficially, gauge theories do not seem to present any difficulties, because the Lie algebra of the gauge group is canonically implemented. However, the Hamiltonian of the theory (which replaces the super-Hamiltonian of a parametrized theory) in general preserves the constraints only via some structure functions. The problem thus reappears at this level. Moreover, the group theoretical origin of the gauge is quite irrelevant in the classical canonical formalism. One can introduce an anholonomic basis in the "big" configuration space and thereby mix the constraints by a linear transformation with coefficients which depend on canonical coordinates. The new constraints close with structure functions instead of structure constants, and the new version of the classical gauge theory is as difficult to quantize as are parametrized theories.

Field systems have an additional factor ordering problem, namely, that associated with an infinite number of degrees of freedom: not all, if any, choices of factor ordering lead to well-defined operators on some Hilbert space. To decouple the problem of renormalization from our original consistency problem of quantized constraints and observables under the commutator algebra, we shall confine the present discussion to finite-dimensional gauge systems. In this paper we build a covariant Hamiltonian formalism of such systems. In the following paper we develop a covariant canonical quantization scheme resolving the factor ordering problem.

B. A preview of results

From our point of view, gauge systems always live on a cotangent bundle to a big N-dimensional configuration manifold $\mathscr{M}$. Coordinate transformations on $\mathscr{M}$ induce point transformations on $T^*\mathscr{M}$. These can be considered as canonical transformations generated by dynamical variables $\mathscr{L}$ which are linear and homogeneous in the canonical momenta. All such variables form a subalgebra of the Poisson algebra of dynamical variables. We take the standpoint that gauge is nothing but a proper subalgebra $\mathscr{V}$ of $\mathscr{L}$. Gauge transformations are those point transformations which are generated by the elements $v \in \mathscr{V}$. The role which the "gauge group" plays in traditional formulations of gauge theories is deemphasized and, in fact, may be profitably forgotten. One can choose an arbitrary basis $\pi_\alpha = \phi_\alpha^A(Q)P_A$, $\alpha = 1, \ldots, C \leq N$ of $\mathscr{V}$. That basis does not need to represent the generators of a Lie group: the Poisson brackets of π_α close, but not necessarily with structure constants.[9] The basis π_α can be subject to an arbitrary regular linear transformation whose coefficients are functions of configuration variables. Gauge transformations generate C-dimensional orbits only if they lie within an $(2N-C)$-dimensional constraint surface $\mathscr{C}$ in $T^*\mathscr{M}$ on which all elements v of $\mathscr{V}$ vanish. These orbits can be considered as points s of a $(2n = 2N - 2C)$-dimensional physical phase space $\mathscr{s}$.

A dynamical variable on $\mathscr{S}$ which is constant along the orbits on $\mathscr{C}$ is called an observable. Two observables F_1 and F_2 which coincide on the constraint surface, $F_1 \approx F_2$, define a physical variable f on $\mathscr{s}$. A physical variable is thus an equivalence class (F) of observables. Poisson brackets between observables respect such equivalence classes and thus define Poisson brackets between physical variables. The general features of this reduction process are, of course, well known and are discussed both in the physical[10] and mathematical[11] literature.

The cotangent bundle structure of $\mathscr{S}$ allows us to classify observables according to the powers of the momenta. All physically important observables are at most quadratic in the momenta. We shall call them special observables. In particular, the Hamiltonian H is of this form. The purely quadratic part of the Hamiltonian endows the configuration manifold $\mathscr{M}$ with a contravariant (possibly degenerate) metric G^{AB}.

The physical coordinates q^a are any $n = N - C$ independent functions $q^a(Q)$ on $\mathscr{M}$ which are constant along the orbits. They may be subject to an arbitrary regular transformation. Their derivatives $Q_A^a \equiv \partial q^a / \partial Q^A$ define projectors into the physical space. In particular, the projection of the metric G^{AB} yields the physical metric g^{ab} which is assumed to be nondegenerate and thus has an inverse g_{ab}. The metrics G^{AB} and g_{ab} allow us to turn the n covectors Q_A^a into n vectors Q_a^A which complement the C vector coefficients ϕ_α^A of π_α into a vector basis in $T\mathscr{M}$. The dual to the basis is a cobasis in $T^*\mathscr{M}$. The projection $p_a \equiv Q_a^A P_A$ of the momentum defines the equivalence class of linear observables (p_a) which is the canonical momentum conjugate to (q^a).

The basis and cobasis elements enable us to split an arbitrary special observable unambiguously into a standard physical part and a gauge part. They also enable us to evaluate explicitly the Poisson brackets between any two special observables. We are then able to show in full detail how the Hamiltonian dynamics in the big phase space reduces to the physical dynamics in $\mathscr{s}$.

While this procedure is just a particular application of the general reduction scheme, the explicit control of the structure of special dynamical variables (with their standard physical and gauge parts expressed in terms of the standard representatives q^a and p_a of the physical canonical variables) allows an equally explicit control of factor ordering, a task which cannot be easily accomplished within the general reduction scheme. This gives us the tool which we need in II for resolving the factor ordering problem.

To summarize, the proposed treatment of gauge systems is built around two basic structures: the gauge algebra $\mathscr{V}$ and the Hamiltonian observable H which provides us with the metric G^{AB}. *Alles anderes ist Menschenwerk.* The classical scheme is covariant under three classes of transformations: (i) point transformations in the big phase space $\mathcal{S}$, (ii) point transformations in the physical phase space $\mathscr{s}$, (iii) change of the basis in $\mathscr{V}$. The main goal of II is to develop a quantum gauge theory with the same features.

II. PHYSICAL SYSTEMS

Let us consider a dynamical system whose instantaneous state is uniquely described by a point q in a configuration space $\mathscr{m}$, $q \in \mathscr{m}$. We assume that, for a system with n degrees of freedom, $\mathscr{m}$ is an n-dimensional manifold, and we introduce the local coordinates $q^a(q)$, $a = 1,2,\ldots,n$. Such coordinates may be subject to an arbitrary regular transformation:

$$q^a \to q^{a'} = q^{a'}(q^b) . \tag{2.1}$$

The Hamiltonian dynamics of the system takes place in the phase space $\mathscr{s} = T^*\mathscr{m}$. A point $s \in \mathscr{s}$ can be characterized by a set of $2n$ canonical coordinates:

$$s^r(s) = (q^a(s), p_a(s)), \quad r = 1,2,\ldots,2n . \tag{2.2}$$

The coordinate transformation (2.1) induces a point transformation of the canonical coordinates (2.2),

$$\begin{aligned} &q^a \to q^{a'} = q^{a'}(q^b) , \\ &p_a \to p_{a'} = Q^b_{a'}(q^c) p_b , \\ &Q^b_{a'} \equiv \partial q^b / \partial q^{a'} , \end{aligned} \tag{2.3}$$

which is linear in the canonical momenta. The emphasis placed on the cotangent bundle structure of the phase space, which enforces the requirement of covariance of the resulting classical and quantum formalisms under point transformations is inherent in our approach to gauge systems.

Information about the state of the system may be obtained by observing assorted dynamical variables. A dynamical variable is a C^∞ function on the phase space, $f(s) \in C^\infty(\mathscr{s}, \mathbb{R})$; in the canonical chart it becomes a function $f(q,p)$ of the canonical coordinates and momenta. Dynamical variables form a Lie algebra (namely, the Poisson algebra) under the Poisson-bracket product:

$$\{e,f\} = \frac{\partial e}{\partial q^a}\frac{\partial f}{\partial p_a} - \frac{\partial e}{\partial p_a}\frac{\partial f}{\partial q^a} . \tag{2.4}$$

The Poisson bracket (2.4) can be considered as the action of a vector field

$$\mathbf{X}_f = \frac{\partial f}{\partial p_a}\frac{\partial}{\partial q^a} - \frac{\partial f}{\partial q^a}\frac{\partial}{\partial p_a} \tag{2.5}$$

on the scalar field e in $\mathscr{s}$:

$$\mathbf{X}_f e = \{e,f\} . \tag{2.6}$$

The vector field $\mathbf{X}_f$ is called the Hamiltonian vector field generated by f. Because of the Jacobi identity, the mapping (2.5) is an antihomomorphism from the Poisson algebra to the Lie-brackets algebra of vector fields on $\mathscr{s}$:

$$[\mathbf{X}_e, \mathbf{X}_f] = -\mathbf{X}_{\{e,f\}} . \tag{2.7}$$

Besides Poisson-bracket multiplication, the dynamical variables are also subject to ordinary multiplication ef. It holds that

$$\mathbf{X}_{ef} = e\mathbf{X}_f + f\mathbf{X}_e . \tag{2.8}$$

The integral curves $s(\tau)$ of $\mathbf{X}_f$ are obtained by solving the differential equation

$$\frac{ds^r(\tau)}{d\tau} = \mathbf{X}_f{}^r(s(\tau)) . \tag{2.9}$$

Because of the Jacobi identity, the Poisson bracket between any two canonical variables s^r is preserved by Eq. (2.9):

$$\{s^{r_1}(\tau), s^{r_2}(\tau)\} = \{s^{r_1}(0), s^{r_2}(0)\} .$$

Hence, $s^r(0) \to s^r(\tau)$ is a canonical transformation.

The cotangent bundle structure of the phase space, with the ensuing restriction of attention to the point transformations (2.3) in $\mathscr{s}$, allows a covariant classification of dynamical variables into polynomials in the momenta. For systems of physical interest, the important dynamical variables are at most quadratic in the momenta. The (inhomogeneous) linear variables provide an invariant way of dealing with the canonical variables while the Hamiltonian of the system is a (in general inhomogeneous) quadratic function of the momenta. We call such variables which are at most quadratic in the momenta *special dynamical variables.* Their building blocks are the following.

(i) Configuration variables $y = y(q)$, $z = z(q)$. Configuration variables are scalar functions on $\mathscr{m}$ and, as such, are possible candidates for the canonical coordinates. Indeed, such coordinates $q^a(q)$ are nothing but a set of n independent scalar functions on $\mathscr{m}$.

(ii) Linear variables $u = u^a(q)p_a$, $v = v^a(q)p_a$, $w = w^a(q)p_a$. Linear variables are in a one-to-one correspondence with vector fields on $\mathscr{m}$: $\mathbf{u} = u^a(q)\partial_a$, etc. They are possible candidates for the canonical momenta: any set v_a of linearly independent and commuting vector fields, $[\mathbf{v}_a, \mathbf{v}_b] = 0$, defines the canonical momenta p_a, with $\{p_a, p_b\} = 0$. More generally, any linear variable u may be considered as the projection of the momentum into the vector field $\mathbf{u}$.

(iii) Quadratic variables $g = g^{ab}(q)p_a p_b$, $k = k^{ab}(q)p_a p_b$. Quadratic variables are in a one-to-one correspondence with symmetric tensor fields on $\mathscr{m}$: $\mathbf{g} = g^{ab}(q)\partial_a \otimes \partial_b$, etc.

The Hamiltonian h of the system has the general form

$$h = \tfrac{1}{2}g + u + y = \tfrac{1}{2}g^{ab}(q)p_a p_b + u^a(q)p_a + y(q) , \tag{2.10}$$

where $\frac{1}{2}g$ is the kinetic energy, u is the vector potential term, and y is the scalar potential term. We assume that the quadratic form g is positive definite. The kinetic energy thus endows the configuration space with a Riemannian structure: $g^{ab}(q)$ and its inverse, $g_{ab}(q)$, are interpreted as the contravariant and covariant forms of the metric. We denote

$$|g| \equiv \det(g_{ab}) > 0 . \tag{2.11}$$

We deliberately associate different ranges of the alphabet with different kinds of dynamical variables: e,f with general variables, y,z with configuration variables, u,v,w with linear variables, and g,k with quadratic variables.

The letter h is reserved for the Hamiltonian (2.10) of the system and g for its kinetic energy piece which defines the metric on m.

The Poisson brackets among special dynamical variables determine everything which we need to know about the system. Those among the configuration and the linear variables describe the kinematical structure of the phase space, whereas the Poisson brackets of these variables with the Hamiltonian fix the dynamical evolution of the system.

The kinematics of the system is thus characterized by the relations

$$\{y,z\}=0\ , \tag{2.12}$$

$$\{y,u\}=\partial_{\mathbf{u}}y=u^a y_{,a}\ , \tag{2.13}$$

$$\{u,v\}=-[\mathbf{u},\mathbf{v}]^a p_a\ . \tag{2.14}$$

These relations are a simple generalization of the fundamental Poisson brackets $\{q^a,q^b\}=0$, $\{q^a,p_b\}=\delta^a_b$, and $\{p_a,p_b\}=0$ among the canonical coordinates and momenta. To explain the notation first, $\mathbf{u}=\partial_{\mathbf{u}}$ is the directional derivative along the vector field $\mathbf{u}$, and $[\mathbf{u},\mathbf{v}]=\pounds_{\mathbf{u}}\mathbf{v}=\mathbf{uv}-\mathbf{vu}$ is the Lie bracket between the vector fields $\mathbf{u}$ and $\mathbf{v}$. The configuration variables form an Abelian algebra, Eq. (2.12). The linear variables also form a subalgebra of the Poisson algebra. By Eq. (2.14), $\mathbf{u}\rightarrow u$ is an anti-isomorphism from the Lie algebra of vector fields into the Poisson algebra of linear dynamical variables. The Poisson brackets relations (2.12)–(2.14) taken together reveal that linear inhomogeneous dynamical variables $f=u+y$ form another subalgebra of the Poisson algebra. This algebra is the generalization of the Lie algebra of the Weyl group which is generated by the canonical coordinates and momenta, together with the unit dynamical variable. A covariant Dirac quantization of the system consists in finding a homomorphic mapping of the Poisson algebra of linear dynamical variables into the commutator algebra of self-adjoint operators on a Hilbert space.

Unlike linear variables, quadratic variables do not form a subalgebra of the Poisson algebra, because the Poisson bracket between two such variables yields in general a cubic variable. The Poisson brackets between a quadratic variable and either a configuration variable or a linear variable stay, however, within the class of special variables:

$$\{g,y\}=-2\partial^{\mathbf{u}}y=-2g^{ab}y_{,b}p_a\ , \tag{2.15}$$

$$\{g,u\}=(\pounds_{\mathbf{u}}\mathbf{g})^{ab}p_a p_b\ . \tag{2.16}$$

These brackets are needed when we study the dynamical evolution of linear inhomogeneous variables.

Indeed, the Hamiltonian dynamics of the system may either be considered as the problem of determining the evolution of an arbitrary dynamical variable with time,

$$\dot{f}=\{f,h\}\ , \tag{2.17}$$

or as addressing the restricted question how linear inhomogeneous variables $f=u+y$ are evolved. The latter formulation is still sufficient to reconstruct dynamical trajectories of the system. By writing "the Heisenberg equation of motion" (2.17) for a set of specific linear inhomogeneous variables, namely, for the canonical coordinates q^a and the canonical momenta p_a, we recover the Hamilton equations

$$\dot{s}^r=\{s^r,h\}\ \text{ or } \dot{q}^a=\{q^a,h\},\ \ \dot{p}_a=\{p_a,h\}\ . \tag{2.18}$$

III. GAUGE TRANSFORMATIONS AND CONSTRAINTS

In gauge theories, one works in bigger-than-real-life spaces. To distinguish the variables which live in such spaces from the corresponding physical variables, we capitalize the symbols. Thus, the instantaneous state of a gauge system is represented by a point Q in the big configuration space $\mathscr{M}$, $Q\in\mathscr{M}$, which is considered to be an N-dimensional manifold. The local coordinates of Q are $Q^A(Q)$, $A=1,2,\ldots,N$. They are subject to arbitrary regular transformations:

$$Q^A\rightarrow Q^{A'}=Q^{A'}(Q^B)\ . \tag{3.1}$$

Any choice of local coordinates Q^A in $\mathscr{M}$ induces a choice of canonical coordinates and momenta

$$S^R=(Q^A,P_A),\ \ R=1,2,\ldots,2N\ , \tag{3.2}$$

in the big phase space $\mathscr{S}=T^*\mathscr{M}$. The change (3.1) of the local coordinates in $\mathscr{M}$ induces a point transformation (3.1) and (3.3),

$$P_A\rightarrow P_{A'}=Q^B_{A'}(Q)P_B\ \text{ where } Q^B_{A'}\equiv\partial Q^B/\partial Q^{A'}\ , \tag{3.3}$$

in the big phase space. The dynamical variables in $\mathscr{S}$, $F(S)\in C^\infty(\mathscr{S},\mathbb{R})$, represented by the functions $F(Q^A,P_A)$, again form a Poisson algebra, and linear inhomogeneous variables $F=U+Y$ form its subalgebra (characterizing the kinematical structure of the big phase space).

To extract physical consequences of a gauge theory, one must restrict big spaces to physical spaces and big dynamical variables to physical variables. This aim is achieved by splitting further the subalgebra of linear variables $U=U^A(Q)P_A$. Let $\boldsymbol{\phi}_\alpha(Q)=\phi^A_\alpha(Q)\partial_A$, $\alpha=1,2,\ldots,C\le N$, be a set of C linearly independent vector fields on $\mathscr{M}$ which are surface forming:

$$[\boldsymbol{\phi}_\alpha,\boldsymbol{\phi}_\beta]=-C^\gamma{}_{\alpha\beta}(Q)\boldsymbol{\phi}_\gamma\ . \tag{3.4}$$

The quantities $C^\gamma{}_{\alpha\beta}(Q)$ entering into the closure relations (3.4) are called the *structure functions* of the set $\boldsymbol{\phi}_\alpha$. Often, the vector fields $\boldsymbol{\phi}_\alpha$ come to us as generators of a Lie group (the gauge group) acting on the big configuration space $\mathscr{M}$. In such a case, $C^\gamma{}_{\alpha\beta}$ are simply the structure constants of this group. However, in the group case the generators $\boldsymbol{\phi}_\alpha$ are not necessarily linearly independent: the simplest example is the three generators of the rotation group SO(3) acting on a three-dimensional configuration space (see Sec. X E). In such a case, we select from among them a set of linearly independent generators and write the structure relations (3.4) for those. In this process, the quantities $C^\gamma{}_{\alpha\beta}$ may cease to be constant. The selection of the vector fields (3.4) does not need to work globally, as the previous example illustrates. All that is needed, however, is a consistent choice of the fields $\boldsymbol{\phi}_\alpha$ in overlapping patches covering the whole configuration space $\mathscr{M}$.

Indeed, what is important is not the choice of one particular set of fields $\boldsymbol{\phi}_\alpha$, but rather the space $\mathscr{V}$ of all vector fields $\boldsymbol{v}=v^\alpha(Q)\boldsymbol{\phi}_\alpha(Q)$ tangential to the family of surfaces spanned by the $\boldsymbol{\phi}_\alpha$. The choice of the basis fields $\boldsymbol{\phi}_\alpha$ is thus subject to a regular linear transformation

$$\boldsymbol{\phi}_{\alpha'}(Q)=\Lambda^{\beta}_{\alpha'}(Q)\boldsymbol{\phi}_\beta(Q) . \tag{3.5}$$

Under such a transformation, the structure functions undergo an inhomogeneous transformation

$$C^{\lambda'}{}_{\mu'\nu'}=\Lambda^{\lambda'}_{\gamma}\Lambda^{\alpha}_{\mu'}\Lambda^{\beta}_{\nu'}C^{\gamma}{}_{\alpha\beta}+\Lambda^{\lambda'}_{\rho}\Lambda^{\rho}_{[\mu',\nu']} . \tag{3.6}$$

(Here, $_{,\alpha}=_{,A}\phi^A_\alpha$ is the directional derivative along the field ϕ_α, and the enclosure of two indices in square brackets signifies their antisymmetrization: $F_{[\alpha\beta]}\equiv F_{\alpha\beta}-F_{\beta\alpha}$.) At least locally, one can always use Eq. (3.6) to transform the structure constants to zero. This fact, however, plays no role in our further considerations.

For every vector field $\boldsymbol{v}\in\mathscr{V}$ one can construct the corresponding linear dynamical variable:

$$\boldsymbol{v}\rightarrow v=v^\alpha(Q)\pi_\alpha=v^A P_A , \tag{3.7}$$

where

$$\pi_\alpha=\phi^A_\alpha P_A \text{ and } v^A=v^\alpha\phi^A_\alpha . \tag{3.8}$$

The mapping (3.7) is an antihomomorphism from the Lie algebra $\mathscr{V}$ of such vector fields $\boldsymbol{v}$ into the Poisson algebra of linear dynamical variables:

$$[\boldsymbol{v},\boldsymbol{\nu}]=-\boldsymbol{\omega}\Longrightarrow\{v,\nu\}=\omega . \tag{3.9}$$

In particular,

$$\{\pi_\alpha,\pi_\beta\}=C^\gamma{}_{\alpha\beta}(Q)\pi_\gamma . \tag{3.10}$$

The linear dynamical variables of the form (3.7) and (3.8) thus form a subalgebra $\mathscr{V}$ of the Poisson algebra of all linear variables. The elements v of $\mathscr{V}$ are called (linear) *gauge variables.* It is the subalgebra $\mathscr{V}$ itself, rather than the particular choice π_α of its basis, which is essential for extracting the physical content from a given gauge theory. The classical and quantum formalism which we are going to develop will be manifestly covariant under any change (3.5) of the basis.

Every dynamical variable generates a one-parameter group of canonical transformations, as in Eq. (2.9). In particular, the linear gauge variables generate *gauge transformations.*

Let us study first what a gauge transformation does to configuration variables. It acts on the configuration coordinates Q^A by

$$\frac{dQ^A}{d\tau}=v^A(Q) . \tag{3.11}$$

Because the fields $\boldsymbol{v}$ are surface forming, the collection of points in $\mathscr{M}$ which can be reached from a given initial point $Q(0)$ by a sequence of gauge transformations (3.11) forms an $(n=N-C)$-dimensional surface in the N-dimensional configuration space $\mathscr{M}$. We call that surface the *orbit* of gauge transformations through $Q(0)$.

Physically, one interprets different points Q within the orbit as different but equivalent descriptions of the same state of the system. One can thus say that the orbits themselves are points $q\in m$ in the physical configuration space m. Any set of n independent functions $q^a(Q)$ on $\mathscr{M}$ which are constant along the orbits,

$$\frac{dq^a(Q)}{d\tau}=\{q^a(Q),v\}=\partial_v q^a(q)=0 \quad \forall v\in\mathscr{V} , \tag{3.12}$$

may serve as physical coordinates in m. Of course, physical coordinates are determined only up to (regular) coordinate transformations (2.1). Because of the arbitrariness of v, Eq. (3.12) characterizing the physical coordinates may be replaced by a set of equations

$$\{q^a(Q),\pi_\alpha\}=0, \text{ or } q^a{}_{,\alpha}(Q)\equiv\phi^A_\alpha q^a{}_{,A}(Q)=0 . \tag{3.13}$$

Any function $Y(Q)$ on $\mathscr{M}$ which is constant along the orbits,

$$\{Y(Q),v\}=0 \quad \forall v\in\mathscr{V} \text{ or } \{Y(Q),\pi_\alpha\}=0 , \tag{3.14}$$

can depend on Q only through the physical coordinates: $Y(Q)=y(q(Q))$. It can thus be interpreted as a physical configuration variable.

When one tries to extend this procedure from the configuration space to the phase space and to identify the physical phase-space points as orbits of gauge transformations in $\mathscr{S}$,

$$\frac{dS^R}{d\tau}=\{S^R,v\}=\mathbf{X}_v S^R, \quad v\in\mathscr{V} , \tag{3.15}$$

one meets an unexpected difficulty: the Hamiltonian vector fields $\mathbf{X}_v$ are not surface forming. Indeed, by the general properties (2.7) and (2.8) of such fields,

$$[\mathbf{X}_v,\mathbf{X}_\nu]=[\boldsymbol{v},\boldsymbol{\nu}]^\alpha\mathbf{X}_{\pi_\alpha}+\pi_\alpha\mathbf{X}_{[\boldsymbol{v},\boldsymbol{\nu}]^\alpha} ; \tag{3.16}$$

the trouble is that the vector fields $\mathbf{X}_{[\boldsymbol{v},\boldsymbol{\nu}]^\alpha}$ do not in general lie in the vector space spanned by the basis $\mathbf{X}_{\pi_\alpha}$. Among the orbits, however, there are such which *are* C-dimensional surfaces: namely, those which lie in the constraint surface $\mathscr{C}$ defined by the equations

$$v=0 \quad \forall v\in\mathscr{V} \text{ or } \pi_\alpha=0 . \tag{3.17}$$

Therefrom stems the necessity of imposing the constraints before reducing the gauge theory to a physical theory. The space of orbits within the constraint surface can then be identified with the physical phase space $\mathscr{s}$ of the gauge system. It is a $(2n=2N-2C)$-dimensional space. The physical coordinates $q^a(S)$, $S\in\mathscr{C}\subset T^*\mathscr{M}$, can serve as canonical coordinates in $\mathscr{s}$. We shall construct the conjugate physical momenta $p_a(S)$, $S\in\mathscr{C}$, in Sec. VII.

IV. OBSERVABLES AND GAUGE VARIABLES

Having constructed the physical phase space, one must ask what dynamical variables $F(S)$ in $\mathscr{S}$ represent physical variables. Of course, any dynamical variable $F(S)$ can be restricted to the constraint surface $\mathscr{C}$. When this restriction happens to vanish, $F(S)=0$ for $S\in\mathscr{C}$, we say that F weakly vanishes and write $F\approx 0$. Such dynamical variables in $\mathscr{S}$ are called *gauge variables.* In particular, the elements v of $\mathscr{V}$ are (linear) gauge variables. Any gauge variable represents a trivial physical variable, namely, 0. Note that for C^∞ variables, $F\approx 0$ implies that there exist dynamical variables $F^\alpha(S)$ such that

$$F=F^\alpha \pi_\alpha \ . \tag{4.1}$$

In general, a dynamical variable $F(S)$ on $\mathscr{S}$ represents a physical variable if its restriction to $\mathscr{C}$ remains constant along the orbits of gauge transformations in $\mathscr{S}$:

$$\{F,v\}\approx 0 \ \ \forall v\in\mathscr{V} \ \text{ or } \ \{F,\pi_\alpha\}\approx 0 \ . \tag{4.2}$$

We call such a dynamical variable F an *observable.* Two observables, F_1 and F_2, represent the same physical variable $f(s)$ when they coincide on the constraint surface $\mathscr{C}$: $F_1\approx F_2$. The physical variables $f(s)$ can be identified with such an equivalence class of observables:

$$f(s)\equiv(F(S))=\{E(S)\colon E(S)\approx F(S)\} \ . \tag{4.3}$$

We denote the class of observables equivalent to $F(S)$ by the symbol $(F(S))$. The condition (4.2) which ensures that F is an observable can be written in the strong form

$$\{F,\pi_\alpha\}=F_\alpha^\beta \pi_\beta \ , \tag{4.4}$$

where F_α^β are some dynamical variables on $\mathscr{S}$.

By the Jacobi identity, when E and F are two observables, their Poisson bracket $\{E,F\}$ is again an observable. Further, by Eq. (4.4), if E_1,E_2 are two representatives of the same physical variable e and F_1,F_2 are two representatives of the same physical variable f, then

$$\{E_1,F_1\}\approx\{E_2,F_2\} \ . \tag{4.5}$$

This allows us to define the physical Poisson bracket $\{e,f\}$ by

$$\{e,f\}\equiv\{(E),(F)\}\equiv(\{E,F\}) \ . \tag{4.6}$$

This endows the physical phase space with a symplectic structure.

V. SPECIAL OBSERVABLES

As in the case of unconstrained systems, those observables which do not depend on the momenta, which are linear in the momenta, or which are quadratic in the momenta deserve special attention.

(i) Configuration observables Y(Q), Z(Q). Because the Poisson bracket $\{Y,\pi_\alpha\}$ depends only on Q but not on P,

$$\{Y,\pi_\alpha\}\approx 0 \Longrightarrow \{Y,\pi_\alpha\}=0\Longrightarrow Y=y(q^a) \ . \tag{5.1}$$

(ii) Linear observables $U=U^A P_A$, $V=V^A P_A$. As in Eq. (2.14),

$$\{U,\pi_\alpha\}=-[\mathbf{U},\boldsymbol{\phi}_\alpha]^A P_A \ . \tag{5.2}$$

By comparing this with the condition

$$\{U,\pi_\alpha\}=U_\alpha^\beta\pi_\beta \tag{5.3}$$

for U to be an observable, we learn that

$$U_\alpha^A\equiv U_\alpha^\beta\phi_\beta^A=-[\mathbf{U},\boldsymbol{\phi}_\alpha]^A \ . \tag{5.4}$$

(iii) Quadratic observables $G=G^{AB}P_A P_B$, $K=K^{AB}P_A P_B$. As in Eq. (2.16),

$$\{G,\pi_\alpha\}=(\pounds_{\phi_\alpha}\mathbf{G})^{AB}P_A P_B \ . \tag{5.5}$$

By comparing this with the condition

$$\{G,\pi_\alpha\}=G_\alpha^\beta\pi_\beta \tag{5.6}$$

for G to be an observable, we learn that G_α^β is a linear dynamical variable (not necessarily an observable), i.e., $G_\alpha^\beta=G_\alpha^{\beta B}(Q)P_B$, and that

$$(\pounds_{\phi_\alpha}\mathbf{G})^{AB}=\tfrac{1}{2}\phi_\beta^{(A}G_\alpha^{\beta B)} \ . \tag{5.7}$$

Here, the enclosure of two indices in round brackets signifies their symmetrization: $F_{(\alpha\beta)}\equiv F_{\alpha\beta}+F_{\beta\alpha}$.

As in Eq. (2.10), the Hamiltonian H of the gauge system is an observable of the generic form

$$H=\tfrac{1}{2}G+U+Y=\tfrac{1}{2}G^{AB}P_A P_B+U^A P_A+Y \ . \tag{5.8}$$

From Eqs. (5.1), (5.3), and (5.6) we get

$$\{H,\pi_\alpha\}=(\tfrac{1}{2}G_\alpha^\beta+U_\alpha^\beta)\pi_\beta \ . \tag{5.9}$$

The quadratic observable G represents the kinetic energy of the system. We assume that the projection g^{ab} of G^{AB} into the physical space,

$$g^{ab}\equiv G^{AB}Q_A^a Q_B^b, \ \ Q_A^a\equiv q^a{}_{,A} \ , \tag{5.10}$$

is nondegenerate,

$$|g|^{-1}\equiv\det(g^{ab})\neq 0 \ , \tag{5.11}$$

and, indeed, positive definite. One can identify g^{ab} with the physical metric tensor. On the other hand, the metric G^{AB} in the big configuration space may be degenerate, though in practice it is often regular. If G^{AB} is regular, one can find its inverse G_{AB} and introduce the determinant

$$|G|\equiv\det(G_{AB}) \ . \tag{5.12}$$

The Hamiltonian dynamics of a gauge system may be conceived as a problem of determining the evolution of an arbitrary observable with time,

$$\dot F=\{F,H\} \ . \tag{5.13}$$

From the Jacobi identity one sees that if $F(t_0)$ is an observable at $t=t_0$, $F(t)$ is an observable at any t. Also, if $F_1(t_0)$ and $F_2(t_0)$ represent the same physical variable at $t=t_0$, $F_1(t)$ and $F_2(t)$ represent the same physical variable $F(t)$ at any t. To study the motion of the system in the physical space, it is sufficient to restrict the observables F in the Heisenberg equation of motion (5.13) to linear inhomogeneous observables $F=V+Z$.

An alternative way of looking at the evolution of the gauge system is through the Hamilton equations for the dynamical trajectory in the big phase space $\mathscr{S}$:

$$\dot S^R=\{S^R,H\} \ . \tag{5.14}$$

Equation (5.9) implies that the constraints are preserved in time, $\dot\pi_\alpha\approx 0$, and hence a point $S(t_0)\in\mathscr{C}$ on the constraint surface keeps moving along the constraint surface, $S(t)\in\mathscr{C}$ $\forall t$. It also implies that if $S_1(t_0)$ and $S_2(t_0)$ lie in the same orbit of the gauge transformations, $S_1(t)$ and $S_2(t)$ will remain in the same orbit for all t. In other words, the dynamics in $\mathscr{S}$ induces the dynamics in $\mathscr{s}$.

VI. PROJECTORS

The gauge properties of a system are most directly described by the Poisson algebra $\mathscr{V}$ of linear gauge variables v. Any choice (3.8) of a basis π_α of $\mathscr{V}$ is arbitrary up to a regular transformation (3.5) which amounts to a *mixing of constraints* (3.17):

$$\pi_{\alpha'}=\Lambda^{\beta}_{\alpha'}(Q)\pi_\beta \,. \tag{6.1}$$

By solving a set of differential equations (3.12) or (3.13), we obtain $n=(N-C)$ independent functions $q^a(Q)$, which play the role of physical coordinates. These are not affected by the change (6.1) of the basis, but they are themselves arbitrary up to a regular transformation (2.1). Finally, the coordinates Q^A in the big configuration space $\mathscr{M}$ are subject to the transformation (3.1). The transformations (6.1), (2.1), and (3.1) can be performed independently of each other, and none of them should affect physical conclusions which are drawn from the formalism.

The gradients

$$Q^a_A\equiv q^a{}_{,A} \tag{6.2}$$

of the physical coordinates enable us to isolate the physical components u^a of the big configuration space vectors U^A:

$$u^a=Q^a_A U^A \,. \tag{6.3}$$

This rule can be extended to contravariant tensors. In particular, from the metric tensor G^{AB} one can obtain the physical metric g^{ab} by Eq. (5.10). Our basic assumption is that g^{ab} is a regular positive-definite metric; as such, it has an inverse g_{ab} which can be used to lower small latin indices. Capital latin indices are raised by the (possibly degenerate) metric G^{AB}. However, unless G^{AB} is regular, there is no natural way of lowering capital latin indices.

There are only $C<N$ basis vector fields ϕ^A_α and only $n=N-C<N$ cobasis vector fields Q^a_A. We want to complement these fields into a full basis and a full cobasis in the big space. This is achieved by using the metric tensor G^{AB} as an auxiliary element. First, by lowering and raising the indices in Q^a_A, we form

$$Q^A_a\equiv g_{ab}Q^b_B G^{BA} \,. \tag{6.4}$$

One can easily check that

$$Q^A_a Q^b_A=\delta^b_a \,. \tag{6.5}$$

From the definition of the physical coordinates,

$$\phi^A_\alpha Q^a_A=0 \,. \tag{6.6}$$

By assumption, ϕ^A_α are linearly independent. From Eqs. (6.5) and (6.6) it follows that the $N=n+C$ vectors Q^A_a,ϕ^A_α are likewise linearly independent. They form a basis in the big space.

The cobasis to this basis is obtained by complementing the n covectors Q^a_A by C covectors ϕ^α_A satisfying the equations

$$\phi^A_\beta\phi^\alpha_A=\delta^\alpha_\beta \,, \tag{6.7}$$

$$Q^A_a\phi^\alpha_A=0 \,. \tag{6.8}$$

These covectors are unique and Q^a_A,ϕ^α_A are linearly independent. Equations (6.5)–(6.8) are the orthogonality relations between the basis Q^A_a,ϕ^A_α and the cobasis Q^a_A,ϕ^α_A.

On the other hand, by summing over the lower case latin and greek indices one can form the mixed tensors

$$\mathscr{P}^A_B\equiv Q^A_a Q^a_B \,, \tag{6.9}$$

$$\Pi^A_B\equiv\phi^A_\alpha\phi^\alpha_B \tag{6.10}$$

with the properties

$$\mathscr{P}^A_C\mathscr{P}^C_B=\mathscr{P}^A_B,\ \ \phi^A_\alpha\mathscr{P}^B_A=0 \,, \tag{6.11}$$

and

$$\Pi^A_C\Pi^C_B=\Pi^A_B,\ \ Q^a_A\Pi^A_B=0 \,. \tag{6.12}$$

Equation (6.11) tells us that $\mathscr{P}^A_B$ is a projector which, when acting on covectors U_A, makes them perpendicular to the orbits of gauge transformations. Similarly, Eq. (6.12) tells us that Π^A_B is a projector which, when acting on a vector U^B, makes it perpendicular to the physical surfaces $q^a=$const.[12] As a counterpart of the orthogonality relations (6.5)–(6.8), we have the completeness relation

$$\mathscr{P}^A_B+\Pi^A_B=\delta^A_B \,. \tag{6.13}$$

This allows us to decompose an arbitrary vector U^A into a physical piece u^A and a gauge piece v^A:

$$U^A=u^A+v^A \,, \tag{6.14}$$

$$u^A=\mathscr{P}^A_B U^B=(U^B Q^a_B)Q^A_a\equiv u^a Q^A_a \,, \tag{6.15}$$

$$v^A=\Pi^A_B U^B=(U^B\phi^\alpha_B)\phi^A_\alpha\equiv v^\alpha\phi^A_\alpha \,. \tag{6.16}$$

Again, this rule can be extended to higher-order tensors. In particular, one can apply it to the metric $\mathbf{G}$ which was used to define the projectors in the first place. One obtains

$$G^{AB}=g^{AB}+\gamma^{AB} \,, \tag{6.17}$$

$$g^{AB}=\mathscr{P}^A_C\mathscr{P}^B_D G^{CD},\ \ \gamma^{AB}=\Pi^A_C\Pi^B_D G^{CD} \,, \tag{6.18}$$

$$g^{AB}=g^{ab}Q^A_a Q^B_b,\ \ g^{ab}=G^{AB}Q^a_A Q^b_B \,, \tag{6.19}$$

$$\gamma^{AB}=\gamma^{\alpha\beta}\phi^A_\alpha\phi^B_\beta,\ \ \gamma^{\alpha\beta}=G^{AB}\phi^\alpha_A\phi^\beta_B \,. \tag{6.20}$$

If the metric G^{AB} is D-times degenerate, $D\le C$, so is the metric $\gamma^{\alpha\beta}$. A regular G^{AB} yields a regular $\gamma^{\alpha\beta}$ and one can define the inverse matrices G_{AB} and $\gamma_{\alpha\beta}$. In this case,

$$\phi^\alpha_A=\gamma^{\alpha\beta}\phi^B_\beta G_{BA} \,, \tag{6.21}$$

which is a counterpart of Eq. (6.4).

The formalism is covariant with respect to all three classes of transformations: (1) Coordinate transformations (3.1) in the big configuration space (which affect the capital latin indices), (2) coordinate transformations (2.1) in the physical configuration space (which affect the lower case latin indices), and (3) the mixing of the constraints (6.1) or (3.5) (which affects the greek indices). The vector or covector character of all quantities introduced in this section is clearly indicated by the position and alphabetical type of the indices which they carry.

Subject to the enumerated transformations, the quantities Q^A_a,ϕ^A_α; Q^a_A,ϕ^α_A and $\mathscr{P}^A_B,\Pi^A_B$ are uniquely determined.

However, the metric G^{AB} enters as an auxiliary element into the construction of the system. The metric comes to us from the kinetic piece G of the Hamiltonian. Physically, it does not matter if we represent the kinetic energy by the observable G or by another observable $\overline{G}$ from the same equivalence class, $\overline{G}\approx G$. But the observable $\overline{G}$ yields a different metric $\overline{G}^{AB}$ which then induces a different set of the quantities Q_a^A and ϕ_A^α. Indeed, if Q_a^A, ϕ_α^A is the basis induced by the old metric G^{AB}, the new metric $\overline{G}^{AB}$ has in general the form

$$\overline{G}^{AB}=G^{AB}+\lambda^{a\alpha}Q^{(A}_a\phi^{B)}_\alpha+\lambda^{\alpha\beta}\phi^A_\alpha\phi^B_\beta \,, \tag{6.22}$$

where $\lambda^{a\alpha}$ and $\lambda^{\alpha\beta}$ are some functions of Q. The physical metric induced by (6.22) is the same as before, $\overline{g}^{ab}=g^{ab}$. On the other hand, the basis vectors Q_a^A are changed,

$$\overline{Q}_a^A=Q_a^A+\lambda_a^\alpha\phi_\alpha^A \,, \tag{6.23}$$

and so are the cobasis vectors ϕ_A^α,

$$\overline{\phi}_A^\alpha=\phi_A^\alpha-\lambda_a^\alpha Q_A^a \,. \tag{6.24}$$

This change, however, does not affect the physical content of the formalism. We can select one representative of the kinetic energy arbitrarily and stick to it for the rest of the paper.

VII. CANONICAL COORDINATES IN THE PHYSICAL PHASE SPACE

In Sec. III we defined the physical phase space $\mathscr{s}$ as the space of orbits (3.15) of gauge transformations in $\mathscr{S}$ which lie on the constraint surface $\mathscr{C}$ and identified its canonical coordinates q^a. Now, we must find the observables p_a representing the canonical momenta in $\mathscr{s}$. We shall see that these are simply the components p_a of the momentum P_A in the cobasis Q_A^a, ϕ_A^α:

$$P_A=Q_A^a p_a+\phi_A^\alpha\pi_\alpha \,, \tag{7.1}$$

$$p_a=Q_a^A P_A, \quad \pi_\alpha=\phi_\alpha^A P_A \,. \tag{7.2}$$

As in Eq. (2.14),

$$\{p_a,p_b\}=-[\mathbf{Q}_a,\mathbf{Q}_b]^A P_A \,. \tag{7.3}$$

Because

$$[\mathbf{Q}_a,\mathbf{Q}_b]^A Q_A^c=0 \,, \tag{7.4}$$

we have

$$\{p_a,p_b\}=C^\alpha{}_{ab}\pi_\alpha \,, \tag{7.5}$$

where we have introduced a new set of structure functions

$$C^\alpha{}_{ab}\equiv-[\mathbf{Q}_a,\mathbf{Q}_b]^A\phi_A^\alpha=-Q_a^A Q_b^B\phi^\alpha_{[A,B]} \,. \tag{7.6}$$

Similarly,

$$\{\pi_\beta,p_a\}=-[\boldsymbol{\phi}_\beta,\mathbf{Q}_a]^A P_A \tag{7.7}$$

and

$$[\boldsymbol{\phi}_\beta,\mathbf{Q}_a]^A Q_A^c=0 \tag{7.8}$$

implies that

$$\{\pi_\beta,p_a\}=C^\alpha{}_{\beta a}\pi_\alpha \,, \tag{7.9}$$

where

$$C^\alpha{}_{\beta a}\equiv-[\boldsymbol{\phi}_\beta,\mathbf{Q}_a]^A\phi_A^\alpha=Q_a^A\phi_\beta^B\phi^\alpha_{[A,B]} \,. \tag{7.10}$$

We complement Eq. (7.9) by a similar equation involving the physical coordinates q^a,

$$\{\pi_\beta,q^a\}=0 \,, \tag{7.11}$$

and write down the trivial counterpart of Eq. (7.5), namely,

$$\{q^a,q^b\}=0 \,. \tag{7.12}$$

Finally,

$$\{q^a,p_b\}=\delta^a_b \,. \tag{7.13}$$

This enables us to draw the desired conclusion. Equations (7.9) and (7.11) tell us that the dynamical variables q^a,p_a are observables. The equivalence classes $(q^a),(p_a)$ of these observables are physical variables. The Poisson brackets of these physical variables are defined by Eq. (4.6); Eqs. (7.12), (7.5), and (7.13) are thereby translated into the statement

$$\begin{aligned}&\{(q^a),(q^b)\}=0 \,,\\&\{(p_a),(p_b)\}=0 \,,\\&\{(q^a),(p_b)\}=\delta^a_b \,.\end{aligned} \tag{7.14}$$

This shows that the physical variables (q^a) and (p_a) are the canonical coordinates and the canonical momenta in the physical phase space $\mathscr{s}$.

Note that the change of the observable which represents the kinetic energy induces by Eq. (6.23) a change

$$\overline{p}_a=p_a+\lambda_a^\alpha\pi_\alpha \tag{7.15}$$

of the observable which represents the physical momentum, but that $\overline{p}_a$ stays in the same equivalence class with p_a: $(\overline{p}_a)=(p_a)$.

VIII. JACOBI IDENTITIES FOR STRUCTURE FUNCTIONS

In addition to the structure functions $C^\alpha{}_{\beta\gamma}$ characterizing the system of constraints, Eq. (3.10), we have just introduced additional structure functions $C^\alpha{}_{\beta a}$ and $C^\alpha{}_{ab}$ characterizing the remaining Poisson brackets (7.9) and (7.5) of the projected momenta. From their definitions (7.6), (7.10), and

$$C^\alpha{}_{\beta\gamma}=-[\boldsymbol{\phi}_\beta,\boldsymbol{\phi}_\gamma]^A\phi_A^\alpha=-\phi_\beta^B\phi_\gamma^C\phi^\alpha_{[B,C]} \tag{8.1}$$

one can read off their transformation properties. First, all structure functions are scalars under coordinate transformations (3.1) in $\mathscr{M}$ and appropriate tensors under coordinate transformations (2.1) in $\mathscr{m}$. Further, $C^\alpha{}_{ab}$ is also a vector under the mixing of constraints (3.5). On the other hand, $C^\alpha{}_{\beta a}$, like $C^\alpha{}_{\beta\gamma}$ in Eq. (3.6), undergoes an inhomogeneous transformation

$$C^{\alpha'}{}_{\beta' a}=\Lambda^{\alpha'}_\mu\Lambda^\nu_{\beta'}C^\mu{}_{\nu a}-\Lambda^\mu_{\beta'}\Lambda^{\alpha'}_{\mu,a} \,. \tag{8.2}$$

The structure functions satisfy certain identities which follow from the Jacobi identities for the Poisson brackets among the projected momenta. Thus

$$\{\{\pi_\alpha,\pi_\beta\},\pi_\gamma\}+\{\{\pi_\gamma,\pi_\alpha\},\pi_\beta\}+\{\{\pi_\beta,\pi_\gamma\},\pi_\alpha\}=0 \quad (8.3)$$

implies

$$C^\delta{}_{\alpha\beta,\gamma}+C^\delta{}_{\beta\gamma,\alpha}+C^\delta{}_{\gamma\alpha,\beta}+C^\mu{}_{\alpha\beta}C^\delta{}_{\mu\gamma}+C^\mu{}_{\beta\gamma}C^\delta{}_{\mu\alpha}+C^\mu{}_{\gamma\alpha}C^\delta{}_{\mu\beta}=0 , \quad (8.4)$$

while

$$\{\{\pi_\beta,p_b\},\pi_\alpha\}+\{\{\pi_\alpha,\pi_\beta\},p_b\}+\{\{p_b,\pi_\alpha\},\pi_\beta\}=0 \quad (8.5)$$

implies

$$C^\gamma{}_{\alpha\beta,b}-C^\gamma{}_{[ab,\beta]}+C^\delta{}_{\alpha\beta}C^\gamma{}_{\delta b}-C^\delta{}_{[ab}C^\gamma{}_{\delta\beta]}=0 . \quad (8.6)$$

Two more identities of this kind may be obtained from the Jacobi identities involving (1) two p's and one π, (2) three p's. We shall not need them in the following, and so we shall not write them down.

Let us finally introduce the contracted forms of the structure functions:

$$C_\alpha\equiv C^\beta{}_{\beta\alpha} \text{ and } C_a\equiv C^\beta{}_{\beta a} . \quad (8.7)$$

Then, by contracting the identities (8.4) and (8.6), we obtain

$$C_{[\alpha,\beta]}-C^\gamma{}_{\alpha\beta,\gamma}+C_\gamma C^\gamma{}_{\alpha\beta}=0 \quad (8.8)$$

and

$$C_{\alpha,a}-C_{a,\alpha}+C^\beta{}_{a\alpha,\beta}-C_\beta C^\beta{}_{a\alpha}=0 . \quad (8.9)$$

IX. POISSON BRACKETS OF SPECIAL OBSERVABLES

For physical systems, the Poisson algebra of linear inhomogeneous variables characterizes the kinematical structure of the phase space, while the Poisson bracket of the Hamiltonian with such variables describes the dynamics of the system through the Heisenberg equations of motion. This statement remains valid for gauge systems, with the proviso that one must limit the attention to Poisson brackets of special *observables*. Even then, the redundancy inherent in the gauge description finds its way into the algebra. If two observables in a Poisson bracket are changed, each within its own equivalence class, the observable representing the Poisson bracket is also changed by a gauge variable. It is a virtue of the gauge formalism that it keeps track of all such changes. In the end, one extracts the physical results by collecting observables into equivalence classes which are identified with the physical variables. We shall see that the algebra of observables reduces thereby to the old algebra of physical variables which we have studied in Sec. II.

The projector formalism which we have developed allows us to split special observables into well-defined pieces.

(1) No splitting is necessary for configuration observables. Because of Eq. (5.1), any configuration observable $Y(Q)$ is entirely physical: $Y=y(q)$.

(2) Any linear observable $U=U^A(Q)P_A$ can be decomposed into a physical piece $u=u^ap_a$ and a gauge variable $v=v^\alpha\pi_\alpha\in\mathscr{V}$ [cf. Eqs. (6.14)–(6.16)]:

$$U=u+v=u^ap_a+v^\alpha\pi_\alpha . \quad (9.1)$$

Direct evaluation of the Poisson brackets of U with the constraints yields

$$\{U,\pi_\alpha\}=u^a{}_{,\alpha}p_a+U^\beta_\alpha\pi_\beta , \quad (9.2)$$

with

$$U^\beta_\alpha=v^\beta{}_{,\alpha}-C^\beta{}_{\alpha\gamma}v^\gamma-C^\beta{}_{\alpha c}u^c . \quad (9.3)$$

In order that U be an observable, $u^a{}_{,\alpha}$ must vanish. By the reasoning (3.14), this implies that

$$u^a=u^a(q^b) , \quad (9.4)$$

i.e., $\mathbf{u}$ is a vector in the physical space $T_q\mathscr{M}$. Equation (9.3) gives an explicit expression for the coefficient U^β_α introduced by Eqs. (5.3) and (5.4).

(3) The kinetic energy observable $G=G^{AB}P_AP_B$ can again be decomposed into a physical piece g and a gauge variable γ [cf. Eqs. (6.17)–(6.20)]:

$$G=g+\gamma=g^{ab}p_ap_b+\gamma^{\alpha\beta}\pi_\alpha\pi_\beta . \quad (9.5)$$

The Poisson bracket of G with the constraints yields

$$\{G,\pi_\alpha\}=g^{ab}{}_{,\alpha}p_ap_b+G^\beta_\alpha\pi_\beta , \quad (9.6)$$

with

$$G^\beta_\alpha=(\gamma^{\beta\gamma}{}_{,\alpha}+C^{(\beta\gamma)}{}_\alpha)\pi_\gamma-2C^\beta{}_\alpha{}^cp_c . \quad (9.7)$$

In order that G be an observable, $g^{ab}{}_{,\alpha}$ must vanish. Again, by reasoning of Eq. (3.14), this implies that

$$g^{ab}=g^{ab}(q^c) , \quad (9.8)$$

i.e., that $\mathbf{g}$ is a tensor in the physical space.[13] We have already identified it with the physical metric tensor. Equation (9.7) gives an explicit expression for the coefficient G^β_α introduced by Eqs. (5.6) and (5.7). The procedure can be applied to other quadratic observables $K=K^{AB}P_AP_B$.

The decompositions (9.1) and (9.5) separate the physical pieces u and g from the gauge variables v and γ. For the purpose of representing physical variables, the gauge variables may be simply omitted. We shall call u and g the *standard representatives* of the observables U and G. Remember, however, that standard representatives depend on the choice (6.22) of the big metric. In the following, we assume that this choice is made and adhered to throughout the argument.

Let us write now the Poisson algebra of linear inhomogeneous observables:

$$\{y,z\}=0 , \quad (9.9)$$

$$\{y,u\}=\partial_{\mathbf{u}}y=u^ay_{,a} , \quad (9.10)$$

$$\{y,v\}=0 , \quad (9.11)$$

$$\{u,v\}=-(\pounds_{\mathbf{u}}\mathbf{v})^\alpha p_a+C^\alpha{}_{ab}u^av^b\pi_\alpha , \quad (9.12)$$

$$\{u,\nu\}=-(\partial_{\mathbf{u}}\boldsymbol{\nu})^\alpha\pi_\alpha , \quad (9.13)$$

$$\{v,\nu\}=-(\pounds_{\mathbf{v}}\boldsymbol{\nu})^\alpha\pi_\alpha . \quad (9.14)$$

Here, $\pounds_{\mathbf{u}}\mathbf{v}$ and $\pounds_{v}\boldsymbol{\nu}$ are abbreviations for the Lie brackets of the physical and the gauge vectors, respectively,

$$(\pounds_{\mathbf{u}}\mathbf{v})^a=[\mathbf{u},\mathbf{v}]^a=v^a{}_{,b}u^b-u^a{}_{,b}v^b\ , \tag{9.15}$$

$$(\pounds_{v}\boldsymbol{\nu})^\alpha=\nu^\alpha{}_{,\beta}v^\beta-v^\alpha{}_{,\beta}\nu^\beta+C^\alpha{}_{\beta\gamma}\nu^\beta v^\gamma\ . \tag{9.16}$$

Because $u^a=u^a(q^b)$ [and $v^a=v^a(q^b)$], the derivatives in Eq. (9.15) can be interpreted as ordinary partial derivatives: $u^a{}_{,b}=\partial u^a(q^c)/\partial q^b$. The structure functions appear in the definition (9.16) of the Lie bracket of two vectors in the gauge space because $\boldsymbol{\phi}_\alpha$ is an anholonomic basis. Similarly, $\boldsymbol{\partial}_{\mathbf{u}}\boldsymbol{\nu}$ is an abbreviation for

$$(\boldsymbol{\partial}_{\mathbf{u}}\boldsymbol{\nu})^\alpha=(\partial_{\mathbf{u}}\boldsymbol{\nu})^\alpha+C^\alpha{}_{\beta a}\nu^\beta u^a=(\nu^\alpha{}_{,a}+C^\alpha{}_{\beta a}\nu^\beta)u^a\ . \tag{9.17}$$

Here, $(\partial_{\mathbf{u}}\boldsymbol{\nu})^\alpha=\nu^\alpha{}_{,a}u^a$ is the ordinary directional derivative of the components ν^α of the gauge vector $\boldsymbol{\nu}$ along the physical direction $\mathbf{u}$. This derivative, however, is not covariant under the mixing of constraints, i.e., $(\partial_{\mathbf{u}}\boldsymbol{\nu})^\alpha$ does not transform as a vector under the transformations (3.5). The term $C^\alpha{}_{\beta a}\nu^\beta$, thanks to the inhomogeneous transformation property (8.2) of the structure functions $C^\alpha{}_{\beta a}$, properly corrects this deficiency so that $(\boldsymbol{\partial}_{\mathbf{u}}\boldsymbol{\nu})^\alpha$ *is* a vector under (3.5). The same is true of the Lie bracket (9.16).

Indeed, these transformation properties are obvious. The variables u,v and v,ν are scalars under the transformations (3.5). Their Poisson brackets (9.12)–(9.14) must therefore also be scalars. This is ensured by the proper transformation behavior of the coefficients on the right-hand sides of Eqs. (9.12)–(9.14): $(\pounds_{\mathbf{u}}\mathbf{v})^a$ is a scalar, whereas $C^\alpha{}_{ab}u^av^b$, $(\boldsymbol{\partial}_{\mathbf{u}}\boldsymbol{\nu})^\alpha$, and $(\pounds_{v}\boldsymbol{\nu})^\alpha$ are vectors under such transformations.

By Eq. (9.9), configuration observables form an Abelian subalgebra of the Poisson algebra of observables. Similarly, by Eqs. (9.12)–(9.14), linear observables form another subalgebra. By collecting linear observables into equivalence classes (U) and (V) and defining the Poisson brackets between the classes by Eq. (4.6), we learn from Eqs. (9.12)–(9.14) that

$$\{(U),(V)\}=\{(u),(v)\}=-[\mathbf{u},\mathbf{v}]^a(p_a)\ . \tag{9.18}$$

In other words, the Poisson bracket of the physical variables (U) and (V) can be evaluated by using their standard representatives u and v. This evaluation leads directly to the ordinary expression (2.14) in terms of the Lie bracket between the physical vector fields $\mathbf{u}=u^a(q^b)\partial/\partial q^a$ and $\mathbf{v}=v^a(q^b)\partial/\partial q^a$.

The same procedure can be applied to the Poisson brackets involving the kinetic energy observables $G=g+\gamma$. The operations $\pounds_{\mathbf{u}}$, $\pounds_{v}$ and $\boldsymbol{\partial}_{\mathbf{u}}$ can be naturally extended from vectors to tensors under the constraint mixing, with the results

$$(\pounds_{\mathbf{u}}\mathbf{g})^{ab}=g^{ab}{}_{,c}u^c-g^{c(a}u^{b)}{}_{,c}\ , \tag{9.19}$$

$$(\pounds_{v}\boldsymbol{\gamma})^{\alpha\beta}=\gamma^{\alpha\beta}{}_{,\gamma}v^\gamma-\gamma^{\gamma(\alpha}v^{\beta)}{}_{,\gamma}+C^{(\alpha\beta)}{}_{\gamma}v^\gamma\ . \tag{9.20}$$

Again, the $,a$ derivative in Eq. (9.19) can be interpreted as an ordinary partial derivative with respect to q^a. With this notation, the Poisson brackets involving G take the simple form

$$\{g,u\}=(\pounds_{\mathbf{u}}\mathbf{g})^{ab}p_ap_b+2C^{\alpha a}{}_bu^bp_a\pi_\alpha\ , \tag{9.21}$$

$$\{g,v\}=-2(\boldsymbol{\partial}^av^\alpha)p_a\pi_\alpha\ , \tag{9.22}$$

$$\{\gamma,u\}=(\boldsymbol{\partial}_{\mathbf{u}}\boldsymbol{\gamma})^{\alpha\beta}\pi_\alpha\pi_\beta\ , \tag{9.23}$$

$$\{\gamma,v\}=(\pounds_{v}\boldsymbol{\gamma})^{\alpha\beta}\pi_\alpha\pi_\beta\ . \tag{9.24}$$

The coefficient $\boldsymbol{\partial}^av^\alpha$ in Eq. (9.22) has the obvious meaning

$$\boldsymbol{\partial}^av^\alpha=g^{ab}(\boldsymbol{\partial}_b\boldsymbol{v})^\alpha=g^{ab}(v^\alpha{}_{,b}+C^\alpha{}_{\beta b}v^\beta)\ . \tag{9.25}$$

By introducing the physical variables (U) and (G), we see that Eqs. (9.21)–(9.24) imply that

$$\{(G),(U)\}=\{(g),(u)\}=(\pounds_{\mathbf{u}}\mathbf{g})^{ab}(p_a)(p_b)\ , \tag{9.26}$$

which is a replica of Eq. (2.16).

Finally,

$$\{g,y\}=-2g^{ab}y_{,b}p_a \tag{9.27}$$

and

$$\{\gamma,y\}=0 \tag{9.28}$$

reproduce Eq. (2.15):

$$\{(G),(Y)\}=\{(g),(y)\}=-2g^{ab}y_{,b}(p_a)\ . \tag{9.29}$$

X. MODEL GAUGE SYSTEMS

To illustrate various points made in the Introduction and in the course of the paper, we shall discuss in some detail a couple of simple gauge systems. In field theories, the gauge constraints typically arise from the action of the gauge group. Although the group-theoretical origin of the constraints may be (sometimes profitably) forgotten in the further development of the formalism, we shall nevertheless introduce the constraints in this way.

We start with models having only one constraint which arises from the action of a one-parameter group on the big configuration space of the system. Globally, the group may be either that of translations $T(1)$ or that of rotations SO(2). For simplicity, we shall pay no attention to the potential terms in the Hamiltonian and shall concentrate only on the free motion of the system described by the kinetic energy. Again for simplicity, we take $\mathscr{M}=E^3$ and endow it with the Euclidean metric:

$$H=\tfrac{1}{2}\delta^{AB}P_AP_B,\quad G^{AB}=\delta^{AB}\ . \tag{10.1}$$

A. Translations $T(1)$ acting on E^3

Let the orbits of a one-parameter translation group acting on E^3, with coordinates $Q^A=(X,Y,Z)$, be

$$X(\tau)=X(0),\quad Y(\tau)=Y(0),\quad Z(\tau)=Z(0)+\tau\ . \tag{10.2}$$

The generator of translations, $\boldsymbol{\phi}=\partial/\partial Z$, yields the constraint $\pi=P_3$. Because translations are isometries of E^3, $\pounds_{\boldsymbol{\phi}}\mathbf{G}=0$ and

$$\{H,\pi\}=0\ . \tag{10.3}$$

One can take $x=X$ and $y=Y$ for the physical coordinates $q^a=(x,y)$. The physical metric is flat and the coordinates q^a are Cartesian: $g^{ab}=\delta^{ab}$. One can picture m as a plane embedded in E^3, perpendicular to the orbits, with the induced metric $g_{ab}=\delta_{ab}$.

Even this trivial example allows us to illustrate two points. The flat metric $G^{AB}=\delta^{AB}$ in the big space is equivalent to a curved metric

$$\bar{G}^{AB}=\delta^{AB}+\Sigma\phi^A\phi^B+\Omega^{(A}\phi^{B)} , \tag{10.4}$$

where $\Sigma(Q)$ is an arbitrary scalar and $\Omega^A(Q)$ an arbitrary vector which we choose to be orthogonal to ϕ^B: $\delta_{AB}\Omega^A\phi^B=0$. Hence comes our first point: a curved metric in the big space may very well induce a flat metric in the physical space. Another illustration of the same fact is any gauge system with $C=N-1$ independent constraints; the physical space is one dimensional, $n=1$ and hence flat.

The new Hamiltonian $\bar{H}=\frac{1}{2}\bar{G}^{AB}P_AP_B$ is now preserved by the constraints only weakly:

$$\{\bar{H},\pi\}=\Sigma_{,3}\pi^2+2\Omega^A{}_{,3}P_A\pi . \tag{10.5}$$

Also, and this is our second point: the physical metric cannot in general be interpreted as the metric induced by $\bar{G}$ on the surfaces perpendicular to the orbits. In fact, with respect to the new standard $\bar{G}$, the field ϕ is no longer necessarily hypersurface orthogonal. For example, the simple choice $\Sigma=0$, $\Omega^A=(-Y,0,0)$ leads to $\bar{\phi}_A=(Y,0,1+2Y^2)$, which does not satisfy the well-known integrability condition for hypersurface orthogonality:

$$\bar{\phi}\cdot\text{curl}\bar{\phi}=\delta^{ABC}\bar{\phi}_A\bar{\phi}_{C,B}=2Y^2-1\neq 0 . \tag{10.6}$$

B. Rotations SO(2) acting on E^3

Let SO(2) act on E^3 by

$$\begin{aligned} X(\tau)&=X(0)\cos\tau+Y(0)\sin\tau ,\\ Y(\tau)&=-X(0)\sin\tau+Y(0)\cos\tau ,\\ Z(\tau)&=Z(0) . \end{aligned} \tag{10.7}$$

The orbits are circles about the Z axis and the points on the Z axis. The vector field $\phi=Y\partial/\partial X-X\partial/\partial Y$ generates the orbits. One can choose $r=(X^2+Y^2)^{1/2}$ and $z=Z$ for the physical coordinates $q^a=(r,z)$. The physical metric $g^{ab}=\delta^{ab}$ is again flat, but the physical configuration space is geodesically incomplete. Due to the limitation $r\geq 0$, it is a half-plane with the boundary $r=0$.

The physical Hamiltonian $h=\frac{1}{2}(p_r{}^2+p_z{}^2)$ generates a uniform rectilinear motion in the half-plane $(r\geq 0,z)$. What happens when the particle reaches the boundary? The motion of the particle in the big space, which is governed by the Hamiltonian $H=\frac{1}{2}(P_X{}^2+P_Y{}^2+P_Z{}^2)$, is also uniform and rectilinear. However, the big space is complete, and the particle moving on the constraint surface $\pi=0$, i.e., with its velocity in a plane through the Z axis, does not meet any obstacles. When the motion is projected into the physical space, it yields an elastic bounce at $r=0$. The gauge origin of the physical motion thus provides a "hard wall" boundary condition at the edge of an incomplete physical space.

C. Translations $T(1)$ acting on E^3 by helical motions

Let the translation group act on E^3 by

$$\begin{aligned} X(\tau)&=X(0)\cos\tau+Y(0)\sin\tau ,\\ Y(\tau)&=-X(0)\sin\tau+Y(0)\cos\tau ,\\ Z(\tau)&=Z(0)+\tau . \end{aligned} \tag{10.8}$$

The orbits (10.8) are helices about the Z axis (and the Z axis itself). The generator of the action,

$$\phi=Y\partial/\partial X-X\partial/\partial Y+\partial/\partial Z , \tag{10.9}$$

is the sum of the rotational Killing vector field of Sec. X B with the translational Killing vector field of Sec. X A. Hence, it is also a Killing vector of the Euclidean metric $G^{AB}=\delta^{AB}$ and $\{H,\pi\}=0$.

In the cylindrical coordinates

$$\begin{aligned} &Q^{A'}=(R,\Theta,Z) ,\\ &X=R\cos\Theta,\quad Y=R\sin\Theta,\quad Z=Z , \end{aligned} \tag{10.10}$$

the metric takes the form

$$G_{A'B'}=\text{diag}(1,R^2,1) , \tag{10.11}$$

and the Killing field (10.9) becomes

$$\phi=\partial/\partial\Theta+\partial/\partial Z . \tag{10.12}$$

The equation

$$\partial q^a/\partial\Theta+\partial q^a/\partial Z=0 \tag{10.13}$$

for the physical coordinates q^a has two functionally independent solutions, e.g.,

$$q^1\equiv r=R,\quad q^2\equiv\theta=\Theta-Z . \tag{10.14}$$

Consider the ranges of r and θ. Because of $r=(X^2+Y^2)^{1/2}$, we have $r\in[0,\infty)$. In principle, $\theta\in(-\infty,\infty)$. However, the points in $\mathcal{M}$ differing in θ by 2π lie on the same helix and hence correspond to the same point in m. The coordinate θ is thus an angle variable and can be assigned the range $\theta\in[0,2\pi)$.

By projecting the metric (10.11) into the physical space, we obtain $g_{ab}=\text{diag}(1,r^2/(1+r^2))$. The line element has the form

$$ds^2=dr^2+\rho^2(r)d\theta^2 , \tag{10.15}$$

$$\rho(r)=r^2/(1+r^2) , \tag{10.16}$$

which indicates that (r,θ) is a cylindrical Gaussian system of coordinates. The line element (10.15) and (10.16) is regular throughout the whole range of coordinates, $r\in[0,\infty)$ and $\theta\in[0,2\pi)$, and describes a geodesically complete Riemannian manifold. For small r's, $r\ll 1$, the line element $ds^2\approx dr^2+r^2d\theta^2$ is flat, without any conical singularity. For large r's, $r\gg 1$, the line element is again flat, $ds^2\approx dr^2+d\theta^2$, corresponding to a cylindrical surface with embedding radius 1. The Gaussian curvature of the physical space, given by the only nonvanishing component

$$R_{1212}=-\tfrac{1}{2}\rho''(r)+\tfrac{1}{4}(\rho'(r))^2/\rho(r) \tag{10.17}$$

of the Riemann curvature tensor, is

$$R_{1212}=3r^2/(1+r^2)^3 . \tag{10.18}$$

Outside the asymptotic limits $r\rightarrow 0$ and $r\rightarrow\infty$, the physical space is curved. This exemplifies the point that a flat metric in the big configuration space may lead to a curved metric in the physical space.[14]

On passing to the curvature coordinates (ρ,θ) by the transformation (10.16), we cast the line element into the form

$$ds^2=(1-\rho^2)^{-3}d\rho^2+\rho^2 d\theta^2 . \tag{10.19}$$

One can visualize the curvature of the physical space by an embedding diagram, requiring that the geometry (10.19) be identical with that on a surface of revolution obtained by rotating the curve $z=f(\rho)$ about the axis Z in a fictitious Euclidean space with cylindrical coordinates (ρ,θ,Z):

$$ds^2=(1+f'^2(\rho))^{-3}d\rho^2+\rho^2 d\theta^2 . \tag{10.20}$$

To obtain $f(\rho)$ in a closed form is messy; the embedding diagram shows the resulting cylindrical vessel curving rather abruptly into a flattened bottom.

Other useful coordinates are those which bring the line element (10.15) into a conformally flat form

$$ds^2=\alpha^2(\sigma)(d\sigma^2+\sigma^2 d\theta^2) . \tag{10.21}$$

This is achieved by the transformation

$$\sigma=r^{-1}((1+r^2)^{1/2}-1)\exp((1+r^2)^{1/2}) , \tag{10.22}$$

which should be inverted, $r=r(\sigma)$, and substituted into

$$\alpha=(1+r^2)^{-1/2}r/\sigma \tag{10.23}$$

to yield the conformal factor $\alpha^2(\sigma)$.

Note that the physical geometry cannot be interpreted as the geometry induced by the flat metric in the big configuration space on surfaces perpendicular to the helical orbits of the translation group. Such surfaces do not exist because the Killing vector field (10.9) is not hypersurface orthogonal.[14] Indeed, by working in the Cartesian coordinates $Q^A=(X,Y,Z)$, one easily checks that

$$\boldsymbol{\phi}\cdot\mathrm{curl}\boldsymbol{\phi}\equiv\delta^{ABC}\phi_A\phi_{C,B}=2\neq 0 . \tag{10.24}$$

D. Translations $T(2)$ acting on E^3

This example is as trivial as that in Sec. X A. Still there are things to learn, because we have now two constraints instead of one.

Take $\boldsymbol{\phi}_1=\partial/\partial X$ and $\boldsymbol{\phi}_2=\partial/\partial Y$ as the generators of translations. The orbits are planes perpendicular to the Z axis. We have two constraints,

$$\pi_1=P_1 \quad\text{and}\quad \pi_2=P_2 , \tag{10.25}$$

which have a vanishing Poisson bracket because $T(2)$ is Abelian:

$$\{\pi_1,\pi_2\}=0 . \tag{10.26}$$

One can choose $q=Z$ for the physical coordinate q^1. The physical metric $g_{11}=1$ is, of course, flat and q is a Cartesian coordinate. Each of the two constraints (10.25) has a vanishing Poisson bracket with the Hamiltonian (10.1).

An interesting point arises, however, when we choose an anholonomic basis $\pi_{1'},\pi_{2'}$ in the algebra $\mathscr{V}$ of linear gauge variables, instead of the holonomic basis (10.25). In particular, let us scale the original basis into

$$\pi_{1'}=e^{Y}\pi_1, \quad \pi_{2'}=\Lambda(q,Y)\pi_2 , \tag{10.27}$$

where $\Lambda(x,Y)$ is some nonvanishing function of the physical coordinate q and the gauge coordinate Y. In the simplest case, Λ can depend only on q, e.g., $\Lambda(q)=1+\frac{1}{2}q^2$. The constraint mixing (10.27) is everywhere regular. The Poisson bracket of the scaled constraints no longer vanishes:

$$\{\pi_{1'},\pi_{2'}\}=\Lambda(q,Y)\pi_{1'} . \tag{10.28}$$

Our new basis has the structure functions

$$C^1{}_{12}=-C^1{}_{21}=\Lambda(q,Y), \quad C^2{}_{12}=-C^2{}_{21}=0 , \tag{10.29}$$

which are not constant. This simulates, at the level of the linear gauge constraints, the situation which we find in general relativity for the quadratic super-Hamiltonian constraints associated with "parametrization." The function $\Lambda(q,Y)$ plays here the same role as the contravariant metric does in the Poisson bracket between two super-Hamiltonians in general relativity. The Poisson bracket of the Hamiltonian with the scaled constraints also does not vanish outside the constraint surface. Therefore, even for this trivial model, a factor ordering problem arises upon quantization.

E. Rotations SO(3) acting on E^3

Let SO(3) act on E^3 by rotations about the origin of Cartesian coordinates $Q^A=(X,Y,Z)$. The generators of the action, viz.,

$$\begin{aligned}\boldsymbol{\phi}_1&=-Z\partial/\partial Y+Y\partial/\partial Z ,\\ \boldsymbol{\phi}_2&=Z\partial/\partial X-X\partial/\partial Z ,\\ \boldsymbol{\phi}_3&=-Y\partial/\partial X+X\partial/\partial Y ,\end{aligned} \tag{10.30}$$

lead to the familiar Poisson brackets

$$\{\pi_\alpha,\pi_\beta\}=\delta^{\gamma}{}_{\alpha\beta}\pi_\gamma \tag{10.31}$$

for the angular momentum π_α. The constraint $\pi_\alpha=0$ allows only motion with zero angular momentum, i.e., the radial lines.

The generators (10.30) are linearly dependent. The space $\mathscr{V}_Q$ of vectors $\boldsymbol{v}=v^\alpha(Q)\boldsymbol{\phi}_\alpha$ at Q is two dimensional (instead of three dimensional), and at the origin $Q^A=0$ it becomes trivial. A two-dimensional basis can be defined only in patches. For example, $\boldsymbol{\phi}_1$ and $\boldsymbol{\phi}_2$ can serve as basis fields everywhere, with the exception of the coordinate plane $Z=0$, on which they become linearly dependent. We have

$$\{\pi_1,\pi_2\}=-\frac{Y}{Z}\pi_1-\frac{X}{Z}\pi_2 . \tag{10.32}$$

The basis π_1,π_2 is thus characterized by structure functions

$$C^1{}_{12}=-C^2{}_{21}=\frac{Y}{Z},\quad C^2{}_{12}=-C^2{}_{21}=\frac{X}{Y}, \qquad (10.33)$$

instead of structure constants. In Sec. X D, structure functions were introduced by mixing the generators of the gauge group. Here, they arise by a selection of linearly independent generators of the group from a redundant set.

The orbits of SO(3) are spheres centered about the origin (and the origin itself). The radius

$$r=(X^2+Y^2+Z^2)^{1/2} \qquad (10.34)$$

can be chosen for the physical coordinate q^1. The projector $Q^1_A=\delta_{AB}Q^B/r$ leads to the flat physical metric $g^{11}=1$; hence, r is a Cartesian coordinate. The physical space is geodesically incomplete, due to the limitation $r\geq 0$ (Ref. 15). The equations of motion in the big space (together with the angular momentum constraint $\pi_\alpha=0$) predict uniform rectilinear motion along straight lines passing through the origin. The projection of this motion into the physical space again yields an elastic reflection at $r=0$.

ACKNOWLEDGMENTS

I want to thank Christopher Stone for reading the paper and suggesting a number of improvements in my formulations. The paper was partially supported by the NSF Grant No. PHY-85-03653 to the University of Utah.

[1]A canonical description of Maxwell's electrodynamics is given e.g., in R. Arnowitt, S. Deser, and C. W. Misner, in *Gravitation: An Introduction to Current Research,* edited by L. Witten (Wiley, New York, 1962); or in K. Kuchař, J. Math. Phys. 17, 801 (1976).

[2]The canonical formulation for the Yang-Mills theories is developed, e.g., in A. J. Hanson, T. Regge, and C. Teitelboim, *Constrained Hamiltonian Systems* (Accademia Nazionale dei Lincei, Rome, 1976); in J. M. Arms, J. Math. Phys. **20**, 443 (1979); or in L. D. Faddeev and A. A. Slavnov, *Gauge Fields: Introduction to Quantum Theory* (Benjamin, London, 1980).

[3]A classical account of the canonical formalism in general relativity is given by R. Arnowitt, S. Deser, and C. W. Misner, in *Gravitation: An Introduction to Current Research* (Ref. 1).

[4]The canonical formulation of the nonlinar σ model is discussed by C. Isham, in *Relativity, Groups, and Topology II,* edited by B. S. DeWitt and R. Stora (North-Holland, Amsterdam, 1984).

[5]This problem has a long history. See, e.g., P. G. Bergmann and A. Komar, Int. J. Theor. Phys. **5**, 15 (1972); and Ref. 6.

[6]C. J. Isham and K. V. Kuchař, Ann. Phys. (N.Y.) **164**, 288 (1985); **164**, 316 (1985).

[7]K. Kuchař, J. Math. Phys. **17**, 792 (1976).

[8]K. Kuchař, following paper, Phys. Rev. D **34**, 3044 (1986).

[9]A general approach to the systems whose constraints close with structure functions was developed, through the path integral formalism, by Faddeev, Popov, Batalin, Fradkin, and Vilkovisky. See, e.g., I. A. Batalin and G. A. Vilkovisky, J. Math. Phys. **26**, 172 (1985); or M. Henneaux, Phys. Rep. **126**, 1 (1985) for the bibliography.

[10]See, e.g., L. D. Faddeev and A. A. Slavnov, *Gauge Fields: Introduction to Quantum Theory* (Ref. 2).

[11]See, e.g., R. Abraham and J. E. Marsden, *Foundations of Mechanics,* 2nd ed. (Benjamin, London, 1978); and J. E. Marsden and T. Ratiu, Lett. Math. Phys. **11**, 161 (1986).

[12]The idea of projecting geometrical objects in general and the metric in particular with respect to a given congruence of lines or orbits recurs in different contexts in the general theory of relativity. Its simplest application is the split of the spacetime metric into the time interval and the spatial metric along the world lines of a family of observers [see, e.g., C. Møller, *The Theory of Relativity* (Clarendon Press, Oxford, 1972)]. In the same spirit, the metric in higher-dimensional Kaluza-Klein-type theories is projected into the physical spacetime metric and matter field components [cf. P. G. Bergmann, *Introduction to the Theory of Relativity* (Prentice-Hall, Engelwood Cliffs, 1942)]. B. S. DeWitt, Phys. Rev. **160**, 1113 (1967), introduced the process by which the metric in the "big space" $\mathscr{M}$ induces a metric in the space of orbits of a group acting on $\mathscr{M}$; this idea is further developed in B. S. DeWitt, in *Relativity: Proceedings of the Relativity Conference in the Midwest,* edited by M. Carmeli, S. I. Fickler, and L. Witten (Plenum, New York, 1970).

[13]Equations (9.6)–(9.8) generalize DeWitt's procedure of Ref. 12 in two respects: (1) The orbits are not necessarily those of a group and (2) the Poisson brackets (9.6) may vanish only weakly, modulo the constraints.

[14]This property of the helical model is mentioned in the footnote on p. 729 of the article by B. S. DeWitt, in *General Relativity: An Einstein Centenary Survey,* edited by S. W. Hawking and W. Israel (Cambridge University Press, Cambridge, 1979).

[15]The subtle points concerning the structure of the physical phase space at the origin $r=0$ will be discussed in a forthcoming paper by J. Arms and M. Gotay.