Stationary axially symmetric exterior solutions in the five-dimensional representation of the Brans-Dicke-Jordan theory of gravitation

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The inverse scattering method of Belinsky and Zakharov is used to investigate axially symmetric stationary vacuum soliton solutions in the five-dimensional representation of the Brans-Dicke-Jordan theory of gravitation, where the scalar field of the theory is an element of a five-dimensional metric. The resulting equations for the spacetime metric are similar to those of solitons in general relativity, while the scalar field generated is the product of a simple function of the coordinates and an already known scalar field solution. A family of solutions is considered that reduce, in the absence of rotation, to the five-dimensional form of a well-known Weyl-Levi Civita axially symmetric static vacuum solution. With a suitable choice of parameters, this static limit becomes equivalent to the spherically symmetric solution of the Brans-Dicke theory. An exact metric, in which the Kerr-scalar McIntosh solution is a special case, is given explicitly.

I. INTRODUCTION

The Brans-Dicke-Jordan (BDJ) scalar-tensor theory was first investigated, in connection with Dirac's large number hypothesis, by Jordan,[1] and, in connection with Mach's principle, By Brans and Dicke.[2,3] The theory was originally expressed in a representation in which the local measurable value of the gravitational "constant" G is a function of a scalar field.[1,2] It can also be put in a form which is Einstein's general relativity with the scalar field of the theory acting as an additional external nongravitational field.[3] Another way of representing the BDJ theory is with a five-dimensional field equation,[1,4,5] where the metric is independent of the fifth coordinate, and the elements $g_{\mu 4}$, where $\mu=(0,1,2,3)$ is a spacetime index, vanish. In a vacuum, this is just a special case of the Klein-Jordan-Thiry[6] theory, where the $g_{\mu 4}$ components are identified with an "electromagnetic" vector potential A_μ. The Klein-Jordan-Thiry theory is in turn a generalization of the original Kaluza-Klein unified theory of gravity and electromagnetism, in which $g_{44}=\text{const}$.

In order to study the physical implications of a theory, we must find solutions to its field equations. Fortunately, it has been possible to find exact solutions of relativistic gravitational theories, such as general relativity and the BDJ theory, when physically reasonable symmetries have been assumed. For instance, many astrophysical systems of interest are approximately stationary and axially symmetric, and can be very well described with metrices having these symmetries. It was recognized by McIntosh[7] that any stationary axially symmetric vacuum metric in general relativity could be used to generate another solution in the BDJ theory. Therefore the well-known Kerr solution, and any other vacuum solution with the same space-time symmetries, can be extended to the BDJ theory. Of particular interest are those solutions that reduce in the static limit to a class of Weyl-Levi Civita metrics, which have been studied by Zipoy[8] and Voorhees,[9] and interpreted as the exterior gravitational fields of oblate and prolate configurations. In general relativity, with the exception of the Schwarzschild case, these solutions are not spherically symmetric; however, the situation is different for their BDJ counterparts, since in these cases it is possible to adjust the parameters to achieve that space symmetry. Hence the generation of new solutions which in the static limit take this Weyl—Levi Civita form could be relevant to the study of exterior gravitational fields of stationary perfect fluids in the BDJ theory.

In this paper the inverse scattering method of Belinsky and Zakharov[10,11] (BZ) is used to construct axially symmetric stationary vacuum solutions to the BDJ field equations. In particular I shall consider solutions that reduce in the absence of rotation to the Zipoy-Voorhees Weyl—Levi Civita metric with a scalar field present. With an appropriate choice of parameters this static limit becomes equivalent to the well-known BDJ spherically symmetric solution. The analysis is carried out in a five-dimensional representation of the BDJ theory, where the application of the BZ technique is generalized in a straightforward way. Furthermore, this representation provides a basis for possible future applications of the BZ formalism to the problem of finding classing solutions of Kaluza-Klein-type field theories of more than four dimensions. We point out, however, that the BZ formalism could be applied directly to a particular four-dimensional representation for the BDJ theory (the Einstein-scalar theory), since for an axially symmetric stationary metric in a vacuum the scalar field decouples from the second-order field equations for the spacetime metric, and therefore these equations are identical to those of general relativity.[7]

In fact, Belinsky[12] studied exact solutions of the Einstein-scalar theory that describe the evolution of gravitational soliton waves against the background of Friedman cosmological models. A very interesting feature of this work is that the energy-momentum tensor of the scalar field is now representing the matter field of a perfect fluid with an equation of state pressure=energy density.

In Sec. II we will describe the Belinsky and Zakharov method in its essentials, without restricting the number of dimensions to four. In Sec. III the BDJ field equations in its five-dimensional representation (BDJ_5) are presented. Next, we applied the BZ technique to the BDJ_5 theory, while in Sec. V the diagonal form of the solution is considered. The BZ technique relies on the knowledge of a seed solution g^0 from which a more general one g_{ph} is generated; hence, in Sec. VI we consider conditions for the BZ solution, g_{ph}, to become equal to g^0. Sections VII and VIII deal with specific applications of the formulas developed in previous sections to the two cases where the background g^0 represents a flat spacetime and a Weyl—Levi Civita metric.

II. THE INVERSE SCATTERING METHOD OF BELINSKY AND ZAKHAROV

This method allows the generation of large classes of new solutions from old, when the metric tensor depends only on two variables. Belinsky and Zakharov[10,11] developed and employed the method to obtain exterior solutions in general relativity. In particular, they have obtained the Kerr solution[11] starting from a flat spacetime metric background. The technique can also be used in relativistic theories of more than four dimensions. This was done by Belinsky and Ruffini[13] in the framework of the five-dimensional Jordan-Thiry-Kaluza-Klein theory, to generate stationary axially symmetric solutions starting from a constant metric.

In what follows we will briefly describe the Belinsky and Zakharov technique. Their notation[11] will be followed in its essentials. For the sake of generality, we will not yet restrict the number of dimensions considered, since the generalization only introduces trivial modifications to the BZ formulas. Hence, let us consider a $(m+2)$-dimensional metric that depends only on two coordinates: ρ and z. The line element can be written in the Lewis form:[14]

$$\begin{aligned} dS^2 &= g_{AB}dX^A dX^B \\ &= f(\rho,z)(d\rho^2+dz^2)+g_{ab}(\rho,z)dX^a dX^b \ , \\ &\qquad\qquad a,b=1-m \ . \end{aligned} \tag{2.1}$$

Furthermore, the source-free Einstein equations in $m+2$ dimensions,

$$G_{AB}=R_{AB}-\tfrac{1}{2}g_{AB}R=0 \ , \tag{2.2}$$

admit the following coordinate condition:

$$\det g \equiv \det g_{ab} = -\rho^2 \ . \tag{2.3}$$

With the metric (2.1) and the condition (2.3) the field equation (2.2) takes the form

$$(\rho g_{,\rho}g^{-1})_{,\rho}+(\rho g_{,z}g^{-1})_{,z}\equiv U_{,\rho}+V_{,z}=0 \ , \tag{2.4}$$

$$(\ln f)_{,\rho}=-1/\rho+(1/4\rho)\mathrm{Tr}(U^2-V^2) \ , \tag{2.5}$$

$$(\ln f)_{,z}=(1/2\rho)\mathrm{Tr}UV \ , \tag{2.6}$$

where we used the conventional notation $(\ )_{,X}=(\partial/\partial X)(\ )$.

Equation (2.4) is the compatibility condition for the following system of linear equations:

$$D_1\psi \equiv \left[\frac{\partial}{\partial z}-\frac{2\lambda^2}{\lambda^2+\rho^2}\frac{\partial}{\partial\lambda}\right]\psi = \left[\frac{\rho V-\lambda U}{\lambda^2+\rho^2}\right]\psi \ , \tag{2.7}$$

$$D_2\psi = \left[\frac{\partial}{\partial \rho}+\frac{2\lambda\rho}{\lambda^2+\rho^2}\frac{\partial}{\partial\lambda}\right]\psi = \left[\frac{\rho U+\lambda V}{\lambda^2+\rho^2}\right]\psi \ , \tag{2.8}$$

where λ is a complex variable. Moreover, when $\lambda=0$, Eqs. (2.7) and (2.8) are just

$$\rho\psi_{,z}\psi^{-1}=V \ , \tag{2.9}$$

$$\rho\psi_{,\rho}\psi^{-1}=U \ , \tag{2.10}$$

which yield, after a suitable choice of the arbitrary constant matrix factor in $\psi(\lambda=0)$,

$$\psi(\lambda=0)=g \ . \tag{2.11}$$

The solitonic solution ψ is given as a function of a particular solution ψ_0, corresponding to a given background metric g^0, in the following way:

$$\psi=\psi_0+\sum_{k=1}^{n}\frac{R_k\psi_0}{\lambda-\mu_k} \ , \tag{2.12}$$

where

$$(R_k)_{ab}\equiv\sum_{l=1}^{n}\frac{\Gamma^{-1}{}_{l\kappa}M_c^k g^0_{ca}M_b^l}{\mu_l} \ , \tag{2.13}$$

$$M_a^k\equiv C_d^k[\psi_0{}^{-1}(k)]_{da} \ ; \tag{2.14}$$

C_d^k are arbitrary constants.

$\psi_0{}^1(k)$ is the inverse of ψ_0 evaluated at

$$\lambda=\mu_k\equiv W_k-z\pm[(W_k-z)^2+\rho^2]^{1/2} \ ; \tag{2.15}$$

W_k are arbitrary constants and $\Gamma^{-1}{}_{lk}$ are the elements of the inverse of the following matrix:

$$\Gamma_{lk}\equiv\frac{M_a^k g^0_{ab}M_b^l}{\mu_k\mu_l+\rho^2} \ . \tag{2.16}$$

The metric g is then given by

$$g=\psi(\lambda=0)=g^0-\sum_{k=1}^{n}\frac{R_k g^0}{\mu_k} \ . \tag{2.17}$$

It also follows, from Eqs. (2.5) and (2.6), that

$$f=C_n f_0\rho^n\prod_{k=1}^{n}(\mu_k)^2\prod_{k=1}^{n}(\mu_k{}^2+\rho^2)^{-1}\det\Gamma \ , \tag{2.18}$$

where f_0 is the solution corresponding to g^0, and C_n is a constant. Even though g satisfies Eq. (2.4), it is not a solution of the field equations (2.2), since now $\det g$ is not equal to $\det g^0=-\rho^2$, but instead equal to

$$\begin{aligned}\det g &= (-1)^n\prod_{k=1}^{n}(\rho/\mu_k)^2\det g^0 \\ &= \left[\prod_{k=1}^{n}(\rho/\mu_k)^2\right](-\rho^2) \ , \end{aligned} \tag{2.19}$$

where n is taken to be an even number in order to preserve the signature of g. However, the new metric

$$g_{\rm ph} \equiv \prod_{k=1}^{n} \left[\frac{\mu_\kappa}{\rho} \right]^{2/m} g \equiv \prod g \ , \tag{2.20}$$

is still a solution of Eq. (2.4) and, furthermore, satisfies the condition

$$\det g_{\rm ph} = -\rho^2 \ . \tag{2.21}$$

The function f is also modified by the transformation (2.20). The new function, $f_{\rm ph}$, is

$$f_{\rm ph} = \rho^{1/m} Q^{-2/m} f \ , \tag{2.22}$$

where

$$Q^{-1} = {\rm const} \times \rho^{-(n+1)^2/2} \prod_{k=1}^{n} (\mu_k)^{n-1} \prod_{k=1}^{n} (\mu_k{}^2 + \rho^2) \prod_{k>l}^{n} (\mu_k - \mu_l) \ . \tag{2.23}$$

The power $1/m$ is the only explicit reference to the dimensionality of the geometry. The Belinsky and Zakharov formulas of Ref. 11 are recovered simply by setting $m=2$.

III. THE BDJ THEORY IN FIVE DIMENSIONS

Let us introduce the following five-dimensional metric:

$$\tilde g_{AB} \equiv \begin{cases} \tilde g_{\mu\nu} = \bar g_{\mu\nu}(X^\mu), \ \mu,\nu = 0,1,2,3 \ , \\ \tilde g_{44} = \bar\phi^{\,2}(X^\mu) \ , \\ \tilde g_{4\nu} = \tilde g_{\nu 4} = 0 \ , \end{cases} \tag{3.1}$$

where $\bar g_{\mu\nu}$ and $\bar\phi$ are a four-dimensional metric tensor and a scalar field, respectively. Note that $\tilde g_{AB}$ is "static" with respect to the additional fifth dimension X^4. It is straightforward to show that[6]

$$\tilde G_{\mu\nu} = \bar G_{\mu\nu} - \frac{1}{\bar\phi} (\bar\phi_{;\mu\nu} - \bar g_{\mu\nu} \bar\phi^{;\alpha}_{;\alpha}) \ , \tag{3.2}$$

$$\tilde G_4{}^4 = \tfrac{1}{2} \bar R^{\mu}_{\mu} \ , \tag{3.3}$$

$$\tilde G_{\mu 4} = \tilde G_{4\mu} = 0 \ , \tag{3.4}$$

where $\tilde G_{AB}$ and $\bar G_{\mu\nu}$ are the Einstein tensors for the $\tilde g_{AB}$ and $\bar g_{\mu\nu}$ metric, respectively. The covariant derivatives are built with $\bar g_{\mu\nu}$.

Furthermore, with the conformal transformation

$$\bar g_{\mu\nu} = \frac{g_{\mu\nu}}{\bar\phi} \ , \tag{3.5}$$

$$\bar\phi = \phi^\alpha, \ \ \alpha = \left[\frac{2\omega+3}{3} \right]^{1/2} \ , \tag{3.6}$$

the BDJ field equations in the Einstein-scalar representation

$$G_{\mu\nu} = 8\pi T_{\mu\nu} + \frac{\omega + \frac{3}{2}}{\phi^2} (\phi_{,\mu}\phi_{,\nu} - \tfrac{1}{2} g_{\mu\nu} \phi^{,\alpha}\phi_{,\alpha}) \ , \tag{3.7}$$

$$(\ln\phi)^{;\alpha}_{;\alpha} = \frac{8\pi}{2\omega+3} T^{\mu}_{\mu} \ , \tag{3.8}$$

can be put in the form

$$\bar G_{\mu\nu} = \frac{8\pi \bar T_{\mu\nu}}{\bar\phi} + \frac{1}{\bar\phi} (\bar\phi_{;\mu\nu} - \bar g_{\mu\nu} \bar\phi^{;\alpha}_{;\alpha}) \ , \tag{3.9}$$

$$\bar\phi^{;\alpha}_{;\alpha} = \frac{8\pi \bar T^{\mu}_{\mu}}{[3(2\omega+3)]^{1/2}} \ , \tag{3.10}$$

where

$$\bar T_{\mu\nu} = \phi^\alpha T_{\mu\nu} \ . \tag{3.11}$$

Thus we have, using Eqs. (3.2) and (3.3),

$$\tilde G_{\mu\nu} = 8\pi \frac{\bar T_{\mu\nu}}{\bar\phi} \ , \tag{3.12}$$

$$\tilde G_4{}^4 = -4\pi (1 - \alpha^{-1}) \frac{\bar T^{\mu}_{\mu}}{\bar\phi} \ . \tag{3.13}$$

Finally, defining

$$\tilde T_{\mu\nu} \equiv \frac{\bar T_{\mu\nu}}{\bar\phi} \ , \tag{3.14}$$

and

$$\tilde T_4{}^4 \equiv -\frac{(1-\alpha^{-1})}{2} \frac{\bar T^{\mu}_{\mu}}{\bar\phi} \ , \tag{3.15}$$

we obtain the concise form

$$\tilde G_{AB} = 8\pi \tilde T_{AB} \ . \tag{3.16}$$

The field equations (3.16) are the five-dimensional representation of the BDJ theory that is equivalent to Einstein's equations for a five-dimensional metric which is "static" with respect to the additional non-space-time dimension (BDJ_5 theory).

For a specific application of the five-dimensional BDJ theory in vacuum, where the effect of a scalar field on the cosmological singularity is studied, see Belinsky and Khalatnikov.[4]

IV. APPLICATION OF THE BZ METHOD TO THE BDJ_5 THEORY

We can apply the BZ technique, described in Sec. II, to a five-dimensional vacuum metric $\tilde g_{AB}$ when this is a function of only two variables. Thus let us assume that the metric $\tilde g_{AB}$, Eq. (3.1), has the form (2.1):

$$d\tilde s^2 = \tilde g_{AB} dX^A dX^B = f(d\rho^2 + dz^2) + g_{ab} dX^a dX^b \ , \quad a,b = 1,2,3 \ , \tag{4.1}$$

where we made the identification

$$g_{3a}=\begin{cases}\tilde{g}_{44} & \text{if } a=3 ,\\ \tilde{g}_{4\mu}=0 & \text{if } a\neq 3 .\end{cases} \tag{4.2}$$

Then it follows that the metrices (2.20) and (2.22), with $m=3$,

$$\begin{aligned}(g_{\rm ph})_{ab}&=\prod_{k=1}^{n}\left[\frac{\mu_k}{\rho}\right]^{2/3} g_{ab}(\rho,z)\\ &=\prod\left[g^0_{ab}-\sum_{k,l=1}^{n}\Gamma^{-1}{}_{kl}\frac{M^k_d g^0_{da}M^l_c g^0_{cb}}{\mu_k\mu_l}\right] ,\end{aligned} \tag{4.3}$$

$$f_{\rm ph}=\rho^{1/3}Q^{-2/3}f , \tag{4.4}$$

are a solution of the BDJ$_5$ theory in a vacuum, $\tilde{G}_{AB}=0$, if the background metric g^0_{ab} is also a solution, and, furthermore,

$$g_{3a}=g^0_{3a}-\sum_{k,l=1}^{n}\Gamma^{-1}{}_{kl}\frac{M^k_d g^0_{da}M^l_c g^0_{c3}}{\mu_k\mu_l}=0, \quad a\neq 3 . \tag{4.5}$$

We now consider a sufficient condition for Eq. (4.5) to be valid. We see that if

$$M^k_3=0, \quad k\leq q , \tag{4.6}$$

$$M^k_a=0, \quad k>q, \quad a\neq 3 , \tag{4.7}$$

then the matrix

$$\Gamma_{kl}=\frac{M^k_c g^0_{cb}M^l_b}{\mu_k\mu_l+\rho^2}$$

takes the block form

$$\Gamma=\begin{bmatrix}\Gamma_1 & 0\\ 0 & \Gamma_2\end{bmatrix} , \tag{4.8}$$

where

$$\Gamma_1\equiv\begin{bmatrix}\Gamma_{11} & \Gamma_{12} & \cdots & \Gamma_{1q}\\ \Gamma_{21} & & & \vdots\\ \vdots & & & \vdots\\ \Gamma_{q1} & \Gamma_{q2} & \cdots & \Gamma_{qq}\end{bmatrix} , \tag{4.9}$$

$$\Gamma_2\equiv\begin{bmatrix}\Gamma_{q+1q+1} & \cdots & \Gamma_{q+1n}\\ \vdots & & \vdots\\ \Gamma_{nq+1} & \cdots & \Gamma_{nn}\end{bmatrix} . \tag{4.10}$$

Hence

$$\Gamma^{-1}=\begin{bmatrix}\Gamma_1{}^{-1} & 0\\ 0 & \Gamma_2{}^{-1}\end{bmatrix} , \tag{4.11}$$

which implies that

$$\Gamma^{-1}{}_{kl}M^k_d g^0_{da}M^l_3=0, \quad a\neq 3 \tag{4.12}$$

or, equivalently,

$$g_{a3}=g^0_{a3}=0, \quad a\neq 3 . \tag{4.13}$$

Therefore, we will assume that M^k_a satisfies Eqs. (4.6) and (4.7).

We now explore further consequences of this assumption. We know that

$$M^k_b\equiv C^k_c\psi_{0cb}{}^{-1} . \tag{4.14}$$

Then the conditions (4.6) and (4.7) can be implemented if we choose

$$\psi_{0c3}=0, \quad c\neq 3 \tag{4.15}$$

and

$$C^k_3=0, \quad k\leq q , \tag{4.16}$$

$$C^k_a=0, \quad k>q, \quad a\neq 3 . \tag{4.17}$$

Note that the assumption in Eq. (4.15) is consistent with Eqs. (2.7) and (2.8), if the matrix g^0 satisfies Eq. (4.13). Hence, we will assume the validity of Eqs. (4.15)–(4.17).

The metric (4.3) takes the form

$$g_{\rm ph}=\prod\begin{bmatrix}g_{11} & g_{12} & 0\\ g_{21} & g_{22} & 0\\ 0 & 0 & g_{33}\end{bmatrix}\equiv\begin{bmatrix}\bar{g} & 0\\ 0 & (g_{\rm ph})_{33}\end{bmatrix} . \tag{4.18}$$

With the metric in the form (4.18), the field equations (2.4) become

$$(\rho\bar{g}_{,\rho}\bar{g}^{-1})_{,\rho}+(\rho\bar{g}_{,z}\bar{g}^{-1})_{,z}=0 , \tag{4.19}$$

$$\{\rho[\ln(g_{\rm ph})_{33}]_{,\rho}\}_{,\rho}+\{\rho[\ln(g_{\rm ph})_{33}]_{,z}\}_{,z}=0 . \tag{4.20}$$

Equation (4.20) is equivalent to Laplace's equation with cylindrical symmetry. Equation (4.19) has the same form as the second-order equation for a metric of the type of Eq. (2.1) in general relativity. However, we see, from Eqs. (2.21) and (4.18), that

$$\det\bar{g}=\frac{-\rho^2}{(g_{\rm ph})_{33}} , \tag{4.21}$$

while a corresponding 2×2 metric in general relativity satisfies

$$\det g_{\rm gr}=-\rho^2 . \tag{4.22}$$

Therefore, a solution $g_{\rm gr}$ in general relativity is not a solution in the BDJ$_5$ theory. Nevertheless, we can use $g_{\rm gr}$ to build a solution $\bar{g}$, in the following way:

$$\bar{g}=\frac{g_{\rm gr}}{(g_{\rm ph})_{33}{}^{1/2}} . \tag{4.23}$$

Then we see that indeed $\bar{g}$, as given by the expression (4.23), is a solution of Eq. (4.19), if $g_{\rm gr}$ is also, since

$$\begin{aligned}(\rho\bar{g}_{,\rho}\bar{g}^{-1})_{,\rho}+(\rho\bar{g}_{,z}\bar{g}^{-1})_{,z}&=(\rho g_{{\rm gr},\rho}g_{\rm gr}{}^{-1})_{,\rho}+(\rho g_{{\rm gr},z}g_{\rm gr}{}^{-1})_{,z}\\ &\quad-\tfrac{1}{2}\{\rho[\ln(g_{\rm ph})_{33}]_{,\rho}\}_{,\rho}\\ &\quad-\tfrac{1}{2}\{\rho[\ln(g_{\rm ph})_{33}]_{,z}\}_{,z}=0 ,\end{aligned} \tag{4.24}$$

where we used Eq. (4.20). Furthermore, Eq. (4.23) implies that

$$\det\bar{g}=\frac{\det g_{\rm ph}}{(g_{\rm ph})_{33}}=\frac{-\rho^2}{(g_{\rm ph})_{33}}\ , \tag{4.25}$$

and therefore

$$\det g_{\rm ph}=(g_{\rm ph})_{33}\det\bar{g}=-\rho^2\ , \tag{4.26}$$

as required by Eq. (2.3).

Summarizing, we have the following result. Given a solution $g_{\rm gr}$ of the general relativity axially symmetric and stationary vacuum field equations, we get a corresponding solution in the BDJ_5 theory

$$g_{\rm ph}=\begin{bmatrix} g_{\rm gr}(g_{\rm ph})_{33}{}^{-1/2} & 0 \\ 0 & (g_{\rm ph})_{33}\end{bmatrix}\ , \tag{4.27}$$

where $\ln(g_{\rm ph})_{33}$ is a solution of Laplace's equation (4.20). The equivalent result in the four-dimensional representation has been given by McIntosh.[7] It is also possible to generalize this conclusion to the case when electromagnetic field sources are present.[5,15]

Let us return to Eq. (4.3) and make use of the implications of conditions (4.15)–(4.17). We obtain

$$(g_{\rm ph})_{ab}=\prod\left[g^0_{ab}-\sum_{k,l=1}^{q}\Gamma_1{}^{-1}{}_{kl}\frac{M^k_c g^0_{ca}M^l_d g^0_{db}}{\mu_k\mu_l}-\sum_{k,l>q}^{n}\Gamma_2{}^{-1}{}_{kl}\frac{M^k_3 g^0_{3a}M^l_3 g^0_{3b}}{\mu_k\mu_l}\right]\ . \tag{4.28}$$

Thus we have

$$(g_{\rm ph})_{ab}=\prod\left[g^0_{ab}-\sum_{k,l=1}^{q}\Gamma_1{}^{-1}{}_{kl}\frac{M^k_c g^0_{ca}M^l_d g^0_{db}}{\mu_k\mu_l}\right]\ ,\quad a,b\neq 3\ , \tag{4.29}$$

$$(g_{\rm ph})_{33}=\prod\left[g^0_{33}-\sum_{k,l>q}^{n}\Gamma_2{}^{-1}{}_{kl}\frac{M^k_3(g^0_{33})^2M^l_3}{\mu_k\mu_l}\right]\ , \tag{4.30}$$

$$(g_{\rm ph})_{3a}=(g_{\rm ph})_{a3}=0,\quad a\neq 3\ . \tag{4.31}$$

The component $(g_{\rm ph})_{33}$ can be reexpressed in a simpler form. In order to do this, we note that

$$\Gamma_{2kl}=\frac{M^k_3 g^0_{33}M^l_3}{\mu_k\mu_l+\rho^2}\ , \tag{4.32}$$

which implies that

$$\Gamma_2{}^{-1}{}_{kl}=\frac{A^{-1}{}_{kl}}{M^k_3 g^0_{33}M^l_3},\quad A_{kl}\equiv\frac{1}{\mu_k\mu_l+\rho^2}\ . \tag{4.33}$$

Consequently, Eq. (4.30) becomes

$$(g_{\rm ph})_{33}=\prod\left[1-\sum_{k,l>q}^{n}\frac{A^{-1}{}_{kl}}{\mu_k\mu_l}\right]g^0_{33}=\prod\left[1-\sum_{k,l>q}^{n}\bar{A}^{-1}{}_{kl}\right]g^0_{33}\ , \tag{4.34}$$

where

$$\bar{A}_{kl}\equiv\frac{\mu_k\mu_l}{\mu_k\mu_l+\rho^2}\ . \tag{4.35}$$

On the other hand, we know the following general result for a nonsingular matrix N_{kl}:

$$\det(1+N_{kl})=\det N_{kl}+(\det N_{kl})\sum_{k,l}N^{-1}{}_{kl}\ , \tag{4.36}$$

from which we obtain

$$1-\sum_{k,l>q}^{n}\bar{A}^{-1}{}_{kl}=\frac{\det(1-\bar{A}_{kl})}{\det(-\bar{A}_{kl})}=(-1)^{n-q}\frac{\det\left[\dfrac{\rho^2}{\mu_k\mu_l+\rho^2}\right]}{\det\left[\dfrac{\mu_k\mu_l}{\mu_k\mu_l+\rho^2}\right]}=(-1)^{n-q}\prod_{n>q}^{n}\frac{\rho^2}{\mu_k{}^2}\ . \tag{4.37}$$

Therefore, substituting Eq. (4.37) into Eq. (4.34), we get

$$(g_{\rm ph})_{33}=\prod(-1)^{n-q}\left[\prod_{k>q}^{n}\frac{\rho^2}{\mu_k{}^2}\right]g^0_{33}=\prod_{k>q}^{n}\left[\frac{\mu_k}{\rho}\right]^{2/3}\left[\prod_{k>q}^{n}\frac{\rho^2}{\mu_k{}^2}\right]g^0_{33}\ , \tag{4.38}$$

where n and q have been chosen as even numbers, in order to preserve the signature of the metric. Thus, we see that the expression for the scalar field $(g_{\rm ph})_{33}$ involves in a very simple way the background g^0_{33}, the poles μ_k, and is independent of the matrix ψ_0.

V. THE DIAGONAL FORM OF THE SOLUTION

In order to study the nonrotational limit of the solutions, we will consider in this section the case where $g_{12}=g_{12}=0$. We can diagonalize the metric g_{ab} using the same procedure that led to the block form (4.18). That is, let us assume that g^0 and ψ_0 are diagonal and, furthermore,

$$C^k_1=0\ \text{ if } k>s\ , \tag{5.1}$$

$$C^k_2=0\ \text{ if } k\leq s\ \text{ or } k>q\ , \tag{5.2}$$

where $s\leq q$, is a positive integer. Then it follows that

$$M^k_1=C^k_c\psi_{0c1}{}^{-1}=C^k_1\psi_{011}{}^{-1}=0\ \text{ if } k>s\ , \tag{5.3}$$

$$M^k_2=C^k_2\psi_{022}{}^{-1}=0,\ \text{ if } k\leq s\ \text{ or } k>q\ , \tag{5.4}$$

and therefore

$$\Gamma_{1kl}=(\Gamma_{11})_{kl}+(\Gamma_{22})_{kl}\ , \tag{5.5}$$

where

$$(\Gamma_{11})_{kl}=\begin{cases}0 & \text{if } k \text{ or } l>s\ ,\\ \Gamma_{1kl} & \text{if } k \text{ and } l\leq s\end{cases} \tag{5.6}$$

and

$$(\Gamma_{22})_{kl}=\begin{cases}0 \text{ if } (k \text{ or } l)\le s \text{ or } (k \text{ or } l)>q \ , \\ \Gamma_{1kl} \text{ if } s<(k \text{ and } l)\le q \ .\end{cases} \tag{5.7}$$

Hence, Γ takes the block form

$$\Gamma=\begin{bmatrix}\Gamma_{11} & 0 & 0\\ 0 & \Gamma_{22} & 0\\ 0 & 0 & \Gamma_{33}\end{bmatrix} , \tag{5.8}$$

where we set

$$\Gamma_2\equiv\Gamma_{33} \ . \tag{5.9}$$

We find, from Eq. (4.29), that $g_{\rm ph}$ simplifies to

$$(g_{\rm ph})_{ab}=0, \quad a\neq b \ , \tag{5.10}$$

$$(g_{\rm ph})_{11}=\prod\left[g_{11}^0-\sum_{k,l=1}^{s}\Gamma_{11}{}^{-1}{}_{kl}\frac{M_1^kM_1^l(g_{11}^0)^2}{\mu_k\mu_l}\right]$$
$$=\prod\left[\prod_{k=1}^{s}\frac{\rho^2}{\mu_k^2}\right]g_{11}^0 \ , \tag{5.11}$$

$$(g_{\rm ph})_{22}=\prod\left[g_{22}^0-\sum_{k,l>s}^{q}\Gamma_{22}{}^{-1}{}_{kl}\frac{M_2^kM_2^l(g_{22}^0)^2}{\mu_k\mu_l}\right]$$
$$=\prod\left[\prod_{k>s}^{q}\frac{\rho^2}{\mu_k^2}\right]g_{22}^0 \ , \tag{5.12}$$

and also, from Eq. (4.38),

$$(g_{\rm ph})_{33}=\prod\left[\prod_{k>q}^{n}\frac{\rho^2}{\mu_k^2}\right]g_{33}^0 \ ,$$

where the expressions at the extreme right in Eqs. (5.11) and (5.12) follow in the same way we obtained Eq. (4.38) in Sec. IV and we have chosen s an even number to preserve the signature of g_{ab}.

To get an explicit expression for the metric component $f_{\rm ph}$, corresponding to the above diagonal metric, we will calculate the determinant of Γ_{kl}. We find

$$\det\Gamma=\det\Gamma_{11}\det\Gamma_{22}\det\Gamma_{33} \ , \tag{5.13}$$

$$\det\Gamma_{11}=\det\left[\frac{M_1^kM_1^lg_{11}^0}{\mu_k\mu_l+\rho^2}\right]$$
$$=\left[\prod_{k=1}^{s}(M_1^k)^2g_{11}^0\right]\det\frac{1}{\mu_k\mu_l+\rho^2} \ , \tag{5.14}$$

$$\det\Gamma_{22}=\det\left[\frac{M_2^kM_2^lg_{22}^0}{\mu_k\mu_l+\rho^2}\right]$$
$$=\left[\prod_{k>s}^{q}(M_2^k)^2g_{22}^0\right]\det\frac{1}{\mu_k\mu_l+\rho^2} \ , \tag{5.15}$$

$$\det\Gamma_{33}=\det\left[\frac{M_3^kM_3^lg_{33}^0}{\mu_k\mu_l+\rho^2}\right]$$
$$=\left[\prod_{k>q}^{n}(M_3^k)^2g_{33}^0\right]\det\frac{1}{\mu_k\mu_l+\rho^2} \ . \tag{5.16}$$

Using the following result, shown in Appendix A,

$$\det\frac{1}{1+A_kB_l}=\frac{\prod_{k>l}(A_k-A_l)(B_l-B_k)}{\prod_{k,l}(1+A_kB_l)} \ , \tag{5.17}$$

we can see that, for m poles μ_k,

$$\det\frac{1}{\mu_k\mu_l+\rho^2}=\frac{\rho^{m(m-1)}\prod_{k\neq l}(\mu_k-\mu_l)}{\prod_{k,l}(\mu_k\mu_l+\rho^2)} \ . \tag{5.18}$$

Therefore

$$\det\Gamma_{11}=\left[\prod_{k=1}^{s}(M_1^k)^2g_{11}^0\right]\frac{\rho^{s(s-1)}\prod_{k\neq l}^{s}(\mu_k-\mu_l)}{\prod_{k,l}^{s}(\mu_k\mu_l+\rho^2)} \ , \tag{5.19}$$

$$\det\Gamma_{22}=\left[\prod_{k>s}^{q}(M_2^k)^2g_{22}^0\right]\times\frac{\rho^{(q-s)(q-s-1)}\prod_{k\neq l>s}^{q}(\mu_k-\mu_l)}{\prod_{k,l>s}^{q}(\mu_k\mu_l+\rho^2)} \ , \tag{5.20}$$

$$\det\Gamma_{33}=\left[\prod_{k>q}^{n}(M_3^k)^2g_{33}^0)\right]\times\frac{\rho^{(n-q)(n-q-1)}\prod_{k\neq l>q}^{n}(\mu_k-\mu_l)}{\prod_{k,l>q}^{n}(\mu_k\mu_l+\rho^2)} \ . \tag{5.21}$$

Substitution of Eqs. (5.19)–(5.21) in Eq. (5.13), and this in Eq. (4.4), gives

$$f_{\rm ph}=C_nf_0\rho^{1/3}Q^{-2/3}\rho^n\left[\prod_{k=1}^{n}\mu_k{}^2\right]\left[\prod_{k=1}^{s}(M_1^k)^2g_{11}^0\right]\left[\prod_{k>s}^{q}(M_2^k)^2g_{22}^0\right]\left[\prod_{k>q}^{n}(M_3^k)^2g_{33}^0\right]$$
$$\times\left[\frac{\rho^{s(s-1)}\prod_{k\neq l}^{s}(\mu_k-\mu_l)}{\prod_{k=1}^{s}(\mu_k{}^2+\rho^2)\prod_{k,l=1}^{s}(\mu_k\mu_l+\rho^2)}\right]\left[\frac{\rho^{(q-s)(q-s-1)}\prod_{k\neq l>s}^{q}(\mu_k-\mu_l)}{\prod_{k>s}^{q}(\mu_k{}^2+\rho^2)\prod_{k,l>s}^{q}(\mu_k\mu_l+\rho^2)}\right]\left[\frac{\rho^{(n-q)(n-q-1)}\prod_{k\neq l>q}^{n}(\mu_k-\mu_l)}{\prod_{k>q}^{n}(\mu_k{}^2+\rho^2)\prod_{k,l>q}^{n}(\mu_k\mu_l+\rho^2)}\right] . \tag{5.22}$$

VI. THE LIMIT $g_{\rm ph} \to g^0$

It is important to study under what circumstances the solution $g_{\rm ph}$ reduces to the background metric g^0. For example, it is useful to investigate the new solution $g_{\rm ph}$ "near" the background when the physical interpretation of g^0 is well known.

We can easily show that the diagonal elements, Eqs. (4.38), (5.11), and (5.12), reduce to g^0 if either one of the following two conditions is satisfied: (i) $s=q-s=n-q$ and $\mu_k=\mu_{s+k}=\mu_{2s+k}$, $k\le s=n/3$; (ii) In each group of poles, $1\le k\le s$, $s<k\le q$, $q<k\le n$, the poles come in pairs $\mu_k,\mu_{k'}$, such that

$$\mu_k=W_k-z+[(W_k-z)^2+\rho^2]^{1/2}\,, \tag{6.1}$$

$$\mu_k=W_k-z-[(W_k-z)^2+\rho^2]^{1/2}\,. \tag{6.2}$$

We see that the condition (i) implies

$$\prod_{k>q}^{n}\frac{\rho^2}{\mu_k{}^2}=\prod_{k>s}^{q}\frac{\rho^2}{\mu_k{}^2}=\prod_{k=1}^{s}\frac{\rho^2}{\mu_k{}^2}=\prod_{k=1}^{n}\left[\frac{\rho}{\mu_k}\right]^{2/3}=\frac{1}{\Pi}\,. \tag{6.3}$$

Also from assumption (ii) it follows that

$$\mu_k\mu_{k'}=-\rho^2\,, \tag{6.4}$$

$$\mu_k+\mu_{k'}=2(W_k-z)\,, \tag{6.5}$$

and therefore, using Eq. (6.4),

$$\prod_{k=1}^{s}\frac{\rho^2}{\mu_k{}^2}=\prod_{k>s}^{q}\frac{\rho^2}{\mu_k{}^2}=\prod_{k>q}^{n}\frac{\rho^2}{\mu_k{}^2}=\Pi=1\,. \tag{6.6}$$

Hence, in both cases (i) and (ii), $g_{\rm ph}=g^0$, since the coefficients of g_{ab} in Eqs. (4.38), (5.11), and (5.12) are unity.

We also expect that, when $g_{\rm ph}=g^0$, then $f_{\rm ph}=f_0$. In Appendix B it is shown explicitly that when $g_{\rm ph}$ is diagonal, then for each one of the conditions (i) and (ii), indeed we have $f_{\rm ph}=f_0$, with a suitable choice of the constant of integration.

VII. SOLUTIONS WITH PSEUDO-EUCLIDEAN BACKGROUND

The simplest application of the Belinsky and Zakharov method to the BDJ$_5$ theory is the generation of solutions using a flat spacetime metric with a constant scalar field as background:

$$g^0=\begin{bmatrix}\rho^2 & 0 & 0\\ 0 & -1 & 0\\ 0 & 0 & 1\end{bmatrix}\,, \tag{7.1}$$

$$f_0=1\,, \tag{7.2}$$

where the constant scalar field g^0_{33} has been normalized to unity. A particular solution ψ_0, corresponding to the metric (7.1), is

$$\psi_0(\lambda)=\begin{bmatrix}-\lambda W & 0 & 0\\ 0 & -1 & 0\\ 0 & 0 & 1\end{bmatrix},\quad \lambda W\equiv\lambda^2+2\lambda z-\rho^2\,. \tag{7.3}$$

We can get new soliton solutions $g_{\rm ph},f_{\rm ph}$ in the BDJ$_5$ theory in a straightforward way, using Eqs. (7.1)–(7.3) in Eqs. (4.4), (4.29), and (4.38).

The problem of creating new soliton solutions in general relativity from a flat spacetime metric has been discussed in detail by Belinsky and Zakharov.[10,11] In particular, they considered the two-soliton case to build the Kerr metric. These solutions can readily be transformed into BDJ$_5$ solutions using the prescription in Eq. (4.27). Hence an alternative way of generating solutions from flat spacetime is to construct them as in general relativity and then use Eq. (4.27), with the scalar field $(g_{\rm ph})_{33}$ given by

$$(g_{\rm ph})_{33}=\left[\prod_{k=1}^{n}\left[\frac{\mu_k}{\rho}\right]^{2/3}\right]\prod_{k>q}^{n}\frac{\rho^2}{\mu_k{}^2},\quad g^0_{33}=1\,, \tag{7.4}$$

or any other solution of Laplace's equation (4.20). For instance, the Kerr solution can be transformed to get a rotating solution in the BDJ theory. However, this solution does not become spherically symmetric in the absence of rotation, if the scalar field is nonconstant, as we shall see below.

The Brans-Dicke-Jordan spherically symmetric static vacuum solution, $g_{\rm sp},f_{\rm sp}$, in the five-dimensional representation (Appendix C) takes the form

$$(g_{\rm sp})_{11}=r^2\left[1-\frac{2\beta}{r}\right]^{1-\delta-\nu}\sin^2\theta\,, \tag{7.5}$$

$$(g_{\rm sp})_{22}=-\left[1-\frac{2\beta}{r}\right]^{\delta-\nu}\,, \tag{7.6}$$

$$(g_{\rm sp})_{33}=\left[1-\frac{2\beta}{r}\right]^{2\nu}\,, \tag{7.7}$$

$$f_{\rm sp}=\left[1-\frac{2\beta}{r}\right]^{1-\delta-\nu}\left[1-\frac{2\beta}{r}+\frac{\beta^2}{r^2}\sin^2\theta\right]^{-1}\,, \tag{7.8}$$

where δ and ν are constants related by

$$\delta^2+3\nu^2=1\,, \tag{7.9}$$

and $(\delta-\nu/\alpha)\beta$ must be identified with the gravitational mass of the system (Appendix C). The spherical coordinates r,θ, are related to the cylindrical coordinates ρ,z via

$$\rho=r\left[1-\frac{2\beta}{r}\right]^{1/2}\sin\theta\,, \tag{7.10}$$

$$z=(r-\beta)\cos\theta\,. \tag{7.11}$$

We can reexpress $g_{\rm sp}$ in terms of ρ and z:

$$(g_{\rm sp})_{11}=\left[\frac{-\mu\bar\mu}{\rho^2}\right]^{-\delta-\nu}\rho^2\,, \tag{7.12}$$

$$(g_{\rm sp})_{22}=\left[\frac{-\mu\bar\mu}{\rho^2}\right]^{\delta-\nu}\,, \tag{7.13}$$

$$(g_{\rm sp})_{33}=\left[\frac{-\mu\bar\mu}{\rho^2}\right]^{2\nu}\,, \tag{7.14}$$

where the functions μ and $\bar\mu$ are just the poles, with $W_k=-\beta$ and $+\beta$:

$$\mu \equiv -(\beta+z)+[(\beta+z)^2+\rho^2]^{1/2} = (r-2\beta)(1-\cos\theta) , \quad (7.15)$$

$$\bar{\mu} \equiv (\beta-z)-[(\beta-z)^2+\rho^2]^{1/2} = -(r-2\beta)(1+\cos\theta) , \quad (7.16)$$

On the other hand, according to Eq. (4.27) the g_{ph} BDJ$_5$ solution generated from the Kerr solution background g_{Kerr} is given by

$$g_{\text{ph}} = \begin{bmatrix} (g_{\text{ph}})_{33}{}^{-1/2} g_{\text{Kerr}} & 0 \\ 0 & (g_{\text{ph}})_{33} \end{bmatrix} , \quad (7.17)$$

and since in the absence of rotation the Kerr solution becomes the Schwarzschild solution

$$g_{\text{Sc}} = \begin{bmatrix} r^2 \sin^2\theta & 0 \\ 0 & -\left[1-\frac{2\beta}{r}\right] \end{bmatrix} , \quad f_{\text{Sc}} = \left[1-\frac{2\beta}{r}+\frac{\beta^2}{r^2}\sin^2\theta\right]^{-1} , \quad (7.18)$$

then we have that, without rotation,

$$g_{\text{ph}} = \begin{bmatrix} (g_{\text{ph}})_{33}{}^{-1/2} g_{\text{Sc}} & 0 \\ 0 & (g_{\text{ph}})_{33} \end{bmatrix} . \quad (7.19)$$

To compare the metrices (7.5)—(7.7) with Eqs. (7.19) we must identify $(g_{\text{ph}})_{33}$ with $(1-2\beta/r)^{2\nu}$. Thus we get

$$g_{\text{ph}} = \begin{bmatrix} r^2\left[1-\frac{2\beta}{r}\right]^{-\nu}\sin^2\theta & 0 & 0 \\ 0 & -\left[1-\frac{2\beta}{r}\right]^{1-\nu} & \\ 0 & 0 & \left[1-\frac{2\beta}{r}\right]^{2\nu} \end{bmatrix} , \quad (7.20)$$

which is just like g_{sp} but with $\delta=1$. Therefore the condition $\delta^2+3\nu^2=1$, can only be satisfied if $\nu=0$, or equivalently when

$$(g_{\text{ph}})_{33} = \text{const} . \quad (7.21)$$

Conversely, if the scalar field $(g_{\text{ph}})_{33}$ is not a constant the solution (7.20) does not represent a spherically symmetric configuration.

More generally, if half of the poles, μ_k (k odd) are chosen equal to μ (or $-\rho^2/\mu$) and the other half equal to $\bar{\mu}$ (or $-\rho^2/\bar{\mu}$) then the diagonal solution, Eqs. (4.38), (5.11), and (5.12), takes the form

$$(g_{\text{ph}})_{11} = \left[\frac{-\mu\bar{\mu}}{\rho^2}\right]^{\pm(n/3-s)} \rho^2 , \quad (7.22)$$

$$(g_{\text{ph}})_{22} = -\left[\frac{-\mu\bar{\mu}}{\rho^2}\right]^{\pm(n/3-q+s)} , \quad (7.23)$$

$$(g_{\text{ph}})_{33} = \left[\frac{-\mu\bar{\mu}}{\rho^2}\right]^{\pm(-2n/3+q)} , \quad (7.24)$$

which is similar to the functional form of g_{sp}. Nevertheless, in order for g_{ph} to be equal to g_{sp} we must identify

$$\delta+\nu = \pm\left[s-\frac{n}{3}\right] , \quad \delta-\nu = \pm\left[\frac{n}{3}-q+s\right] , \quad 2\nu = \pm\left[-\tfrac{2}{3}n+q\right] , \quad (7.25)$$

or, equivalently,

$$\delta = \pm(s-\tfrac{1}{2}q) , \quad \nu = \pm(\tfrac{1}{2}q-\tfrac{1}{3}n) , \quad (7.26)$$

but again, $\delta^2+3\nu^2$ cannot be unity unless ν vanishes.

Thus it seems that to obtain the spherically symmetric solution as a diagonal limit of a soliton metric we must start with a nonflat metric g^0. In particular, if we put $g^0=g_{\text{sp}}$, it is possible to obtain the reduction $g_{\text{ph}}\to g_{\text{sp}}$ when either of the conditions (i) or (ii), discussed in Sec. VI, are satisfied. Another possibility is to start with a static axially symmetric but not necessarily spherical background solution, and, instead of imposing conditions (i) or (ii), require that the diagonal metric generated is equal to g_{sp}. The study of solutions of this type, where g^0 is given by the BDJ$_5$ generalization of a well-known Weyl—Levi Civita solution of general relativity, is discussed in the next section.

VIII. SOLUTIONS WITH A WEYL—LEVI CIVITA BACKGROUND

A solution of the BDJ$_5$ field equations that describe the exterior gravitational field of a static axially symmetric configuration is given by (Appendix C)

$$g^0_{11} = \rho^2\left[1-\frac{2\beta}{r}\right]^{-\delta-\nu} = r^2\left[1-\frac{2\beta}{r}\right]^{1-\delta-\nu}\sin^2\theta , \quad (8.1)$$

$$g^0_{22}=-\left[1-\frac{2\beta}{r}\right]^{\delta-\nu}, \tag{8.2}$$

$$g^0_{33}=\left[1-\frac{2\beta}{r}\right]^{2\nu}, \tag{8.3}$$

$$f^0=\frac{\left[1-\frac{2\beta}{r}\right]^{\sigma^2-\delta-\nu}}{\left[1-\frac{2\beta}{r}+\frac{\beta^2}{r^2}\sin^2\theta\right]^{\sigma^2}}, \tag{8.4}$$

where

$$\sigma^2=\delta^2+3\nu^2 . \tag{8.5}$$

When $\nu=0$, this solution is a well-known Weyl—Levi Civita static metric.[16–18]

We can verify that a particular solution of Eqs. (2.7) and (2.8) when $g=g^0$ is

$$\psi_{011}=-\lambda W\left[\frac{(\lambda-\mu)(\lambda-\bar{\mu})}{\lambda W}\right]^{-\delta-\nu}, \tag{8.6}$$

$$\psi_{022}=-\left[\frac{(\lambda-\mu)(\lambda-\bar{\mu})}{\lambda W}\right]^{\delta-\nu}, \tag{8.7}$$

$$\psi_{033}=\left[\frac{(\lambda-\mu)(\lambda-\bar{\mu})}{\lambda W}\right]^{2\nu}. \tag{8.8}$$

The evaluation of ψ_0 at $\lambda=\mu_k$ gives

$$\psi_{011}(k)=-2W_k\mu_k\left[\frac{(\mu_k-\mu)(\mu_k-\bar{\mu})}{2W_k\mu_k}\right]^{-\delta-\nu}, \tag{8.9}$$

$$\psi_{022}(k)=-\left[\frac{(\mu_k-\mu)(\mu_k-\bar{\mu})}{2W_k\mu_k}\right]^{\delta-\nu}, \tag{8.10}$$

$$\psi_{033}(k)=\left[\frac{(\mu_k-\mu)(\mu_k-\bar{\mu})}{2W_k\mu_k}\right]^{2\nu}. \tag{8.11}$$

Note that

$$\lim_{\lambda\to 0}\psi_0(\lambda)\to g^0 , \tag{8.12}$$

as required by Eq. (2.11). An alternative form for the solution can be obtained using the expression

$$\lambda W=\lambda^2+2\lambda z-\rho^2=(\lambda-\mu_0)(\lambda-\bar{\mu}_0) , \tag{8.13}$$

where the functions μ_0 and $\bar{\mu}_0$ are poles with $W_k=0$:

$$\mu_0\equiv z+(z^2+\rho^2)^{1/2} , \tag{8.14}$$

$$\bar{\mu}_0\equiv -z-(z^2+\rho^2)^{1/2} . \tag{8.15}$$

It is worth noticing that, as follows from Eqs. (8.1)—(8.3),

$$\lim_{\beta\to 0} g^0\to\begin{bmatrix}\rho^2 & 0 & 0\\ 0 & -1 & 0\\ 0 & 0 & 1\end{bmatrix}, \tag{8.16}$$

and furthermore, using Eqs. (8.6)—(8.8) and (7.15), (7.16),

$$\lim_{\beta\to 0}\psi_0(\lambda)\to\begin{bmatrix}-\lambda W & 0 & 0\\ 0 & -1 & 0\\ 0 & 0 & 1\end{bmatrix}. \tag{8.17}$$

Thus, when $\beta=0$, the seed solutions g^0,ψ_0 are the same as for the pseudo-Euclidean case (Sec. VII).

Let us consider the diagonal form, Eqs. (5.11), (5.12), and (4.38). We see that a Weyl—Levi Civita-type solution can be obtained when the poles are chosen in the following way:

$$\mu_k=\begin{cases}\mu_1=-\dfrac{\rho^2}{\bar{\mu}}=r(1-\cos\theta), & k \text{ odd} ,\\[2mm] \mu_2=-\dfrac{\rho^2}{\mu}=-r(1+\cos\theta), & k \text{ even} ,\end{cases} \tag{8.18}$$

since, in this case,

$$(g_{\rm ph})_{11}=\rho^2\left[1-\frac{2\beta}{r}\right]^{-\bar{\delta}-\bar{\nu}}, \tag{8.19}$$

$$(g_{\rm ph})_{22}=-\left[1-\frac{2\beta}{r}\right]^{\bar{\delta}-\bar{\nu}}, \tag{8.20}$$

$$(g_{\rm ph})_{33}=\left[1-\frac{2\beta}{r}\right]^{2\bar{\nu}}, \tag{8.21}$$

where

$$\bar{\delta}\equiv\delta+\frac{q}{2}-s , \tag{8.22}$$

$$\bar{\nu}\equiv\nu+\frac{n}{3}-\frac{q}{2} . \tag{8.23}$$

In other words, if we start with $g^0(\delta,\nu)$ as the background, the new diagonal metric will be $g^0(\bar{\delta},\bar{\nu})$. Furthermore, if the parameters $\bar{\delta}$ and $\bar{\nu}$ are chosen such that $\bar{\sigma}^2\equiv\bar{\delta}^2+3\bar{\nu}^2=1$, we will obtain $g^0(\bar{\delta},\bar{\nu})=g_{\rm sp}$ and $f^0(\bar{\delta},\bar{\nu})=f_{\rm sp}$.

For these types of solutions, the functions $\psi_0(k)$, Eqs. (8.9)—(8.11) take the form, for k odd,

$$\psi^{(1)}_{011}=2\beta r(1-\cos\theta)\left[2\left[1-\frac{\beta}{r}(1+\cos\theta)\right]\right]^{-\delta-\nu}, \tag{8.24}$$

$$\psi^{(1)}_{022}=-\left[2\left[1-\frac{\beta}{r}(1+\cos\theta)\right]\right]^{\delta-\nu}, \tag{8.25}$$

$$\psi^{(1)}_{033}=\left[2\left[1-\frac{\beta}{r}(1+\cos\theta)\right]\right]^{2\nu}, \tag{8.26}$$

and, for k even,

$$\psi^{(2)}_{011}=2\beta r(1+\cos\theta)\left[2\left[1-\frac{\beta}{r}(1-\cos\theta)\right]\right]^{-\delta-\nu}, \tag{8.27}$$

$$\psi^{(2)}_{022}=-\left[2\left[1-\frac{\beta}{r}(1-\cos\theta)\right]\right]^{\delta-\nu}, \tag{8.28}$$

$$\psi^{(2)}_{033}=\left[2\left[1-\frac{\beta}{r}(1-\cos\theta)\right]\right]^{2\nu}. \tag{8.29}$$

Thus we can construct generalizations of the Weyl–Levi Civita metric by substituting the above matrix ψ_0 and the metric (8.1)–(8.14) in Eqs. (4.29), (4.38), and (4.4). The case $n=q=s=2$, is given explicitly below

$$(g_{\rm ph})_{11}=\frac{1}{\Omega}\left[2\Omega+\frac{4\gamma^2R^2}{\Delta}-4\frac{(b-a\cos\theta)^2}{\sin^2\theta}+a^2\sin^2\theta-\Delta\right]g^0_{11}(\delta,\bar\nu)\ , \tag{8.30}$$

$$(g_{\rm ph})_{22}=\frac{1}{\Omega}(\Delta-a^2\sin^2\theta)g^0_{22}(\delta,\bar\nu),\quad \bar\nu\equiv\nu-\tfrac{1}{3}\ , \tag{8.31}$$

$$(g_{\rm ph})_{12}=\frac{-2\beta}{\Omega\gamma}[\Delta(a-b\cos\theta)-a\sin^2\theta(a^2-b^2-MR)]\times\left[1-\frac{2\beta}{r}\right]^{-\bar\nu}, \tag{8.32}$$

$$(g_{\rm ph})_{33}=\left[1-\frac{2\beta}{r}\right]^{2\bar\nu}, \tag{8.33}$$

$$f_{\rm ph}=\frac{\Omega f^0(\bar\delta,\bar\nu)}{(r-2\beta)^2},\quad \bar\delta\equiv\delta-1\ , \tag{8.34}$$

where

$$\gamma\equiv\frac{\beta-M_0}{2}+\frac{\beta+M_0}{2}\left[\frac{1-\frac{2\beta}{r}}{1-\frac{2\beta}{r}+\frac{\beta^2}{r^2}\sin^2\theta}\right]^{2\delta}, \tag{8.35}$$

$$M\equiv\frac{M_0-\beta}{2}+\frac{M_0+\beta}{2}\left[\frac{1-\frac{2\beta}{r}}{1-\frac{2\beta}{r}+\frac{\beta^2}{r^2}\sin^2\theta}\right]^{2\delta}, \tag{8.36}$$

$$a\equiv\left[1-\frac{2\beta}{r}\right]^{\delta}\left[\frac{a_0-b_0}{2}\left[1-\frac{\beta}{r}(1-\cos\theta)\right]^{-2\delta}+\frac{a_0+b_0}{2}\left[1-\frac{\beta}{r}(1+\cos\theta)\right]^{-2\delta}\right], \tag{8.37}$$

$$b\equiv\left[1-\frac{2\beta}{r}\right]^{\delta}\left[\frac{b_0-a_0}{2}\left[1-\frac{\beta}{r}(1-\cos\theta)\right]^{-2\delta}+\frac{b_0+a_0}{2}\left[1-\frac{\beta}{r}(1+\cos\theta)\right]^{-2\delta}\right], \tag{8.38}$$

$$R\equiv\frac{\gamma}{\beta}(r-\beta)+M\ , \tag{8.39}$$

$$\Omega\equiv R^2+(b-a\cos\theta)^2\ , \tag{8.40}$$

$$\Delta\equiv\frac{\gamma^2}{\beta^2}[r(r-2\beta)]=(R-M)^2-\gamma^2\ . \tag{8.41}$$

The constants a_0, b_0, M_0, and β have been chosen such that $M_0{}^2-\beta^2=a_0{}^2-b_0{}^2$, in order to obtain the Kerr-NUT (Newman-Unti-Tamburino) solution when $\delta=\bar\nu=0$, as we shall see below.

The study of the physical interpretation of the above solution is under way, but we can already state the following results.

(i) If $\delta=\bar\nu=0$, then $\gamma=\beta$, $M=M_0$, $a=a_0$, $b=b_0$, and the solution becomes equivalent to the Kerr-NUT metric, since one can verify that the time coordinate transformation

$$\tau=t+2a_0\varphi$$

leads to the Boyer-Lindquist Kerr-NUT line element

$$ds^2=-\frac{1}{\Omega}\{(\Delta-a^2\sin^2\theta)d\tau^2-4[\Delta b\cos\theta-a\sin^2\theta(MR+b^2)]d\tau\,d\varphi+[\Delta(a\sin^2\theta+2b\cos\theta)^2-\sin^2\theta(R^2+b^2+a^2)^2]d\varphi^2\}+\frac{\Omega}{\Delta}(dR^2+\Delta\,d\theta^2)\ . \tag{8.42}$$

If $b_0=0$, we get the Kerr solution with mass M and geometric angular momentum Ma.

(ii) If $a_0=b_0=0$, we have $(g_{\rm ph})_{12}=0$, and therefore zero angular momentum. The metric (8.30)–(8.34) becomes a Weyl–Levi Civita solution

$$(g_{\rm ph})_{11}=r^2\left[1-\frac{2\beta}{r}\right]^{1-\bar\delta-\bar\nu}\sin^2\theta\ , \tag{8.43}$$

$$(g_{\rm ph})_{22}=-\left[1-\frac{2\beta}{r}\right]^{\delta-\bar\nu}, \tag{8.44}$$

$$(g_{\rm ph})_{33}=\left[1-\frac{2\beta}{r}\right]^{2\bar{v}}, \tag{8.45}$$

$$f_{\rm ph}=f^0(\tilde{\delta},\bar{v})$$
$$=\left[1-\frac{2\beta}{r}\right]^{\tilde{\sigma}^2-\tilde{\delta}-\bar{v}}\left[1-\frac{2\beta}{r}+\frac{\beta^2}{r^2}\sin^2\theta\right]^{-\tilde{\sigma}^2}, \tag{8.46}$$

where $\tilde{\delta}\equiv\delta\pm 1$, depending on the choice $M_0=\pm\beta$. If, furthermore, $\tilde{\sigma}^2=\tilde{\delta}^2+3\bar{v}^2=1$, we have spherical symmetry.

The solution (8.30)–(8.34) has been verified with the algebraic computing program REDUCE.[19]

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APPENDIX A

In this appendix it is shown that

$$\det\left[\frac{1}{1+A_kB_l}\right]=\prod_{k>l}(A_k-A_l)(B_l-B_k)\times\left[\prod_{k,l}(1+A_kB_l)\right]^{-1}. \tag{A1}$$

First note that

$$\det\prod_m\left[\frac{(1+A_kB_m)}{1+A_kB_l}\right]\equiv\det\prod_{\substack{m\\ m\neq l}}(1+A_kB_m)=\left[\prod_{k,m}(1+A_kB_m)\right]\det\left[\frac{1}{(1+A_kB_l)}\right], \tag{A2}$$

where we used the fact that $\det(b_ka_{kl})=(\prod_k b_k)\det a_{kl}$. Hence Eq. (A1) is equivalent to

$$\det\prod_{\substack{m\\ m\neq l}}(1+A_kB_m)=\prod_{k>l}(A_k-A_l)(B_l-B_k). \tag{A3}$$

The proof of Eq. (A3) will be by induction. Hence let us assume that Eq. (A3) is valid for a $(n-1)\times(n-1)$ matrix:

$$C_{kl}^{n-1}\equiv\prod_{\substack{m\\ m\neq l}}^{n-1}(1+A_kB_m), \tag{A4}$$

$$\det C_{kl}^{n-1}=\prod_{k>l}^{n-1}(A_k-A_l)(B_l-B_k). \tag{A5}$$

Then we have

$$\begin{aligned}\prod_{k>l}^{n}(A_k-A_l)(B_l-B_k)&=\left[\prod_{l}^{n-1}(A_n-A_l)(B_l-B_n)\right]\prod_{k>l}^{n-1}(A_k-A_l)(B_l-B_k)\\&=\left[\prod_{l}^{n-1}(A_n-A_l)(B_l-B_n)\right]\det C_{kl}^{n-1},\end{aligned} \tag{A6}$$

where we used Eq. (A5). We know that

$$\det C_{kl}^{n-1}=\epsilon_{i_1,i_2,\ldots,i_{n-1}}C_{1i_1}^{n-1}C_{2i_2}^{n-1}\cdots C_{n-1i_{n-1}}^{n-1}\equiv\epsilon_{i_1,i_2,\ldots,i_{n-1}}\prod_{l}^{n-1}C_{1i_l}^{n-1}. \tag{A7}$$

Also, it is easy to show that

$$\begin{aligned}\prod_{l}^{n-1}(A_n-A_l)(B_l-B_n)&=\prod_{l}^{n-1}(A_n-A_l)(B_{i_l}-B_n)\\&=\prod_{l}^{n-1}[(1+A_lB_n)(1+A_nB_{i_l})-(1+A_nB_n)(1+A_lB_{i_l})],\end{aligned} \tag{A8}$$

for any particular choice of $n-1,i_l$. Consequently, using Eqs. (A7) and (A8),

$$\prod_{k>l}^{n}(A_k-A_l)(B_l-B_k)=\epsilon_{i_1,i_2,\ldots,i_{n-1}}\prod_{l}^{n-1}C_{li_l}^{n-1}[(1+A_lB_n)(1+A_nB_{i_l})-(1+A_nB_n)(1+A_lB_{i_l})]$$
$$=\epsilon_{i_1,i_2,\ldots,i_{n-1}}\prod_{l}^{n-1}[C_{li_l}^{n-1}(1+A_lB_n)(1+A_nB_{i_l})-(1+A_nB_n)(1+A_lB_{i_l})C_{li_l}^{n-1}]$$
$$\equiv\epsilon_{i_1,i_2,\ldots,i_{n-1}}\prod_{l}^{n-1}[C_{li_l}^{n-1}(1+A_lB_n)(1+A_nB_{i_l})-K_l]\ . \tag{A9}$$

The terms

$$K_l\equiv(1+A_nB_n)(1+A_lB_{i_l})C_{li_l}^{n-1}=(1+A_nB_n)\prod_{k}^{n-1}(1+A_lB_k) \tag{A10}$$

are independent of i_l. Hence any term on the right-hand side of Eq. (A9) that contains a factor such as $K_lK_m\cdots K_s$ is symmetrical under permutations of $i_l,i_m,\ldots,i_s$. Therefore these terms do not contribute to the sum in Eq. (A9). Thus, the only surviving terms are either independent of K_l, or linear in K_l. Consequently, Eq. (A9) becomes

$$\prod_{k>l}^{n}(A_k-A_l)(B_l-B_k)=\epsilon_{i_1,i_2,\ldots,i_{n-1}}\left[\prod_{l}^{n-1}C_{li_l}^{n-1}(1+A_lB_n)(1+A_nB_{i_l})-\sum_{m}^{n-1}K_m\prod_{\substack{l\\ l\neq m}}^{n-1}C_{li_l}^{n-1}(1+A_lB_n)(1+A_nB_{i_l})\right]\ . \tag{A11}$$

Using Eq. (A10), we obtain, after some rearrangement,

$$K_m\prod_{\substack{l\\ l\neq m}}^{n-1}C_{li_l}^{n-1}(1+A_NB_n)(1+A_nB_{i_l})=\left[\prod_{k}^{n-1}(1+A_mB_k)\right]\left[(1+A_nB_n)\prod_{\substack{l\\ l\neq m}}^{n-1}(1+A_nB_{i_l})\right]\left[\prod_{\substack{l\\ l\neq m}}^{n-1}C_{li_l}^{n-1}(1+A_lB_n)\right]$$
$$=C_{mn}^{n}C_{ni_m}^{n}\prod_{\substack{l\\ l\neq m}}^{n-1}C_{li_l}^{n}=\frac{C_{mn}^{n}C_{ni_m}^{n}}{C_{mi_m}^{n}}\prod_{l}^{n-1}C_{li_l}^{n}\ , \tag{A12}$$

where we use the definitions

$$C_{mn}^{n}\equiv\prod_{\substack{k\\ k\neq n}}^{n}(1+A_mB_k)=\prod_{k}^{n-1}(1+A_mB_k)\ , \tag{A13}$$

$$C_{ni_m}^{n}\equiv\prod_{k\neq i_m}^{n}(1+A_nB_k)=(1+A_nB_n)\prod_{k\neq i_m}^{n-1}(1+A_nB_k)=(1+A_nB_n)\prod_{l\neq m}^{n-1}(1+A_nB_{i_l})\ , \tag{A14}$$

$$C_{li_l}^{n}\equiv\prod_{k\neq i_l}^{n}(1+A_lB_k)=(1+A_lB_n)\prod_{k\neq i_l}^{n-1}(1+A_lB_k)=(1+A_lB_n)C_{li_l}^{n-1}\ . \tag{A15}$$

Substituting Eq. (A12) and using again the definitions (A13)–(A15) in Eq. (A11) we get

$$\prod_{k>l}^{n}(A_k-A_l)(B_l-B_k)=\epsilon_{i_1,i_2,\ldots,i_{n-1}}\left[\prod_{l}^{n-1}C_{li_n}^{n}C_{nn}^{n}-\sum_{m}^{n-1}\frac{C_{mn}^{n}C_{ni_m}^{n}}{C_{mi_m}^{n}}\prod_{l}^{n-1}C_{li_l}^{n}\right]\ . \tag{A16}$$

On the other hand, we know that

$$\det C_{kl}^{n}=\epsilon_{i_1,i_2,\ldots,i_n}\prod_{k}^{n}C_{ki_k}^{n}\ . \tag{A17}$$

Equation (A17) can be reexpressed as

$$\det C_{kl}^{n}=\epsilon_{i_1,i_2,\ldots,n}\prod_{k}^{n-1}C_{ki_k}^{n}C_{nn}^{n}+\epsilon_{i_1,i_2,\ldots,n,i_n}\prod_{\substack{k\\ k\neq n-1}}^{n}C_{ki_k}^{n}C_{n-1n}^{n}$$
$$+\epsilon_{i_1,i_2,\ldots,n,i_{s+1},\ldots,i_n}\prod_{\substack{k\\ k\neq s}}^{n}C_{ki_k}^{n}C_{sn}^{n}+\cdots+\epsilon_{n,i_2,\ldots,i_n}\prod_{\substack{k\\ k\neq 1}}^{n}C_{ki_k}^{n}C_{1n}^{n}$$
$$=\epsilon_{i_1,i_2,\ldots,n}\prod_{k}^{n-1}C_{ki_k}^{n}C_{nn}^{n}-\epsilon_{i_1,i_2,\ldots,i_n,n}\prod_{\substack{k\\ k\neq n-1}}^{n}C_{ki_k}^{n}C_{n-1n}^{n}$$
$$-\epsilon_{i_1,i_2,\ldots,i_n,i_{s+1},\ldots,n}\prod_{\substack{k\\ k\neq s}}^{n}C_{ki_k}^{n}C_{sn}^{n}-\cdots-C_{i_n,i_2,\ldots,n}\prod_{\substack{k\\ k\neq 1}}^{n}C_{ki_k}^{n}C_{1n}^{n}\ . \tag{A18}$$

We see that

$$\epsilon_{i_1,i_2,\ldots,i_n,i_s+1,\ldots,n}\prod_{\substack{k\\k\neq s}}^{n} C^n_{ki_k}C^n_{sn}=\epsilon_{i_1,i_2,\ldots,i_n,i_s+1,\ldots,n}\prod_{k}^{n-1}\frac{C^n_{ki_k}C^n_{ni_n}C^n_{sn}}{C^n_{si_n}}$$

$$=\epsilon_{i_1,i_2,\ldots,i_s,i_s+1,\ldots,n}\prod_{k}^{n-1}\frac{C^n_{ki_k}C^n_{ni_s}C^n_{sn}}{C^n_{si_s}}\ . \tag{A19}$$

Therefore, applying Eq. (A19) to Eq. (A18), we obtain

$$\det C^n_{kl}=\epsilon_{i_1,i_2,\ldots,n}\prod_{k}^{n-1}C^n_{ki_k}C^n_{nn}-\epsilon_{i_1,i_2,\ldots,i_{n-1},n}\prod_{k}^{n-1}\frac{C_{ki_k}C^n_{ni_{n-1}}C^n_{n-1n}}{C^n_{n-1i_{n-1}}}$$

$$-\cdots-\epsilon_{i_1,i_2,\ldots,i_s,i_s+1,\ldots,n}\prod_{k}^{n-1}\frac{C^n_{ki_k}C^n_{ni_s}C^n_{sn}}{C^n_{si_s}}-\cdots-\epsilon_{i_1,i_2,\ldots,n}\prod_{k}^{n-1}\frac{C^n_{ki_k}C^n_{ni_1}C^n_{1n}}{C^n_{1i_1}}$$

$$=\epsilon_{i_1,i_2,\ldots,i_{n-1},n}\left[\prod_{k}^{n-1}C^n_{ki_k}C^n_{nn}-\sum_{m}^{n-1}\prod_{k}^{n-1}\frac{C^n_{ki_k}C^n_{ni_m}C^n_{mn}}{C^n_{mi_m}}\right]\ . \tag{A20}$$

Comparing Eq. (A20) with Eq. (A16) we see that

$$\det C^n_{kl}\equiv\det\prod_{\substack{m\\m\neq l}}^{n}(1+A_kB_m)$$

$$=\prod_{k>l}^{n}(A_k-A_l)(B_l-B_k)\ , \tag{A21}$$

or, equivalently, from (A2)

$$\det\left[\frac{1}{1+A_kB_l}\right]=\prod_{k>l}^{n}\frac{(A_k-A_l)(B_l-B_k)}{\prod_{k,l}(1+A_kB_l)}\ . \tag{A22}$$

Since (A22) is valid for a 2×2 matrix, then we have shown that it is valid for all $n\geq2$. Q.E.D. In particular, if $A_k=B_k=\mu_k$, we have

$$\det\left[\frac{1}{1+\mu_k\mu_l}\right]=\frac{\prod_{k>l}(\mu_k-\mu_l)(\mu_l-\mu_k)}{\prod_{k,l}(1+\mu_k\mu_l)}$$

$$=\frac{\prod_{k\neq l}(\mu_k-\mu_l)}{\prod_{k,l}(1+\mu_k\mu_l)}\ . \tag{A23}$$

APPENDIX B

It is shown below that when $g_{\rm ph}$ is diagonal, then for each one of the conditions (i) or (ii) in Sec. VI, we have

$$f_{\rm ph}={\rm const}\times f_0\ . \tag{B1}$$

First consider case (i). In this situation Eq. (5.22) becomes

$$f_{\rm ph}=C_n\rho^{1/3}Q^{-2/3}f_0\rho^{3s}\left[\prod_{k=1}^{s}\mu_k{}^6\right]\times\left[\prod_{k=1}^{s}(M^k_1M^k_2M^k_3)^2g^0_{11}g^0_{22}g^0_{33}\right]\times\left[\frac{\rho^{\rho(\rho-1)}\prod_{k\neq l=1}^{s}(\mu_k-\mu_l)}{\prod_{k=1}^{s}(\mu_k{}^2+\rho^2)\prod_{k,l}^{s}(\mu_k\mu_l+\rho^2)}\right]^3\ . \tag{B2}$$

We know that

$$g^0_{11}g^0_{22}g^0_{33}=\det g^0=-\rho^2\ , \tag{B3}$$

and

$$M^k_1M^k_2M^k_3=C^k_1C^k_2C^k_3\psi_{011}{}^{-1}(k)\psi_{022}{}^{-1}(k)\psi_{033}{}^{-1}(k)$$

$$=C^k_1C^k_2C^k_3\det\psi_0{}^{-1}(k)\ . \tag{B4}$$

It can be shown[20] that

$$\det\psi(\lambda)=\lambda G(W)\ , \tag{B5}$$

where $G(W)$ is only a function of

$$W(\lambda)\equiv\frac{\lambda^2+2\lambda z-\rho^2}{\lambda}\ . \tag{B6}$$

Since, using Eq. (2.15),

$$W(\mu_k)=\frac{\mu_k{}^2+2\mu_kz-\rho^2}{\mu_k}=2W_k\ , \tag{B7}$$

we have

$$\det\psi(\mu_k)={\rm const}\times\mu_k\ . \tag{B8}$$

Furthermore, we will also use the following relation, which is shown in Appendix D:

$$\prod_{k\neq l}^{m}(\mu_k-\mu_l)=\left[\prod_{k\neq l}^{m}\frac{2(W_k-W_l)}{(\mu_k\mu_l+\rho^2)}\right]\prod_{k}^{m}(\mu_k)^{2(m-1)} . \tag{B9}$$

Substituting Eqs. (B3), (B4), (B8), and (B9) in Eq. (B2) we obtain

$$f_{\rm ph}=\frac{\text{const}\times\rho^{1/3}Q^{-2/3}f_0\rho^{3s^2+2s}\prod_{k=1}^{s}\mu_k^{6s-2}}{\prod_{k,l}^{s}(\mu_k\mu_l+\rho^2)^6} . \tag{B10}$$

On the other hand, from Eq. (2.23), we have

$$Q^{-1}=\text{const}\times\rho^{-(n+1)^2/2}\left[\prod_{k=1}^{n}\mu_k^{1-n}\right]\times\prod_{k,l}^{n}(\mu_k\mu_l+\rho^2) , \tag{B11}$$

where we used Eq. (B9). The last factor of Eq. (B11) can be reexpressed in the form

$$\prod_{k,l=1}^{n}(\mu_k\mu_l+\rho^2)=\prod_{k,l=1}^{s}(\mu_k\mu_l+\rho^2)^9 , \tag{B12}$$

where we use condition (i). Thus we have, putting $n=3s$,

$$Q^{-1}=\left[\text{const}\times\rho^{-(3s+1)^2/3}\left[\prod_{k=1}^{s}\mu_k^{2(1-3s)}\right]\times\left[\prod_{k,l=1}^{s}(\mu_k\mu_l+\rho^2)\right]^6\right]^{3/2} . \tag{B13}$$

Substitution of Eq. (B13) into Eq. (B10) gives the desired result:

$$f_{\rm ph}=\text{const}\times f_0 . \tag{B14}$$

We now consider condition (ii). Let us recall Eqs. (6.4) and (6.5):

$$\mu_k\mu_{k'}=-\rho^2 ,$$
$$\mu_k+\mu_{k'}=2(W_k-z) .$$

Some consequences of Eqs. (6.4) and (6.5) will be derived below. First consider

$$\prod_{k\neq l}(\mu_k-\mu_l)=\left[\prod_{l}(\mu_l-\mu_{l'})(\mu_{l'}-\mu_l)\right]\times\prod_{k\neq l,l'}(\mu_k-\mu_l)(\mu_k-\mu_{l'}) , \tag{B15}$$

but

$$\begin{aligned}(\mu_k-\mu_l)(\mu_k-\mu_{l'})&=\mu_k{}^2-\mu_k(\mu_l+\mu_{l'})+\mu_l\mu_{l'}\\&=\mu_k{}^2-2\mu_k(W_l-z)-\rho^2\\&=2\mu_k(W_k-z)-2\mu_k(W_l-z)\\&=2\mu_k(W_k-W_l) ,\end{aligned} \tag{B16}$$

where we used Eqs. (6.4), (6.5), and (B7). Consequently,

$$\prod_{k\neq l}(\mu_k-\mu_l)=\prod_{l}\left[(\mu_l-\mu_{l'})(\mu_{l'}-\mu_l)\prod_{k\neq l,l'}2(W_k-W_l)\mu_k\right]=\text{const}\times\prod_{l}\left[\frac{(\mu_l-\mu_{l'})^2}{\rho^2}\prod_{k}\mu_k\right] . \tag{B17}$$

Also it can easily be shown that

$$\prod_{k}(\mu_k{}^2+\rho^2)=\prod_{l}(\mu_l{}^2+\rho^2)(\mu_{l'}{}^2+\rho^2)=\prod_{l}\rho^2(\mu_l-\mu_{l'})^2 . \tag{B18}$$

Therefore we have, for m poles μ_k, or equivalently $m/2$ pairs $\mu_l,\mu_{l'}$

$$\frac{\prod_{k\neq l}(\mu_k-\mu_l)}{\prod_{k}(\mu_\kappa{}^2+\rho^2)}=\text{const}\times\prod_{l}\left[\frac{\prod_k\mu_k}{\rho^4}\right]=\text{const}\times\frac{\prod_k\mu_k^{m/2}}{\rho^{2m}}=\text{const}\times\rho^{m^2/2-2m} , \tag{B19}$$

where we use Eq. (6.4).

We can now simplify the last three factors of Eq. (5.22) using Eq. (B19), since we have, for m poles μ_k,

$$\frac{\rho^{m(m-1)}\prod_{k\neq l}(\mu_k-\mu_l)}{\left[\prod_{k}(\mu_k{}^2+\rho^2)\right]\left[\prod_{k,l}(\mu_k\mu_l+\rho^2)\right]}=\frac{\text{const}\times\rho^{m(m-1)}}{\prod_k\mu_k^{2(m-1)}}\left[\frac{\prod_{k\neq l}(\mu_k-\mu_l)}{\prod_k(\mu_k{}^2+\rho^2)}\right]^2=\text{const}\times\rho^{-3m} , \tag{B20}$$

where we used Eqs. (B9), (B19), and Eq. (6.4). On the other hand, with Eqs. (B19) and (6.4), the expression for Q^{-1} also simplifies to

$$Q^{-1}=\text{const}\times\rho^{-1/2} . \tag{B21}$$

Using the results (B20) and (B21) in Eq. (5.22), we get

$$f_{\rm ph}=\text{const}\times f_0\left[\prod_{k=1}^{s}(M_1^k)^2g_{11}^0\right]\left[\prod_{k>s}^{q}(M_2^k)^2g_{22}^0\right]\left[\prod_{k>q}^{n}(M_3^k)^2g_{33}^0\right] . \tag{B22}$$

Finally, if g and ψ are diagonal then it follows that

$$\psi(\lambda)g^{-1}\psi\left[\frac{-\rho^2}{\lambda}\right]=F(W)\ , \tag{B23}$$

is only a function of $W(\lambda)$ (Ref. 20) and therefore

$$\psi^{-1}(\mu_l)g\psi^{-1}(\mu_{l'})=\psi^{-1}(\mu_l)g\psi^{-1}\left[\frac{-\rho^2}{\mu_l}\right]$$

$$=F^{-1}[W(\mu_l)]=\text{const}\ , \tag{B24}$$

or, equivalently, for $a=1,2,3$

$$M_a^l g_{aa} M_a^{l'}=C_a^l C_a^{l'}\psi_{aa}{}^{-1}(\mu_l)\psi_{aa}{}^{-1}(\mu_{l'})g_{aa}^0$$

$$=\text{const} \tag{B25}$$

(no sum over a'). This implies that

$$\left[\prod_{k=1}^{s}(M_1^k)^2 g_{11}^0\right]\left[\prod_{k>s}^{q}(M_2^k)^2 g_{22}^0\right]\left[\prod_{k>q}^{n}(M_3^k)^2 g_{33}^0\right]=\text{const}\ . \tag{B26}$$

Hence again

$$f_{\text{ph}}=\text{const}\times f_0\ .$$

Q.E.D.

APPENDIX C

A solution of the Einstein scalar field equations (3.7) and (3.8) that represents the exterior gravitational field of a static axially symmetry configuration is

$$(g_{\text{ES}})_{11}=r^2\left[1-\frac{2\beta}{r}\right]^{1-\delta}\sin^2\theta\ , \tag{C1}$$

$$(g_{\text{ES}})_{22}=-\left[1-\frac{2\beta}{r}\right]^{\delta}, \tag{C2}$$

$$f_{\text{ES}}=\frac{\left[1-\dfrac{2\beta}{r}\right]^{\sigma^2-\delta}}{\left[1-\dfrac{2\beta}{r}+\dfrac{\beta^2}{r^2}\sin^2\theta\right]^{\sigma^2}},\quad \sigma^2\equiv\delta^2+3\nu^2\ , \tag{C3}$$

$$\phi=\left[1-\frac{2\beta}{r}\right]^{\nu/\alpha},\quad \alpha=\left[\frac{2\omega+3}{3}\right]^{1/2}, \tag{C4}$$

where β and δ are constants, and the coordinates r,θ are related to ρ and z by Eqs. (7.10) and (7.11). If $\nu=0$, the above equations transform into a well-known Weyl—Levi Civita metric in general relativity. The total gravitational mass of the system is $(\delta-\nu/\alpha)\beta$, which follows from the requirement that the equations of motions become Newtonian in the limit $r\to\infty$.

The case $\sigma^2=1$ is of particular interest, since we get a spherically symmetric gravitational field. The line element takes the form

$$ds^2=-\left[1-\frac{2\beta}{r}\right]^{\delta}dt^2+\left[1-\frac{2\beta}{r}\right]^{-\delta}dr^2$$

$$+\bar{r}^2\left[d\theta^2+\sin^2\theta\, d\varphi^2\right], \tag{C5}$$

where

$$\bar{r}^2\equiv r^2\left[1-\frac{2\beta}{r}\right]^{1-\delta}. \tag{C6}$$

A solution of the BDJ$_5$ field equation $\tilde{g}$ is built, from the metric (C1)—(C3) and the scalar field (C4), using the transformations (3.1), and Eqs. (3.5) and (3.6):

$$\tilde{g}_{\mu\nu}=\phi^{-\alpha}(g_{\text{ES}})_{\mu\nu}\ , \tag{C7}$$

$$\tilde{g}_{44}=\phi^{2\alpha}=\left[1-\frac{2\beta}{r}\right]^{2\nu}, \tag{C8}$$

or, in the canonical form (4.1),

$$g_{ab}=\left[1-\frac{2\beta}{r}\right]^{-\nu}(g_{\text{ES}})_{ab},\quad a,b=1,2\ , \tag{C9}$$

$$g_{33}=\left[1-\frac{2\beta}{r}\right]^{2\nu}, \tag{C10}$$

$$f=\left[1-\frac{2\beta}{r}\right]^{-\nu}f_{\text{ES}}\ . \tag{C11}$$

APPENDIX D

We will see below that

$$\prod_k \mu_k^{2(n-1)}=\prod_{\substack{k,l\\k\neq l}}\frac{(\rho^2+\mu_k\mu_l)(\mu_k-\mu_l)}{2(W_k-W_l)}\ , \tag{D1}$$

where n is the number of poles, μ_k. Thus consider two poles μ_k, μ_l. They satisfy Eq. (2.15), which in turn implies

$$\mu_k{}^2-2\mu_k(W_k-z)-\rho^2=0\ , \tag{D2}$$

$$\mu_l{}^2-2\mu_l(W_l-z)-\rho^2=0\ . \tag{D3}$$

The difference of these two equations gives, after some rearrangement,

$$W_k-z=\frac{\mu_k+\mu_l}{2}-\frac{\mu_l(W_k-W_l)}{\mu_k-\mu_l}\ . \tag{D4}$$

Substitution of Eq. (D4) in Eq. (D2) yields

$$\rho^2=-\mu_k\mu_l+\frac{2\mu_k\mu_l(W_k-W_l)}{\mu_k-\mu_l}\ , \tag{D5}$$

or, equivalently,

$$\mu_k\mu_l=\frac{(\rho^2+\mu_k\mu_l)(\mu_k-\mu_l)}{2(W_k-W_l)}\ . \tag{D6}$$

Thus we have

$$\prod_{\substack{k,l\\k\neq l}}\mu_k\mu_l=\prod_k \mu_k^{2n-2}=\prod_{\substack{k,l\\k\neq l}}\frac{(\rho^2+\mu_k\mu_l)(\mu_k-\mu_l)}{2(W_k-W_l)}\ . \tag{D7}$$

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