Double-null coordinates for the Vaidya metric

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Einstein's equations with spherical symmetry are formulated in double-null coordinates, and the high-frequency approximation to a unidirectional radial flow of unpolarized radiation (the Vaidya metric) is studied in detail. For this case the Einstein equations reduce to a single first-order nonlinear partial differential equation. Integration of this equation introduces an arbitrary function (of one null variable) which must be chosen so as to regularize the metric across horizons. Although the problem is, in general, not analytically solvable, we are able to extend the class of known analytic solutions from the constant-mass case (Kruskal-Szekeres metric) to linear and exponential mass functions. In the linear case we give the first explicit regular covering of a spacetime with a naked shell-focusing singularity.

I. INTRODUCTION

The Vaidya metric gives the general-relativistic field associated with the high-frequency (eikonal) approximation to a unidirectional radial flow of unpolarized radiation. The metric has seen considerable use: at the semiclassical level for the study of evaporating black holes,[1] at the classical level as a model for the exterior of spherical radiating objects,[2] and as a model of a spacetime which develops a naked singularity.[3]

Despite the usefulness of the Vaidya metric, relatively little attention has been paid to the coordinates used to describe it. Most often, the metric is considered in radiation coordinates (r,θ,ϕ,w), where it takes the form

$$ds^2=2c\,dr\,dw-\left[1-\frac{2m(w)}{r}\right]dw^2+r^2d\Omega^2$$

with $d\Omega^2=d\theta^2+\sin^2\theta\,d\phi^2$. For $c=+1$ the field is ingoing and m is monotone increasing in w (advanced time), and for $c=-1$, the field is outgoing and m is monotone decreasing in w (retarded time). It has long been known that the radiation coordinates are defective.[4] Also long ago, a more useful, but less intuitive, set of coordinates was given by Israel.[5] These have seen too little application.[6]

It is natural to ask if double-null coordinates can be constructed for the Vaidya metric. A little experimentation with the metric shows that looking for transformations from radiation coordinates to double-null coordinates is fruitless.[7] In this paper we retreat from coordinate transformations, and consider the Einstein equations with spherical symmetry *ab initio* in double-null coordinates.[8] We find that the Einstein equations, for the radiation field under consideration, reduce to a single first-order nonlinear partial differential equation. Regularization of the metric across horizons is achieved by a suitable choice for the function which arises from the integration of this equation. This procedure offers little hope for many analytic solutions, although we are able to extend the Kruskal-Szekeres case ($m=$const) to linear and exponential mass functions. In the linear case, our coordinates give the first explicit regular covering of a spacetime with a shell-focusing singularity.

II. EINSTEIN'S EQUATIONS

A. Development

The spherically symmetric metric in double-null coordinates is

$$ds^2=-2f(u,v)du\,dv+r^2(u,v)d\Omega^2 . \tag{1}$$

The algebra relevant to the metric (1) is summarized in Appendix A.

The energy tensor is taken to be of the form

$$T_{\alpha\beta}=h(u,v)k_\alpha k_\beta/8\pi , \tag{2}$$

where k_α is a radial null vector. This is the high-frequency approximation to a unidirectional radial flow of unpolarized radiation. We can choose either $k^\alpha=(\dot u,0,0,0)$ (flow along the v direction) or $k^\alpha=(0,0,0,\dot v)$ (flow along the u direction) where a dot denotes $d/d\lambda$, λ an affine parameter. Since the u and v directions are unspecified, it is not in fact necessary to consider flows both along the u and v directions for energy tensors of the form (2). In what follows we consider, without loss of generality, a flow along the v direction only. From Eq. (A3) it follows that the radial null geodesic equation for u gives

$$f\dot u=\text{const.} \tag{3}$$

For a flow along the v direction then

$$T_{\alpha\beta}=h(u,v)\delta_\alpha^v\delta_\beta^v/8\pi , \tag{4}$$

where the constant has been absorbed into λ.

The Einstein equations with Eq. (4) give a zero Ricci scalar so that either from Eqs. (A5) or (A8) we have

$$2(f_1r_1/f-r_{11})/r=0 , \tag{5}$$

$$2(r_1r_4+rr_{14})/f+1=0 , \tag{6}$$

$$2(f_4 r_4/f - r_{44})/r = h(u,v) , \tag{7}$$

and

$$(f_1 f_4/f - f_{14})/f - 2r_{14}/r = 0 . \tag{8}$$

B. Simplification

Differentiating Eq. (6) with respect to u and using Eq. (5) gives

$$r^2 r_{14}/f + A(v) = 0 \tag{9}$$

after integration with respect to u, where A is an arbitrary function of v. Similarly, differentiating Eq. (6) with respect to v and using Eqs. (7) and (9) yields

$$h = 2A_4 r_{14}/A r_1 . \tag{10}$$

Integrating Eq. (5) with respect to u gives

$$f = 2B(v) r_1 , \tag{11}$$

where B is a second arbitrary function of v. Equations (8) and (11) reproduce Eq. (9). With the substitution of Eqs. (6) and (11), Eq. (9) yields

$$r_4 = -B(v)[1 - 2A(v)/r] . \tag{12}$$

Substitution of Eq. (12) into Eq. (10) gives

$$h = -4B(v) A_4/r^2 . \tag{13}$$

In summary, Einstein's equations (5)–(8) follow from Eqs. (11)–(13).

C. Interpretation of free functions

If we define the "mass" according to[9]

$$m \equiv \tfrac{1}{2} r^3 R_{23}{}^{23} , \tag{14}$$

it follows from Eqs. (A2) and (A4) that

$$m = r/2 + r r_1 r_4/f . \tag{15}$$

Equations (6) and (9) then give

$$A(v) = m . \tag{16}$$

We take $m \geq 0$.

The metric (1) and Eq. (11) show that B defines the scale for v which, of course, is required for a proper definition of m. From the weak-energy condition and Eq. (4) we note that

$$B(v) m_4 \leq 0 . \tag{17}$$

If we suppose that the metric (1) is asymptotically flat we can choose v such that the $v=$const null geodesics terminate on $\mathscr{I}$. [That is, say, on $\mathscr{I}^-$ for an ingoing field ($m_4 > 0$) and on $\mathscr{I}^+$ for an outgoing field ($m_4 < 0$), see below.] Then, v is the proper time as measured in the rest frame at infinity as long as

$$B = -m_4/2|m_4| , \quad m_4 \neq 0 . \tag{18}$$

Equation (18) offers a "natural," though not required, choice for B as long as $m_4 \neq 0$. [With $m_4 = 0$, the form of $B(v)$ is dictated by regularity conditions, see below.] With $m_4 \neq 0$, it follows from Eqs. (3), (11), and (18) that

$$\dot{r} = -\frac{|m_4|}{m_4} \tag{19}$$

along $v=$const. As a result, we say that the radiation field is "ingoing" ($\dot{r} < 0$) for $m_4 > 0$ and "outgoing" ($\dot{r} > 0$) for $m_4 < 0$. In both cases the null geodesics with $u=$const represent a "backscattered" test field.

III. SOLUTIONS

The problem at hand may be viewed in the following way: given $T_{\alpha\beta}$ (that is, given m, m_4, and B), the Einstein equations reduce to the single first-order nonlinear partial differential equation (12), the solution to which yields the metric (1) via Eq. (11). Unfortunately, Eq. (12) is, in general, not analytically solvable. Moreover, the arbitrary function of u which enters the integration of Eq. (12) must be chosen so that f, from Eq. (11), is regular. There is, unfortunately, no algorithm for this choice.

A. Review of the Schwarzschild solution (Ref. 8)

With $m=$const the function $B(v)$ is not specified *a priori*. However, Eq. (12) is immediately separable, giving

$$\int \frac{dr}{1 - 2m/r} = -\int B\, dv + C(u) , \tag{20}$$

where C is an arbitrary function of u, so that Eq. (11) gives

$$f = 2BC_1(1 - 2m/r) . \tag{21}$$

Let

$$B = -\frac{2m}{v} \tag{22}$$

and

$$C_1 = \frac{2m}{u} . \tag{23}$$

Equations (20), (22), and (23) then give

$$(1 - r/2m) e^{r/2m} = uv , \tag{24}$$

where we have oriented $uv > 0$ for $r < 2m$. Equations (21)–(24) then give

$$f = \frac{16m^3}{r} e^{-r/2m} . \tag{25}$$

B. Linear mass function

We make use of Eq. (18) and consider mass functions of the form

$$m = \lambda c v , \tag{26}$$

where λ is a constant > 0 and $c \equiv -2B$. With

$$g = \frac{r}{cv} \tag{27}$$

it follows that Eq. (12) can be written as

$$\ln cv + \int \frac{g\,dg}{g^2 - g/2 + \lambda} = D(u) , \tag{28}$$

where D is an arbitrary function of u. From Eqs. (11) and (28) we find

$$f = -cD_1(r^2 - \tfrac{1}{2}cvr + \lambda v^2)/r . \tag{29}$$

It remains to choose $D(u)$ [that is, via Eq. (28), to define $r(u,v)$] so as to remove zeros in f.

With $\lambda > \frac{1}{16}$, f is without zeros and so, for example, with

$$D = -cu \tag{30}$$

we have

$$f = (r^2 - \tfrac{1}{2}cvr + \lambda v^2)/r . \tag{31}$$

From Eqs. (28) and (30) we have

$$-cu = \tfrac{1}{2}\ln|r^2 - \tfrac{1}{2}cvr + \lambda v^2| + \frac{1}{\delta}\arctan\left[\frac{4r - cv}{cv\delta}\right] , \tag{32}$$

where $\delta \equiv \sqrt{16\lambda - 1}$. Equation (32) defines $r(u,v)$. The spacetime diagram is shown in Fig. 1.

For $\lambda = \frac{1}{16}$, we have

$$f = -\frac{cD_1(4r - cv)^2}{16r} , \tag{33}$$

where, from Eq. (28),

$$D(u) = \frac{cv}{cv - 4r} + \ln\left|\frac{4r - cv}{4}\right| \equiv L . \tag{34}$$

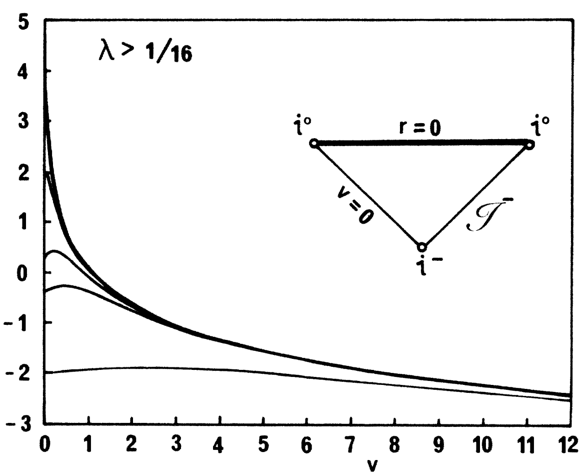

FIG. 1. u-v diagram for a linear mass function $m = c\lambda v$ with $\lambda > \frac{1}{16}$ and $c = +1$ (ingoing field). The curves represent surfaces of constant r. [The particular case shown is $\lambda = 1$, and the values of r shown are (from top to bottom) $r = 0$, 0.1, $\frac{1}{2}$, 1, and 5.] The future is to the right and up. The general characteristics of this diagram are summarized in Appendix B. The Penrose diagram is inserted. Note that the outgoing case ($c = -1$) is obtained by reflection about a horizontal axis (so, e.g., $\mathscr{I}^- \to \mathscr{I}^+$).

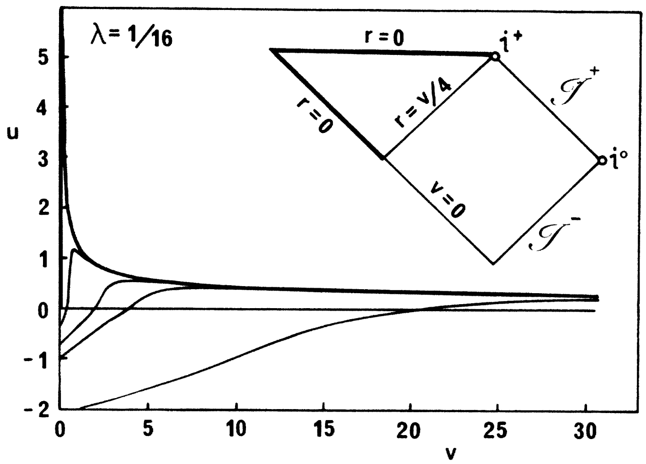

FIG. 2. As in Fig. 1 but for $\lambda = \frac{1}{16}$. The u axis ($u > 0$) is a "shell-focusing" singularity.

With

$$D = c\left[\frac{1}{u} - u\right] \tag{35}$$

we find

$$f = \frac{(4r - cv)^2}{32r}[L^2 + 4 + cL(L^2 + 4)^{1/2}] \tag{36}$$

which, on the horizon ($4r = cv > 0$, $u = 0$) reduces to

$$f(u = 0,\ v > 0) = \frac{cv}{4} . \tag{37}$$

The spacetime diagram is shown in Fig. 2.

For $\lambda < \frac{1}{16}$ we have

$$f = \frac{-cD_1(r - {}_0r/4)(r - {}_1r/4)}{r} , \tag{38}$$

where ${}_0r \equiv cv(1 - \Delta)$, ${}_1r \equiv cv(1 + \Delta)$, and $\Delta \equiv \sqrt{1 - 16\lambda}$. From Eq. (28) we find

$$D(u) = \ln\left|(r - {}_0r/4)^{(\Delta - 1)/2\Delta}(r - {}_1r/4)^{(\Delta + 1)/2\Delta}\right| . \tag{39}$$

With

$$D(u) = \frac{1 + \Delta}{2\Delta}\ln|cu| \tag{40}$$

and $cu > 0$ for ${}_1r/4 > r$, we find

$$f = \frac{1 + \Delta}{2\Delta r}(r - {}_0r/4)^{2/(1+\Delta)} \tag{41}$$

so that the choice (40) is useful for $r > {}_0r/4$. Similarly, with

$$D(u) = \frac{\Delta - 1}{2\Delta}\ln|cu| \tag{42}$$

and $cu > 0$ for ${}_0r/4 > r$, we find

$$f = \frac{1 - \Delta}{2\Delta r}({}_1r/4 - r)^{2/1-\Delta} \tag{43}$$

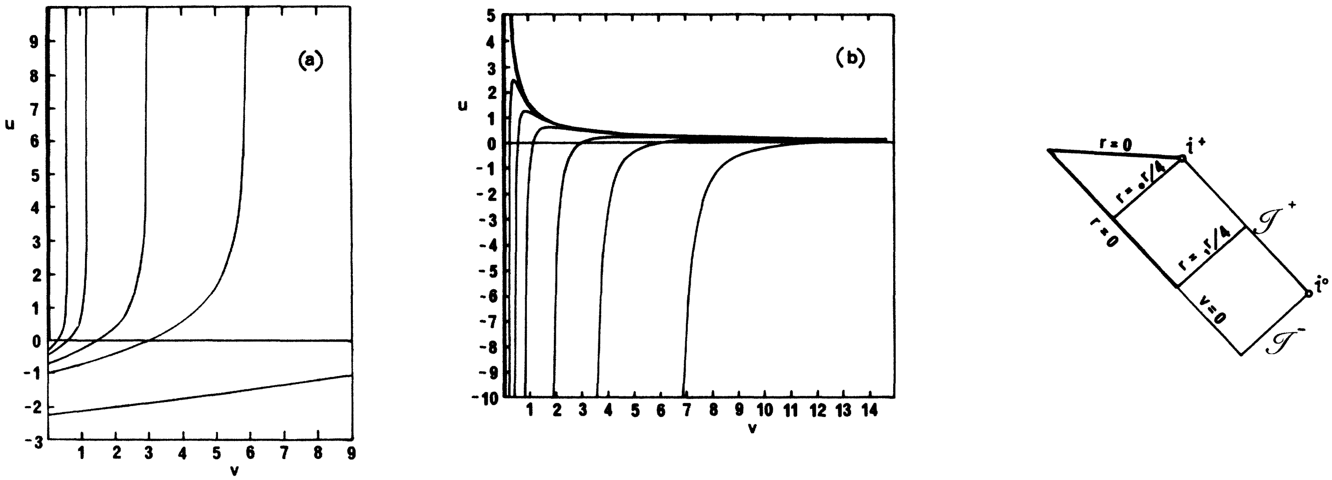

FIG. 3. u-v diagram for a linear mass function $m = c\lambda v$ with $\lambda < \frac{1}{16}$ and $c = +1$ (ingoing field). (The particular case shown has $\lambda = \frac{1}{18}$.) (a) A portion of the spacetime obtained from Eq. (40) ($r > {}_0r/4$, $r = {}_0r/4$ at $u \to \infty$). The curves represent surfaces of constant r, where the values of r shown are (from top to bottom) 0.1, 0.2, $\frac{1}{2}$, 1, and 5. $r = 0$ is given by the positive u axis. (b) A portion of the spacetime obtained from Eq. (41) ($r < {}_1r/4$, $r = {}_1r/4$ at $u \to -\infty$). The curves represent surfaces of constant r, where the values of r shown are (from top to bottom) $r = 0$, 0.05, 0.1, 0.2, $\frac{1}{2}$, 1, and 2. The u axis also gives $r = 0$. In both diagrams the future is to the right and up. The general characteristics of these diagrams are summarized in Appendix B. The Penrose diagram is shown. Again, the outgoing case ($c = -1$) is obtained by reflection about a horizontal axis.

so that the choice (42) is useful for $r < {}_1r/4$. The spacetime diagram is shown in Fig. 3.

C. Exponential mass function

Finally we consider

$$m = \frac{1}{\beta}[\alpha \exp(\beta c v/2) + 1] , \tag{44}$$

where α and β are constants > 0, $c = -2B$ as above, and $-\infty < v < \infty$. With the source function (44), Eq. (12) can be integrated to give

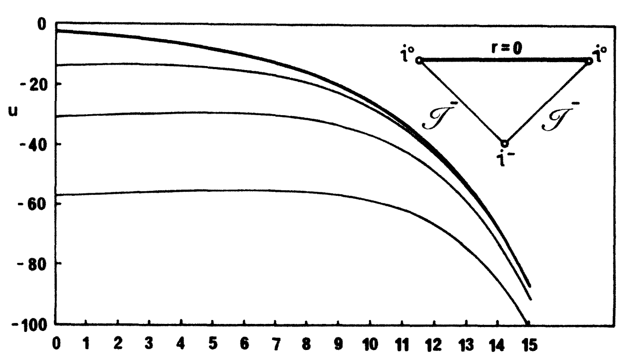

FIG. 4. u-v diagram for the exponential mass function (44) with $\alpha = \beta = 1$. We have chosen $c = +1$. The curves represent surfaces of constant r, where the values of r shown are (from top to bottom) 0, 10, 25, and 50. The future is to the right and up. Although only positive values of v are shown, v can be negative as well. The Penrose diagram is inserted. Again, the outgoing case ($c = -1$) is obtained by reflection about a horizontal axis.

$$\beta x + 2\ln|r - 2/\beta + x| - \beta c v/2 = P(u) , \tag{45}$$

where P is an arbitrary function of u, and x is defined by

$$x = (r^2 - 4r/\beta + 4m/\beta)^{1/2} . \tag{46}$$

From Eqs. (11) and (45) we find

$$f = -\frac{cP_1 x}{\beta r} . \tag{47}$$

Since $x > 0$, we can simply choose $P = -cu$ to define r. The spacetime diagram is shown in Fig. 4.

IV. SUMMARY AND DISCUSSION

We have examined the construction of double-null coordinates for the Vaidya metric, and reduced the problem to the integration of a single first-order nonlinear partial differential equation [Eq. (12)]. This equation has been integrated for linear and exponential mass functions [Eqs. (26) and (44)] and a complete regular covering of the associated spacetimes has been given. In the linear case we have given the first explicit regular covering of a spacetime with a naked shell-focusing singularity.

Note added in proof. Recently it has been shown[10] that shell-focusing singularities in the ingoing Vaidya spacetimes are strong curvature singularities (in the sense of Tipler[11]) *only* for mass functions which are initially linear functions of the advanced time.

ACKNOWLEDGMENTS

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APPENDIX A: THE ALGEBRA ASSOCIATED WITH A GENERAL SPHERICALLY SYMMETRIC METRIC IN DOUBLE-NULL COORDINATES

The following appendix gives the algebra for a general spherically symmetric metric in double-null coordinates. Although some of this has been given previously by Synge,[8] it is repeated here for convenience. The calculations were carried out with the aid of MACSYMA. Only nonvanishing terms are given.

The general form of a spherically symmetric metric in double-null coordinates (u,θ,ϕ,v) is

$$ds^2=-2f(u,v)du\,dv+r^2(u,v)d\Omega^2 , \tag{A1}$$

where

$$d\Omega^2 \equiv d\theta^2+\sin^2\theta\, d\phi^2 ,$$

and the contravariant metric tensor is given by

$$\begin{aligned} g^{14}&=-1/f(u,v) ,\\ g^{22}&=\sin^2\theta g^{33}=1/r^2(u,v) . \end{aligned} \tag{A2}$$

The associated Christoffel symbols of the second kind are

$$\begin{aligned} \Gamma^1_{11}&=f_1/f ,\\ \Gamma^1_{22}&=\sin^{-2}\theta\Gamma^1_{33}=rr_4/f ,\\ \Gamma^2_{33}&=-\sin\theta\cos\theta ,\\ \Gamma^2_{12}&=\Gamma^3_{13}=r_1/r ,\\ \Gamma^2_{42}&=\Gamma^3_{43}=r_4/r ,\\ \Gamma^3_{23}&=\cos\theta/\sin\theta ,\\ \Gamma^4_{22}&=\sin^{-2}\theta\Gamma^4_{33}=rr_1/f ,\\ \Gamma^4_{44}&=f_4/f . \end{aligned} \tag{A3}$$

The Riemann-Christoffel tensor as calculated from (A3) is

$$\begin{aligned} R_{1212}&=\sin^{-2}\theta R_{1313}=r(f_1r_1/f-r_{11}) ,\\ R_{1224}&=\sin^{-2}\theta R_{1334}=rr_{14} ,\\ R_{1414}&=-f_{14}+f_1f_4/f ,\\ R_{2323}&=r^2\sin^2\theta(1+2r_1r_4/f) ,\\ R_{2424}&=\sin^{-2}\theta R_{3434}=r(f_4r_4/f-r_{44}) . \end{aligned} \tag{A4}$$

The Ricci tensor then reduces to

$$\begin{aligned} R_{11}&=2(f_1r_1/f-r_{11})/r ,\\ R_{22}&=\sin^{-2}\theta R_{33}=2(r_1r_4+rr_{14})/f+1 ,\\ R_{44}&=2(f_4r_4/f-r_{44})/r ,\\ R_{14}&=(f_1f_4/f-f_{14})/f-2r_{14}/r . \end{aligned} \tag{A5}$$

From the components (A2) and (A5) we find that the Ricci scalar is given by

$$R=R^\alpha_\alpha=2\{(f_{14}-f_1f_4/f)/f^2 +[1+2(r_1r_4+2rr_{14})/f]/r^2\} \tag{A6}$$

and that

$$\begin{aligned} R^\beta_\alpha R^\alpha_\beta=&\,2[2f^2r_{14}+r(ff_{14}-f_1f_4)]^2/r^2f^6\\ &+2(f+2rr_{14}+2r_1r_4)^2/r^4f^2\\ &+8(fr_{11}-r_1f_1)(f_{44}-r_4f_4)/r^2f^4 . \end{aligned} \tag{A7}$$

From (A5) and (A6) it follows that the components of the Einstein tensor are

$$\begin{aligned} G_{11}&=2(f_1r_1/f-r_{11})/r ,\\ G_{22}&=\sin^{-2}\theta G_{33}=r^2(f_1f_4/f-f_{14})/f^2-2rr_{14}/f ,\\ G_{44}&=2(f_4r_4/f-r_{44})/r ,\\ G_{14}&=[f+2(r_1r_4+rr_{14})]/r^2 . \end{aligned} \tag{A8}$$

The Weyl tensor is given by

$$\begin{aligned} C_{1224}&=r^2(f_1f_4-ff_{14})/6f^2 -f/6+(rr_{14}-r_1r_4)/3 ,\\ C_{1334}&=\sin^2\theta C_{1224} ,\\ C_{1414}&=\frac{2f}{r^2}C_{1224} ,\\ C_{2323}&=\frac{-2r^2}{f}\sin^2\theta C_{1224} . \end{aligned} \tag{A9}$$

As usual, the Weyl scalar is given by

$$C_{\alpha\beta\gamma\delta}C^{\alpha\beta\gamma\delta}=K-2R^\beta_\alpha R^\alpha_\beta+\tfrac{1}{3}R^2 , \tag{A10}$$

where K is the Kretschmann scalar $(R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta})$. For the metric (1) we find

$$K=4(f+2r_1r_2)^2/r^4f^2+16(fr_{11}-r_1f_1)(fr_{44}-r_4f_4)/r^2f^4+16r_{14}{}^2/r^2f^2+4(f_1f_4-ff_{14})^2/f^6 . \tag{A11}$$

[For the case considered in this paper, from Eqs. (11), (12), (13), and (15) we find $K=48m^2/r^6$.]

APPENDIX B: CHARACTERISTICS OF THE u-v DIAGRAM FOR LINEAR MASS FUNCTIONS

For $\lambda>\frac{1}{16}$, from relation (32), we find that the singularity $r=0$ is given by

$$cu=\ln\left[\frac{1}{\sqrt{\lambda}cv}\right]+\frac{1}{\delta}\arctan\left[\frac{1}{\delta}\right] . \tag{B1}$$

The surface $r=\text{const}\equiv r_*>0$ intersects with the u axis $(v=0)$ at $u=u_*$, where

$$cu_*=-\left[\ln r_*+\frac{c\pi}{2\delta}\right] , \tag{B2}$$

and u is extremal (maximal for $c=+1$, minimal for $c=-1$) at

$$cv_m=\frac{r_*}{2\lambda} \tag{B3}$$

where

$$cu_m=-\left[\ln r_*+\frac{1}{\delta}\arctan\left[\frac{8\lambda-1}{\delta}\right]\right]. \tag{B4}$$

For $\lambda=\frac{1}{16}$, from relation (34), we find that $r=0$ is given by

$$c\left[\frac{1}{u}-u\right]=1+\ln\left[\frac{cv}{4}\right], \tag{B5}$$

so $cu\to0^+$ as $cv\to\infty$. The surface $r=\text{const}\equiv r_*>0$ intersects the v axis ($u=0$) at $cv=4r_*$ and intersects the u axis ($v=0$) at $u=u_*$, where

$$c\left[\frac{1}{u^*}-u^*\right]=\ln r_* . \tag{B6}$$

The curve $u(v)$ (defined by $r=r_*$) is extremal (maximal for $c=+1$, minimal for $c=-1$) at

$$cv_m=8r_* \tag{B7}$$

and $cu_m>0$, where

$$c\left[\frac{1}{u_m}-u_m\right]=2+\ln r_* . \tag{B8}$$

For $\lambda<\frac{1}{16}$, we must consider relation (39) defined by both Eqs. (40) and (42). With the definition (40) we have $r>{}_0r/4$. In particular, from Eqs. (39) and (40)

$$cu=({}_1r/4-r)(r-{}_0r/4)^{(\Delta-1)/(\Delta+1)}, \tag{B9}$$

where we have orientated $cu>0$ for $r<r_1/4$. Since $\Delta<1$, $cu\to\infty$ as $r\to{}_0r/4^+$. That is, given the surface $r=\text{const}=r_*>0$, the associated curve $u(v)$ has $cu\to\infty$ as $cu\to cv_m^-$, where

$$cv_m=\frac{4r_*}{1-\Delta} . \tag{B10}$$

Moreover, $u=0$ for

$$cv=\frac{4r_*}{1+\Delta} \tag{B11}$$

and $v=0$ for

$$cu=-r_*^{2\Delta/(\Delta+1)} . \tag{B12}$$

To cover the region $r<{}_1r/4$ we make use of Eqs. (39) and (42) from which

$$cu=({}_0r/4-r)({}_1r/4-r)^{(\Delta+1)/(\Delta-1)}, \tag{B13}$$

where we have orientated $cu>0$ for $r<{}_0r/4$. It follows that $r=0$ is given by

$$cu=\frac{(1-\Delta)}{(1+\Delta)^{(1+\Delta)/(1-\Delta)}}\left[\frac{4}{cv}\right]^{2\Delta/(1-\Delta)} . \tag{B14}$$

The surface $r\equiv r_*>0$ intersects the v axis ($u=0$) at $cv=4r_*/(1-\Delta)$ and $cu\to-\infty$ as $cv\to[4r_*/(1+\Delta)]^+$. The curve is extremal again at

$$cv_m=\frac{r_*}{2\lambda}, \tag{B15}$$

where

$$cu_m=\frac{(1-\Delta-8\lambda)}{(1+\Delta-8\lambda)^{(1+\Delta)/(1-\Delta)}}\left[\frac{8\lambda}{r_*}\right]^{2\Delta/(1-\Delta)} . \tag{B16}$$

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[2]See, for example, V. Hamity and R. Gleiser, Astrophys. Space Sci. **58**, 353 (1978); S. Bayin, Phys. Rev. D **19**, 2838 (1979); **21**, 2433 (1980); L. Herrera, J. Jiminez, and G. Ruggeri, *ibid.* **22**, 2305 (1980); M. Cosenza, L. Herrera, M. Esculpi, and L. Witten, *ibid.* **25**, 2527 (1982); M. Castagnino and N. Umerez, Gen. Relativ. Gravit. **15**, 625 (1983); V. Hamity and R. Spinosa, *ibid.* **16**, 9 (1984); R. Pim and K. Lake, Phys. Rev. D **31**, 233 (1985).

[3]These spacetimes fall into two categories: radiating objects which collapse to zero mass, and transitions from (say) Minkowski space to Schwarzschild space. For the former see, for example, M. Demianski and J. Lasota, Astrophys. Lett. **1**, 250 (1968); B. Steinmüller, A. King, and J. Lasota, Phys. Lett. **51A**, 191 (1975); K. Lake and C. Hellaby, Phys. Rev. D **24**, 3019 (1981); K. Lake, *ibid.* **26**, 518 (1982); J. Zhang and K. Lake, *ibid.* **26**, 1479 (1982); N. Santos, Phys. Lett. **106A**, 296 (1984). For the latter, see, for example, W. Hiscock, L. Williams, and D. Eardley (Ref. 1); A. Papapetrou, in, *A Random Walk in Relativity and Cosmology, Essays in Honour of P. C. Vaidya and A.K. Raychaudhuri*, edited by N. Dadhich *et al.* (Wiley-Eastern, New Delhi, 1985); Y. Kuroda, Prog. Theor. Phys. **72**, 63 (1984); K. Lake, Phys. Lett. **116A**, 17 (1986). An analysis of the backscattered radiation shows that radiating collapse to zero mass is (classically) unstable. See B. Waugh and K. Lake, Phys. Lett. **116A**, 154 (1986).

[4]See R. Lindquist, R. Schwartz, and C. Misner, Phys. Rev. **137**, 1364 (1965). The radiation coordinates are defective at horizons where $w^2\to\infty$, and at shell-focusing singularities where backscattered rays focus (see Ref. 3).

[5]W. Israel, Phys. Lett. **24A**, 184 (1967). It is worth noting here that for the linear function $m=\lambda v$ studied in this paper Israel's coordinates $(\tilde u,\theta,\phi,\tilde w)$ (see Ref. 6) have $U(\tilde u)=\tilde u^{-1/(1+4\lambda)}$ and it follows that $g_{\tilde u\tilde u}$ is irregular at $m=0$ for all finite $\tilde w$.

[6]The only application of Israel coordinates for the Vaidya metric of which we are aware is the consideration of radiating shells by Pim and Lake (Ref. 2).

[7]Indeed, Lindquist, Schwartz, and Misner (Ref. 4) pointed out that they were unable to find such transformations. Hiscock,

Williams, and Eardley (Ref. 1) give transformations to double-null coordinates for the case $m=\lambda v,\lambda$ and $v>0$. However, the resultant metrics are defective at the horizons for $\lambda\leq\frac{1}{16}$, the cases of interest for shell-focusing singularities. We find that the Penrose diagrams they give for $\lambda\leq\frac{1}{16}$ are, however, correct. The only regular double-null representation of the Vaidya metric of which we are aware is the $\lambda>\frac{1}{16}$ case given by Hiscock, Williams, and Eardley.

[8]This approach follows J. Synge, Ann. Mat. Pura. Appl. **98**, 239 (1974) who considered the Schwarzschild case, $m=$const and obtained the Kruskal-Szekeres metric directly from Einstein's equations.

[9]See, for example, C. Hernandez and C. Misner, Astrophys. J. Phys. **143**, 452 (1966); M. Cahill and G. McVittie, J. Math. Phys. **11**, 1382 (1970).

[10]K. Rajagopal and K. Lake, Queen's University Report, 1986 (unpublished).

[11]See, for example, F. J. Tipler, C. J. S. Clarke, and G. F. R. Ellis in, *General Relativity and Gravitation,* edited by A. Held (Plenum, New York, 1980).