Interacting many-gluon systems within the MIT bag model

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Based on quantum chromodynamics in a cavity, we calculate the spectra of exotic hadrons consisting of up to four interacting $M1$ or $E1$ gluons in the lowest eigenmodes of a spherical bag. The states that satisfy Bose-Einstein statistics are classified according to the group chain $U(24)\supset U(8)\otimes U(3)\supset SU(3)_{color}\otimes SU(2)_{spin}$, and the coefficients of fractional parentage are evaluated up to four gluons in a color-singlet state. The two-gluon interaction includes the one-gluon exchange, the Compton or annihilation graph, and the elementary four-gluon vertex. Using the parameters of the MIT bag model, both the two-gluon and the four-gluon states of lowest energy with the quantum numbers of the vacuum turn out to be degenerate with the perturbative vacuum state.

I. INTRODUCTION

Quantum chromodynamics, the leading candidate for a strong-interaction theory of quarks and gluons, has been with us for more than a decade[1] and the supporting evidence for this theory must still be considered as rather circumstantial. At low energies, for instance, the non-Abelian gauge theory that is based on the symmetry group $SU(3)_{color}$ has been fairly successful in the description of the low-lying hadron states.[2] Quantum chromodynamics, however, predicts also the existence of exotic hadronic matter, i.e., matter that is not composed primarily of three quarks or a quark-antiquark pair. In spite of the tremendous effort put into the clarification of this crucial issue, the experimental evidence for the existence of these exotic hadrons is still rather weak.[3-5] Sooner or later, however, we will have to understand the properties of these states thoroughly, if they exist. Alternatively, if they do not exist, we must answer the pertinent question: why, out of the many possible states given, e.g., by the state vectors

$$|q^{3A+M},\bar{q}^{M},g^{N}\rangle \quad \text{with } A,M,N=0,1,2,\ldots, \tag{1.1}$$

can we only observe in nature a tiny subset of these, i.e., the baryons with $(A,M,N)=(1,0,0)$ and the mesons with $(A,M,N)=(0,1,0)$? The existence of such exotic states would, of course, shed some light on the most interesting interaction, that of two quarks in the color $\{6\}$ state, and a quark-antiquark pair in a color $\{8\}$ state, which have not yet been observed in normal hadronic matter.

This article is devoted to the study of a particular case of exotic hadronic matter, i.e., the study of many-gluon systems with a fixed number of interacting gluons with $(A,M,N)=(0,0,N)$ in the framework of the MIT bag model.[2] Of course, the number of gluons is not a conserved quantity in quantum chromodynamics. However, in the "bagged" version of the gauge theory we are dealing with, the gluons can only occupy discrete energy levels that correspond to the cavity modes. Thus the ground-state energy of a noninteracting many-gluon system rises linearly with the number of gluons. Now, if the spacing between the energy levels and in particular the energy of the first cavity mode is large, which is the case for bag radii that correspond to the usual size of the hadron, states with a different number of gluons are not likely to be mixed. This argument does not hold once the strong interaction is turned on, and, due to the strong splitting of the many-gluon states, some states consisting of a different number of gluons may become quasidegenerate. A possible strategy on how to tackle these problems of degenerate levels, is to evaluate the interacting many-gluon spectra for different but fixed numbers of gluons and then identify those states that are quasidegenerate in energy and carry the same quantum numbers; one would then have to take these states as a basis for a coupled-channel treatment in order to obtain the correct spectrum. Thus we consider this article as a contribution to the first step in this direction.

The outline of the paper is as follows. In Sec. II, we discuss the classification and quantum numbers of a general many-gluon system satisfying Bose-Einstein statistics. Then, in Sec. III, we turn to the evaluation of the coefficients of fractional parentage which describe the two-gluon content of a general wave function for a noninteracting many-gluon system completely. Finally, in Sec. IV, after having introduced the two-gluon interaction via one-gluon exchange, the Compton effect, and the elementary four-gluon vertex, we discuss the many-gluon spectra for a fixed number of identical transverse-magnetic or -electric gluons of spin 1.

II. CLASSIFICATION OF A MANY-GLUON SYSTEM

In this section we want to construct the $SU(3)_{color}$ and $SU(2)_{spin}$ representations of a system consisting of N identical gluons that carry color $\{8\}$ and spin-1 quantum numbers and satisfy Bose-Einstein statistics. For example, a two-gluon system in the color $\{1\}$ representation can exist only with total spin 0 or 2, since the wave function with spin 1 would violate Bose-Einstein statistics. Thus the color and spin parts of the wave function must be selected such that the total wave function is symmetric with respect to the interchange of any two gluons. The

solution of this nontrivial many-gluon problem is much more complicated than the corresponding many-quark problem. The reason for this lies in the fact that the gluons, rather than being in the fundamental representation like the quarks, are in the adjoint (or regular) representation of $SU(3)_{color}$ and $SU(2)_{spin}$.

Our aim is to construct the irreducible representations (irreps) of the $SU(3)_{color} \otimes SU(2)_{spin}$ group that are compatible with Bose-Einstein statistics. As a first step toward this goal we must find the group which is suitable for the description of the internal structure of a many-gluon system. To guide our ideas let us introduce creation and annihilation operators $b^{\dagger}_{\alpha s}$ and $b_{\alpha s}$ which describe a gluon with color index $\alpha = 1, \ldots, 8$ and spin component $s = 1, \ldots, 3$. We can now write down the bilinear operators that conserve the number of gluons and form the algebra of U(24):

$$C_{\alpha s,\beta t} = b^{\dagger}_{\alpha s} b_{\beta t} \ . \tag{2.1}$$

The generators of the U(8) and U(3) subgroups can be easily obtained from (2.1) by contracting the spin and color indices, respectively, i.e.,

$$\mathscr{C}_{\alpha\beta} = \sum_{s=1}^{3} C_{\alpha s,\beta s} \ , \tag{2.2}$$

$$\mathrm{C}_{st} = \sum_{\alpha=1}^{8} C_{\alpha s,\alpha t} \ . \tag{2.3}$$

The generators of the subgroups $SU(3)_{color}$ and $SU(2)_{spin}$ are linear combinations of these operators and given by

$$F_{\gamma} = i \sum_{\alpha,\beta=1}^{8} f_{\alpha\beta\gamma} \mathscr{C}_{\alpha\beta} \ , \tag{2.4}$$

$$S_u = i \sum_{s,t=1}^{3} \epsilon_{stu} \mathrm{C}_{st} \ , \tag{2.5}$$

where $f_{\alpha\beta\gamma}$ and ϵ_{stu} denote the structure constants of SU(3) and SU(2), respectively.

We have found in U(24) the group that is appropriate for the description of the internal structure of a many-gluon system. However, in order to describe this system completely, we need to know the group chain and quantum numbers of the corresponding representations. These are given by

$$\begin{array}{ccccc} [N\dot{0}] & & [h_1h_2h_3\dot{0}] & & [h_1h_2h_3] \\ \mathrm{U}(24) & \supset & \mathrm{U}(8) & \otimes & \mathrm{U}(3) \\ & & \cup \quad \delta & & \cup \quad \Omega \\ & & \mathrm{SU}(3)_{color} & & \mathrm{SU}(2)_{spin} \\ & & (\lambda,\mu)Y,T,T_3 & & S,S_3 \end{array} \tag{2.6}$$

where the irreps of U(24), U(8), and U(3) are represented by Young diagrams.[6] Thus the (totally symmetric) irrep of U(24) is described by N boxes in the first row and zero boxes in the next 23 which is indicated by a dot above the 0. The most general irrep of the U(3) group can have three rows with h_1, h_2, and h_3 boxes in the first, second, and third rows, respectively. Finally, the most general irrep of the U(8) group could have in principle eight rows; in our many-gluon problem with spin-1 gluons, however, the irreps of U(8) are restricted to three rows. In fact the Young diagrams describing the irreps of U(8) must be identical to the diagrams of U(3) through the condition that the product representation of $U(8) \otimes U(3)$ must be symmetric.[7] The symbols (λ,μ) and S label the $SU(3)_{color}$ and $SU(2)_{spin}$ representations and the quantum numbers Y,T,T_3, and S_3 characterize the individual members of the $SU(3)_{color}$ and $SU(2)_{spin}$ multiplets. The multiplicity indices are denoted by δ and Ω and N is the total number of gluons which must satisfy the relation

$$N = h_1 + h_2 + h_3 \ . \tag{2.7}$$

We now turn to the important question: which irrep of $SU(3)_{color}$ [or $SU(2)_{spin}$] is contained in a given irrep of U(8) [or U(3)]? The $U(3) \supset SU(2)_{spin}$ reduction problem has been solved by Elliott[8] many years ago, using the method of plethysm which was developed by Littlewood.[9] A detailed description of this quite general method has been given by Wybourne[10] and an elementary introduction can be found in Ref. 11. In this paper we merely quote the results for the $U(3) \supset SU(2)_{spin}$ reduction. With $p = h_1 - h_2$ and $q = h_2 - h_3$ the multiplicity index Ω takes the values

$$\Omega = \min(p,q), \ \min(p,q)-2, \ldots, \ 0 \text{ or } 1 \tag{2.8}$$

and the allowed spin values for $\Omega = 0$ are

$$S = \max(p,q,) \ \max(p,q)-2, \ldots, \ 0 \text{ or } 1 \ , \tag{2.9}$$

whereas for $\Omega \neq 0$ we obtain

$$S = \Omega, \Omega+1, \Omega+2, \ldots, \Omega+\max(p,q) \ . \tag{2.10}$$

To illustrate these equations let us discuss two examples. For the irrep $[h_1h_2h_3] = [210]$ of U(3) we have $(p,q) = (1,1)$. The only possible value for the multiplicity is $\Omega = \min(p,q) = 1$ according to Eq. (2.8). Thus the spins contained in the [210] irrep of U(3) are $S = 1$ and $S = 2$. For the irrep [300] we have $(p,q) = (3,0)$ and $\Omega = 0$. Thus only spin values of $S = 1$ and 3 will be allowed. In Table I we have listed all spin states contained in a particular irrep of U(3) up to five gluons.

The results for the $U(8) \supset SU(3)_{color}$ reduction problem up to five gluons are also given in Table I. These representations can be found in a paper by Butler and Wybourne[12] or can be determined using the general formula

$$[h]_{U(8)} = [h_1h_2h_3\dot{0}]_{U(8)} = \sum_{\substack{r[\alpha] \\ [\beta][\gamma]}} (-1)^r \Gamma_{\alpha 1 rh} g_{\beta\gamma\alpha} [\beta]_{U(3)_{color}} \times [\gamma]^{*}_{U(3)_{color}} \tag{2.11}$$

which is derived in Appendix A. Here the sum over r runs from 0 to 3 and $[\alpha] = [\alpha_1\alpha_2\alpha_3]$ is a partition of $N-r$. The symbols $[\beta] = [\beta_1\beta_2\beta_3]$ and $[\gamma] = [\gamma_1\gamma_2\gamma_3]$ denote the irreps of $U(3)_{color} \supset SU(3)_{color}$ and $[\gamma]^{*}$ stands for the complex-conjugate irrep of $[\gamma]$. The symbols $\Gamma_{\alpha 1 rh}$ are by definition the expansion coefficients of the direct product of the representations $[\alpha]$ and $[1^r]$ (Refs. 6

TABLE I. $SU(3)_{color} \otimes SU(2)_{spin}$ irreps contained in a given $U(8) \otimes U(3)$ irrep. $\langle d \rangle$ describes the dimension of the U(8) irrep and $[d']$ the dimension of the U(3) irrep. Two notations for the $SU(3)_{color}$ irrep are used: $\{d''\}$ denotes the dimension of the irrep and (λ,μ) are the ordinary $SU(3)_{color}$ labels.

U(8)	U(3)	$SU(3)_{color}$	$SU(2)_{spin}$
$[1\dot{0}]$		$(1,1)=\{8\}$	1
$\langle 8 \rangle$	[3]		
$[2\dot{0}]$		$(2,2)+(1,1)+(0,0)=\{27\}+\{8\}+\{1\}$	0,2
$\langle 36 \rangle$	[6]		
$[1^2\dot{0}]$		$(1,1)+(3,0)+(0,3)=\{8\}+\{10\}+\{\overline{10}\}$	1
$\langle 28 \rangle$	$[\bar{3}]$		
$[3\dot{0}]$		$(3,3)+(2,2)+(3,0)+(0,3)+(1,1)+(0,0)=\{64\}+\{27\}+\{10\}+\{\overline{10}\}+\{8\}+\{1\}$	1,3
$\langle 120 \rangle$	[10]		
$[21\dot{0}]$		$(4,1)+(1,4)+(2,2)_2+(3,0)+(0,3)+(1,1)_3=\{35\}+\{\overline{35}\}+\{27\}_2+\{10\}+\{\overline{10}\}+\{8\}_3$	1,2
$\langle 168 \rangle$	[8]		
$[1^3\dot{0}]$		$(2,2)+(3,0)+(0,3)+(1,1)+(0,0)=\{27\}+\{10\}+\{\overline{10}\}+\{8\}+\{1\}$	0
$\langle 56 \rangle$	[1]		
$[4\dot{0}]$		$(4,4)+(3,3)+(4,1)+(1,4)+(2,2)_2+(1,1)_2+(0,0)=\{125\}+\{64\}+\{35\}+\{\overline{35}\}$	0,2,4
$\langle 330 \rangle$	[15]	$+\{27\}_2+\{8\}_2+\{1\}$	
$[21^2\dot{0}]$		$(3,3)+(4,1)_2+(1,4)_2+(2,2)_3+(3,0)_3+(0,3)_3+(1,1)_4+(0,0)$	1
$\langle 378 \rangle$	[3]	$=[64]+\{35\}_2+\{\overline{35}\}_2+\{27\}_3+[10]_3+\{\overline{10}\}_3+\{8\}_4+\{1\}$	
$[31\dot{0}]$		$(5,2)+(2,5)+(3,3)_2+(4,1)_2+(1,4)_2+(2,2)_4+(3,0)_3+(0,3)_3+(1,1)_4$	1,2,3
$\langle 630 \rangle$	[15′]	$=\{81\}+\{\overline{81}\}+\{64\}_2+\{35\}_2+\{\overline{35}\}_2+\{27\}_4+\{10\}_3+\{\overline{10}\}_3+\{8\}_4$	
$[2^2\dot{0}]$		$(6,0)+(0,6)+(3,3)+(4,1)+(1,4)+(2,2)_4+(3,0)+(0,3)+(1,1)_2+(0,0)_2$	0,2
$\langle 336 \rangle$	$[\bar{6}]$	$=\{28\}+\{\overline{28}\}+\{64\}+\{35\}+\{\overline{35}\}+\{27\}_4+\{10\}+\{\overline{10}\}+\{8\}_2+\{1\}_2$	
$[5\dot{0}]$		$(5,5)+(4,4)+(5,2)+(2,5)+(3,3)_2+(4,1)+(1,4)+(2,2)_2+(3,0)+(0,3)+(1,1)_2$	1,3,5
$\langle 792 \rangle$	[21]	$+(0,0)=\{216\}+\{125\}+\{81\}+\{\overline{81}\}+\{64\}_2+\{35\}+\{\overline{35}\}+\{27\}_2+\{10\}+\{\overline{10}\}+\{8\}_2$ $+\{1\}$	
$[41\dot{0}]$		$(6,3)+(3,6)+(4,4)_2+(5,2)_2+(2,5)_2+(6,0)+(0,6)+(3,3)_5+(4,1)_4+(1,4)_4$	1,2,3,4
$\langle 1848 \rangle$	[24]	$+(2,2)_7+(3,0)_4+(0,3)_4+(1,1)_5+(0,0)=\{154\}+\{\overline{154}\}+\{125\}_2+\{81\}_2+\{\overline{81}\}_2$ $+\{28\}+\{\overline{28}\}+\{64\}_5+\{35\}_4+\{\overline{35}\}_4+\{27\}_7$ $+\{10\}_4+\{\overline{10}\}_4+\{8\}_5+\{1\}$	
$[32\dot{0}]$		$(1,7)+(7,1)+(4,4)+(5,2)_2+(2,5)_2+(6,0)+(0,6)+(3,3)_5+(4,1)_5+(1,4)_5+(2,2)_8$	1,2,3
$\langle 1680 \rangle$	$[\overline{15}']$	$+(3,0)_4+(0,3)_4+(1,1)_6+(0,0)=\{80\}+\{\overline{80}\}+(125\}+\{81\}_2+\{\overline{81}\}_2+\{28\}+\{\overline{28}\}$ $+\{64\}_5+\{35\}_5+\{\overline{35}\}_5+\{27\}_8+\{10\}_4+\{\overline{10}\}_4+\{8\}_6+\{1\}$	
$[31^2\dot{0}]$		$(4,4)+(5,2)_2+(2,5)_2+(6,0)+(0,6)+(4,1)_5+(1,4)_5+(3,3)_4+(2,2)_9+(3,0)_5$	0,2
$\langle 1512 \rangle$	[6]	$+(0,3)_5+(1,1)_7+(0,0)_2=\{125\}+\{81\}_2+\{\overline{81}\}_2+\{28\}+\{\overline{28}\}+\{35\}_5+\{\overline{35}\}_5+\{64\}_4$ $+\{27\}_9+\{10\}_5+\{\overline{10}\}_5+\{8\}_7+\{1\}_2$	
$[2^21\dot{0}]$		$(5,2)+(2,5)+(6,0)+(0,6)+(3,3)_3+(4,1)_4+(1,4)_4+(2,2)_7+(3,0)_4+(0,3)_4$	1
$\langle 1008 \rangle$	$[\bar{3}]$	$+(1,1)_6+(0,0)=\{81\}+\{\overline{81}\}+\{28\}+\{\overline{28}\}+\{64\}_3+\{35\}_4+\{\overline{35}\}_4+\{27\}_7+\{10\}_4$ $+\{\overline{10}\}_4+\{8\}_6+\{1\}$	

and 10)

$$[\alpha]\times[1^r]=\sum_{[h]}\Gamma_{\alpha 1rh}[h] \tag{2.12}$$

and the $g_{\beta\gamma\alpha}$ are the expansion coefficients of the inner product in the symmetric group[6,10]

$$[\beta]\circ[\gamma]=\sum_{[\alpha]}g_{\beta\gamma\alpha}[\alpha]\,. \tag{2.13}$$

The coefficients $\Gamma_{\alpha 1rh}$ and $g_{\beta\gamma\alpha}$ can be determined readily with the techniques given in Refs. 6, 7, and 10. In order to evaluate the irrep $[\alpha]$, we have to subtract r boxes from

the Young diagram $[h_1h_2h_3]$ with the restriction that only one box can be removed from each row. The coefficient $\Gamma_{\alpha 1 rh}$ is always equal to unity, whereas the symbols $g_{\beta\gamma\alpha}$ are more difficult to evaluate. To this end we need the characters of the symmetric group, but fortunately there exist elementary techniques to calculate these.[13]

III. COEFFICIENTS OF FRACTIONAL PARENTAGE

The strong interactions of a many-gluon system can be described in second-order perturbation theory by the one-gluon-exchange interaction, the annihilation or Compton graph, and the elementary four-gluon vertex[14] (Fig. 1). For a fixed number of gluons the effective interaction can therefore be written as a sum of two-body interactions:

$$H_{\text{int}} = \sum_{j<k}^{N} V(j,k) . \tag{3.1}$$

Since the N gluons are assumed to be identical, each term of the interaction operator (3.1) will have the same matrix element, provided it is taken between states that satisfy the Bose-Einstein statistics. We can thus work out the summation trivially and the problem is reduced to the evaluation of the interaction between the first two gluons:

$$\left\langle [N]a \left| \sum_{j<k}^{N} V(j,k) \right| [N]a' \right\rangle = \frac{N(N-1)}{2} \langle [N]a \mid V(1,2) \mid [N]a' \rangle . \tag{3.2}$$

Since the operator $V(1,2)$ is a two-body operator, we do not need the full information on the N-gluon wave function for the evaluation of this matrix element. In fact we only need the two-particle content of the many-particle wave function. We thus decompose the N-particle wave function into 2- and $(N-2)$-particle pieces introducing the (real) coefficients of fractional parentage (CFP's) through the equation

$$| [N]a\alpha \rangle = \sum_{bc} C_{bc}^{a} [\, | [2]b \rangle \times | [N-2]c \rangle]_{a\alpha} . \tag{3.3}$$

Here a, b, and c denote the representations of $\mathrm{SU(3)_{color}} \otimes \mathrm{SU(2)_{spin}}$, including the multiplicity indices, and the α refers to all quantum numbers (Y, T, T_3, and S_3) that characterize the individual member of the $\mathrm{SU(3)_{color}} \otimes \mathrm{SU(2)_{spin}}$ multiplet a. The symbol

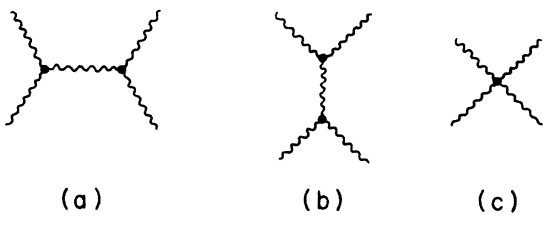

FIG. 1. The contributions of second order in the strong coupling constant to the interaction of two gluons: (a) the one-gluon-exchange interaction; (b) the annihilation or Compton graph; (c) the elementary four-gluon vertex.

$$[\, | [2]b \rangle \times | [N-2]c \rangle]_{a\alpha} = \sum_{\beta\gamma} \langle b\beta c\gamma \mid a\alpha \rangle \, | [2]b\beta \rangle \times | [N-2]c\gamma \rangle \tag{3.4}$$

denotes a linear combination of products of the 2- and $(N-2)$-particle wave functions which are coupled to the quantum numbers $a\alpha$ using the usual Clebsch-Gordan coefficients of SU(2) and SU(3) (Refs. 15 and 16). Thus the CFP's do not depend on the "magnetic" quantum numbers Y, T, T_3, and S_3 any more. Since the operator $V(1,2)$ must obviously conserve color and spin, we can write down the matrix element (3.2) immediately as

$$\left\langle [N]a \left| \sum_{j<k}^{N} V(j,k) \right| [N]a' \right\rangle = \frac{N(N-1)}{2} \sum_{b,c} C_{bc}^{a} C_{bc}^{a'} \langle [2]b \mid V(1,2) \mid [2]b \rangle \tag{3.5}$$

which can be readily evaluated once the CFP's are known. However, for particular many-gluon states which are discussed in Appendix B, one can work out the diagonal interaction matrix elements without actually knowing the CFP's.

We now turn to the evaluation of the CFP's for three and four gluons in a color $\{1\}$ state. In this context it is important to note that CFP's are an intrinsic property of the noninteracting many-gluon wave function, and therefore do not depend on the particular choice of two-body interaction. A possible calculation scheme for the CFP's, though probably not the most efficient one, is to construct the explicit wave functions for three and four gluons and work out their overlap with the two-gluon wave functions. Thus we have to consider all two-gluon wave functions with the quantum numbers listed in Table I. For the three- and four-gluon wave functions, however, we can restrict ourselves to the color $\{1\}$ states given in Table I, since we only want to study three- and four-gluon states that are color $\{1\}$.

The two-gluon wave functions with arbitrary color and spin are readily determined in terms of one-gluon wave functions using the SU(3) and SU(2) Clebsch-Gordan coefficients[15,16] and Table I. The three-gluon wave functions are obtained by first coupling the two-gluon and one-gluon wave functions to a total spin S and color $\{1\}$. Then we construct the totally symmetric three-gluon wave functions by symmetrizing a linear combination of product wave functions which describe the same total spin S and color $\{1\}$. Technically, the symmetrization is achieved by first diagonalizing the permutation operator P_{23} in the Hilbert space of the product wave functions of two and one gluons. Then we pick out the wave function to the eigenvalue 1 and disregard the remaining state. The resulting coefficients are listed in Table II.

In order to construct the four-gluon wave functions we can proceed in a similar way. First the products of the general two-gluon wave functions are coupled to a total spin S and color $\{1\}$. Then the permutation operator P_{23} is diagonalized in the space of the product wave functions and the states with the eigenvalue 1 are picked out. To separate the color (or spin) symmetry [4] and $[2^2]$ we also diagonalize the permutation operator P_{23}^{c} that acts on the

TABLE II. Coefficients of fractional parentage (CFP) for three gluons in a color $\{1\}$ state.

g^3			g^2			g			
Symmetry	Color	Spin	Symmetry	Color	Spin	Symmetry	Color	Spin	CFP
$[1^3]$	$\{1\}$	0	$[1^2]$	$\{8\}$	1	[1]	$\{8\}$	1	1
[3]	$\{1\}$	1	[2]	$\{8\}$	0	[1]	$\{8\}$	1	$\sqrt{5/9}$
			[2]	$\{8\}$	2	[1]	$\{8\}$	1	$\frac{2}{3}$
[3]	$\{1\}$	3	[2]	$\{8\}$	2	[1]	$\{8\}$	1	1

TABLE III. Coefficients of fractional parentage (CFP) for four gluons in a color $\{1\}$ state.

g^4			g^2			g^2			
Symmetry	Color	Spin	Symmetry	Color	Spin	Symmetry	Color	Spin	CFP
$[2^2]$	$\{1\}$	0	[2]	$\{1\}$	0	[2]	$\{1\}$	0	$-1/3\sqrt{5/6}$
			[2]	$\{1\}$	2	[2]	$\{1\}$	2	$5/6\sqrt{1/6}$
			[2]	$\{8\}$	0	[2]	$\{8\}$	0	$-4/3\sqrt{1/15}$
			[2]	$\{8\}$	2	[2]	$\{8\}$	2	$2/3\sqrt{1/3}$
			[2]	$\{27\}$	0	[2]	$\{27\}$	0	$1/3\sqrt{1/10}$
			[2]	$\{27\}$	2	[2]	$\{27\}$	2	$-1/6\sqrt{1/2}$
			$[1^2]$	$\{8\}$	1	$[1^2]$	$\{8\}$	1	0
			$[1^2]$	$\{10\}$	1	$[1^2]$	$\{\overline{10}\}$	1	$-\frac{1}{2}$
			$[1^2]$	$\{\overline{10}\}$	1	$[1^2]$	$\{10\}$	1	$\frac{1}{2}$
$[2^2]$	$\{1\}$	0	[2]	$\{1\}$	0	[2]	$\{1\}$	0	$-1/3\sqrt{1/3}$
			[2]	$\{1\}$	2	[2]	$\{1\}$	2	$1/6\sqrt{5/3}$
			[2]	$\{8\}$	0	[2]	$\{8\}$	0	$1/3\sqrt{2/3}$
			[2]	$\{8\}$	2	[2]	$\{8\}$	2	$-1/3\sqrt{5/6}$
			[2]	$\{27\}$	0	[2]	$\{27\}$	0	$\frac{1}{3}$
			[2]	$\{27\}$	2	[2]	$\{27\}$	2	$-\sqrt{5/6}$
			$[1^2]$	$\{8\}$	1	$[1^2]$	$\{8\}$	1	$-\sqrt{1/2}$
			$[1^2]$	$\{10\}$	1	$[1^2]$	$\{\overline{10}\}$	1	0
			$[1^2]$	$\{\overline{10}\}$	1	$[1^2]$	$\{10\}$	1	0
[4]	$\{1\}$	0	[2]	$\{1\}$	0	[2]	$\{1\}$	0	$-5/6\sqrt{1/3}$
			[2]	$\{1\}$	2	[2]	$\{1\}$	2	$-1/3\sqrt{5/3}$
			[2]	$\{8\}$	0	[2]	$\{8\}$	0	$1/3\sqrt{2/3}$
			[2]	$\{8\}$	2	[2]	$\{8\}$	2	$2/3\sqrt{2/15}$
			[2]	$\{27\}$	0	[2]	$\{27\}$	0	$-\frac{1}{2}$
			[2]	$\{27\}$	2	[2]	$\{27\}$	2	$-\sqrt{1/5}$
$[21^2]$	$\{1\}$	1	$[1^2]$	$\{10\}$	1	$[1^2]$	$\{\overline{10}\}$	1	$-\sqrt{1/6}$
			$[1^2]$	$\{\overline{10}\}$	1	$[1^2]$	$\{10\}$	1	$-\sqrt{1/6}$
			[2]	$\{8\}$	0	$[1^2]$	$\{8\}$	1	$-1/3\sqrt{1/2}$
			[2]	$\{8\}$	2	$[1^2]$	$\{8\}$	1	$-1/3\sqrt{5/2}$
			$[1^2]$	$\{8\}$	1	[2]	$\{8\}$	0	$-1/3\sqrt{1/2}$
			$[1^2]$	$\{8\}$	1	[2]	$\{8\}$	2	$-1/3\sqrt{5/2}$
$[2^2]$	$\{1\}$	2	[2]	$\{1\}$	0	[2]	$\{1\}$	2	$1/6\sqrt{5/6}$
			[2]	$\{1\}$	2	[2]	$\{1\}$	0	$1/6\sqrt{5/6}$
			[2]	$\{1\}$	2	[2]	$\{1\}$	2	$-1/6\sqrt{35/6}$
			[2]	$\{8\}$	0	[2]	$\{8\}$	2	$2/3\sqrt{1/15}$

TABLE III. (*Continued*).

g^4			g^2			g^2			
Symmetry	Color	Spin	Symmetry	Color	Spin	Symmetry	Color	Spin	CFP
			[2]	{8}	2	[2]	{8}	0	$2/3\sqrt{1/15}$
			[2]	{8}	2	[2]	{8}	2	$-2/3\sqrt{7/15}$
			[2]	{27}	0	[2]	{27}	2	$-1/6\sqrt{1/10}$
			[2]	{27}	2	[2]	{27}	0	$-1/6\sqrt{1/10}$
			[2]	{27}	2	[2]	{27}	2	$1/6\sqrt{7/10}$
			$[1^2]$	{8}	1	$[1^2]$	{8}	1	0
			$[1^2]$	{10}	1	$[1^2]$	$\{\overline{10}\}$	1	$-\frac{1}{2}$
			$[1^2]$	$\{\overline{10}\}$	1	$[1^2]$	{10}	1	$\frac{1}{2}$
$[2^2]$	{1}	2	[2]	{1}	0	[2]	{1}	2	$1/6\sqrt{1/3}$
			[2]	{1}	2	[2]	{1}	0	$1/6\sqrt{1/3}$
			[2]	{1}	2	[2]	{1}	2	$-1/6\sqrt{7/3}$
			[2]	{8}	0	[2]	{8}	2	$-1/3\sqrt{1/6}$
			[2]	{8}	2	[2]	{8}	0	$-1/3\sqrt{1/6}$
			[2]	{8}	2	[2]	{8}	2	$1/3\sqrt{7/6}$
			[2]	{27}	0	[2]	{27}	2	$-\frac{1}{6}$
			[2]	{27}	2	[2]	{27}	0	$-\frac{1}{6}$
			[2]	{27}	2	[2]	{27}	2	$\sqrt{7/6}$
			$[1^2]$	{8}	1	$[1^2]$	{8}	1	$-\sqrt{1/2}$
			$[1^2]$	{10}	1	$[1^2]$	$\{\overline{10}\}$	1	0
			$[1^2]$	$\{\overline{10}\}$	1	$[1^2]$	{10}	1	0
[4]	{1}	2	[2]	{1}	0	[2]	{1}	2	$1/6\sqrt{35/6}$
			[2]	{1}	2	[2]	{1}	0	$1/6\sqrt{35/6}$
			[2]	{1}	2	[2]	{1}	2	$1/3\sqrt{5/6}$
			[2]	{8}	0	[2]	{8}	2	$-1/3\sqrt{7/15}$
			[2]	{8}	2	[2]	{8}	0	$-1/3\sqrt{7/15}$
			[2]	{8}	2	[2]	{8}	2	$-2/3\sqrt{1/15}$
			[2]	{27}	0	[2]	{27}	2	$1/2\sqrt{7/10}$
			[2]	{27}	2	[2]	{27}	0	$1/2\sqrt{7/10}$
			[2]	{27}	2	[2]	{27}	2	$\sqrt{1/10}$
[4]	{1}	4	[2]	{1}	2	[2]	{1}	2	$1/2\sqrt{5/3}$
			[2]	{8}	2	[2]	{8}	2	$-\sqrt{2/15}$
			[2]	{27}	2	[2]	{27}	2	$3/2\sqrt{1/5}$

color part only. The resulting CFP's are listed in Table III. In contrast with the three-gluon case, a multiplicity occurs in the state with color (or spin) symmetry $[2^2]$. Thus there exist two (orthonormal) states with color {1}, $S=0,2$ and color (or spin) symmetry $[2^2]$. Finally, we apply an additional rotation in the plane of these two states in order to get CFP's that are square roots of rational numbers.

The CFP's listed in Tables II and III can be used to calculate any kind of two-body interaction in the three- and four-gluon system. Thus they do not depend on the interaction parameters used in our model which will be specified in Sec. IV.

IV. THE MANY-GLUON SYSTEM

We now turn to the evaluation of many-gluon spectra in the framework of the MIT bag model.[2] For simplicity, the Hilbert space of the one-gluon states has been truncated such that it includes only the lowest eigenmodes of positive and negative parity with spin 1. Thus all the gluons will occupy either the lowest transverse magnetic ($M1$) or transverse electric ($E1$) mode of a spherical cavity. If we restrict the Fock space to the subspace generated by the N-gluon states, we can describe the many-gluon system in a fixed cavity by the "effective" Hamilton operator

$$H=H_0+H_{\rm int}=\sum_{i=1}^{N}T(i)+\sum_{j<k}^{N}V(j,k)\ , \tag{4.1}$$

where the first term denotes the kinetic energy operator and the latter stands for the gluon-gluon interaction. Since we restrict our study to identical gluons occupying the same shell, we easily arrive, using Eqs. (3.3) and (3.4), at the matrix elements

$$\langle [N]a\mid H\mid [N]a'\rangle = N\frac{\hbar c\omega}{R}\delta_{aa'} + \frac{N(N-1)}{2}\sum_{bc}C_{bc}^{a}C_{bc}^{a'}\langle [2]b\mid V(1,2)\mid [2]b\rangle\ , \tag{4.2}$$

where $\hbar c\omega/R$ denotes the single-particle energy of the gluon in a cavity of the radius R.

We now turn to the explicit form of the two-gluon interaction $V(1,2)$ that describes the one-gluon exchange, the Compton graph, and the elementary four-gluon vertex in second-order perturbation theory. This two-body interaction can be written in the form

$$V(1,2)=\frac{\hbar c\alpha_s}{R}\mathbf{F}_1\cdot\mathbf{F}_2[A+B\mathbf{S}_1\cdot\mathbf{S}_2+C(\mathbf{S}_1\cdot\mathbf{S}_2)^2]\ , \tag{4.3}$$

where $\mathbf{F}_i$ and $\mathbf{S}_i$ $(i=1,2)$ denote the color and spin operator of the ith gluon, respectively, and α_s is the fine-structure constant of the strong interactions. The parameters ω, A, B, and C, listed in Table IV separately for the one-gluon-exchange interaction (a), the Compton graph (b), and the four-gluon vertex (c) of Fig. 1, have been calculated by Buser and Viollier[14] based on "bagged" quantum chromodynamics in the Feynman gauge. The numerical results agree very well with the calculations by Chanowitz and Sharpe,[17] Carlson, Hansson, and Peterson[18] ($M1$ and $E1$ gluons), and Barnes, Close, and Monaghan[19] ($M1$ gluons), for the transverse part of the one-gluon-exchange interaction and the four-gluon vertex. However, the Coulomb ($\equiv$scalar + longitudinal) part of the one-gluon-exchange and Compton interactions does not agree with the Coulomb contributions calculated by these authors in the Coulomb gauge. To illustrate our two-gluon interaction parameters, the normalized matrix elements of the effective interaction in an arbitrary two-gluon state

$$\Delta_{[2]b}=\langle [2]b\mid \mathbf{F}_1\cdot\mathbf{F}_2[A+B\mathbf{S}_1\cdot\mathbf{S}_2+C(\mathbf{S}_1\cdot\mathbf{S}_2)^2]\mid [2]b\rangle \tag{4.4}$$

have been plotted in Fig. 2 for $M1$ gluons and in Fig. 3 for $E1$ gluons in the lowest cavity modes.

Following the MIT group[2] the total energy matrix of an N-gluon bag is given by

$$E_{[N]a,[N]a'}(R)=\left[\frac{4\pi}{3}B_0R^3-\frac{Z_0\hbar c}{R}\right]\delta_{aa'} + \langle [N]a\mid H\mid [N]a'\rangle\ , \tag{4.5}$$

where B_0 is the bag constant and Z_0 is the sum of the zero-point energy and center-of-mass correction. For all these parameters and for the fine-structure constant of the strong interactions α_s we have chosen the original MIT bag values[2]

$$B_0=(145\ \text{MeV})^4/(\hbar c)^3\ , \qquad Z_0=1.842,\ \ \alpha_s=2.2\ , \tag{4.6}$$

for simplicity. The energy eigenvalues of the bag $E_{[N]n}(R)$ are obtained by diagonalizing the energy matrix (4.4). We must also consider the quadratic boundary condition of the MIT bag model which ensures the exact pressure balance at the boundary. States with nonzero spin obviously exert a nonspherical pressure on the bag surface and therefore the bag will deform. The study of deformed bags, however, lies beyond the scope of this article.[20] In order to satisfy the quadratic boundary condition at least in an average fashion for these nonzero spin states, we minimize the energy eigenvalues $E_{[N]n}(R)$ with respect to R. The equilibrium radii $R_{[N]n}$ are, of course, state dependent and given by the solutions of the equation

$$\frac{dE_{[N]n}(R)}{dR}=0\ , \tag{4.7}$$

TABLE IV. The single-particle energy parameter ω and the interaction parameters A, B, and C for two interacting $M1$ or $E1$ gluons in the lowest cavity modes of a spherical bag.

Gluons	ω	Graph		A	B	C
$M1$	2.7437	Fig. 1(a)	(a) Coulomb	−0.1046	0.0407	0.0814
			Transverse	0	−0.3399	0
$J^{\pi C}=1^{+-}$		Fig. 1(b)	(b) Coulomb	−0.1962	0.0981	0.0981
			Transverse	0	0	0
		Fig. 1(c)	(c)	−0.1549	0.1549	0.0774
			(a) + (b) + (c)	−0.4556	−0.0462	0.2569
$E1$	4.4934	Fig. 1(a)	(a) Coulomb	0.1381	0.0072	0.0144
			Transverse	0	−0.3277	0
$J^{\pi C}=1^{--}$		Fig. 1(b)	(b) Coulomb	−0.2072	0.1036	0.1036
			Transverse	0	0	0
		Fig. 1(c)	(c)	−0.1616	0.1616	0.0807
			(a) + (b) + (c)	−0.2307	−0.0553	0.1988

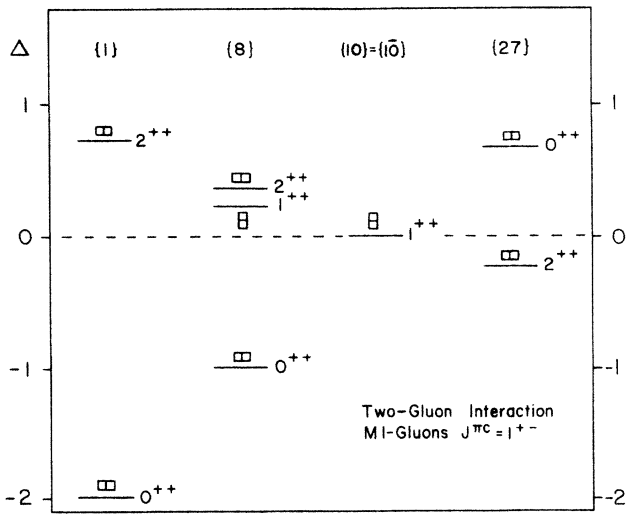

FIG. 2. The interaction of two $M1$ gluons with spin, parity, and charge conjugation $J^{PC}=1^{+-}$ in the lowest cavity mode for various color states. The symmetry of the color (or spin) part of the wave function is denoted by a Young diagram.

while the masses of the exotic hadrons are given by the value at the minimum

$$M_{[N]n}=E_{[N]n}(R_{[N]n}). \tag{4.8}$$

In Figs. 4 and 5 the spectra of the many-gluon systems (or glueballs) are shown for two, three, and four interacting gluons in a color $\{1\}$ state. In general, the interaction (4.3) does not conserve the permutation symmetry of the color part of the many-gluon wave function. Thus, the four-gluon states with $S=0$ and $S=2$ are linear combinations of the $[2^2]$ and $[4]$ symmetries of the color wave function. The spectra for $M1$ and $E1$ gluons are similar:

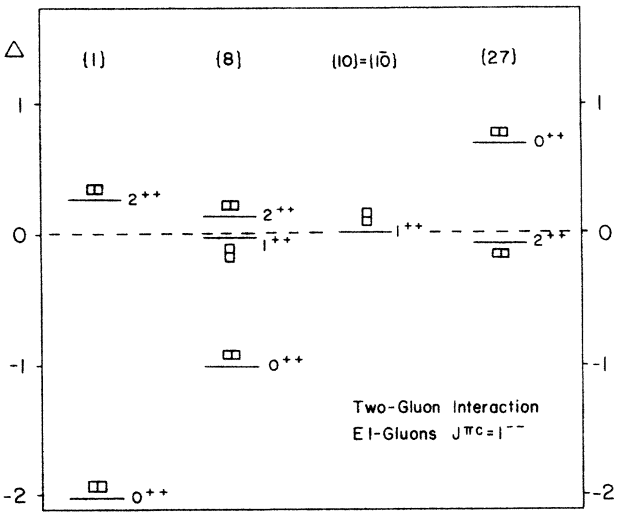

FIG. 3. The interaction of two $E1$ gluons with spin, parity and charge conjugation $J^{PC}=1^{--}$ in the lowest cavity mode for various color states. The symmetry of the color (or spin) part of the wave function is denoted by a Young diagram.

there is a large splitting of the various states due to the strong gluon-gluon interaction. In fact, the interaction energy may occasionally cancel the kinetic energy of the gluons, if

$$\Omega_{[N]n}=\omega N-Z_0+\alpha_s\Delta_{[N]n}\leq 0\ , \tag{4.9}$$

where $\Delta_{[N]n}$ is defined as

$$\Delta_{[N]n}=\frac{N(N-1)}{2}\langle [N]n\mid \mathbf{F}_1\cdot\mathbf{F}_2[A+B\mathbf{S}_1\cdot\mathbf{S}_2 + C(\mathbf{S}_1\cdot\mathbf{S}_2)^2]\mid [N]n\rangle\ , \tag{4.10}$$

consistent with Eq. (4.4). This cancellation occurs in the lowest states $|[2]\{1\}0^{++}\rangle$ and $|[4]\{1\}0^{++}\rangle$ consisting of $M1$ gluons, where we have

$$\begin{aligned}\Delta_{[2]\{1\}0^{++}}&=-1.9932\ ,\\ \Delta_{[4]\{1\}0^{++}}&=-4.4732\ ,\end{aligned} \tag{4.11}$$

and thus $\Omega_{[2]\{1\}0^{++}}$ and $\Omega_{[4]\{1\}0^{++}}$ turn negative for $\alpha_s>1.8289$ and $\alpha_s>2.0417$, respectively. Indeed, using the MIT value $\alpha_s=2.2$, we obtain

$$\begin{aligned}\Omega_{[2]\{1\}0^{++}}&=-0.7396\ ,\\ \Omega_{[4]\{1\}0^{++}}&=-0.7082\ ,\end{aligned} \tag{4.12}$$

which indicates that for a fixed bag radius the two-gluon state is lower in energy than the four-gluon state. The interaction energy per gluon

$$\begin{aligned}\tfrac{1}{2}\Delta_{[2]\{1\}0^{++}}&=-0.9966\ ,\\ \tfrac{1}{4}\Delta_{[4]\{1\}0^{++}}&=-1.1183\ ,\end{aligned} \tag{4.13}$$

is slightly larger in the four-gluon than in the two-gluon case.

For negative $\Omega_{[N]n}$'s, the minimization procedure (4.7) ceases to be meaningful, since there is no minimal value for the energy any more. Thus these states will become degenerate with the (perturbative) vacuum. In other words, in these states gluons can be created spontaneously, a fact which could cause the perturbative vacuum to

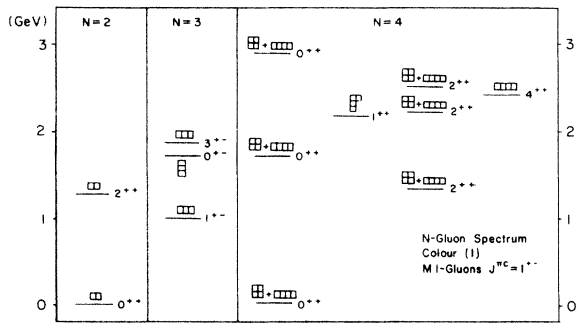

FIG. 4. The spectrum of $N=2$, 3, and 4 interacting $M1$ gluons with $J^{PC}=1^{+-}$ in the lowest cavity mode for a color $\{1\}$ state. The symmetry of the color (or spin) part of the wave function is denoted by a Young diagram.

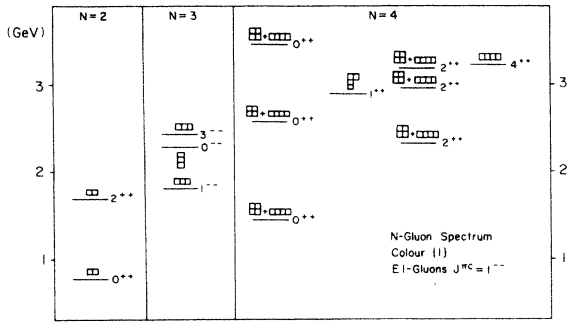

FIG. 5. The spectrum of $N=2$, 3, and 4 interacting $E1$ gluons with $J^{PC}=1^{--}$ in the lowest cavity mode for a color $\{1\}$ state. The symmetry of the color (or spin) part of the wave function is denoted by a Young diagram.

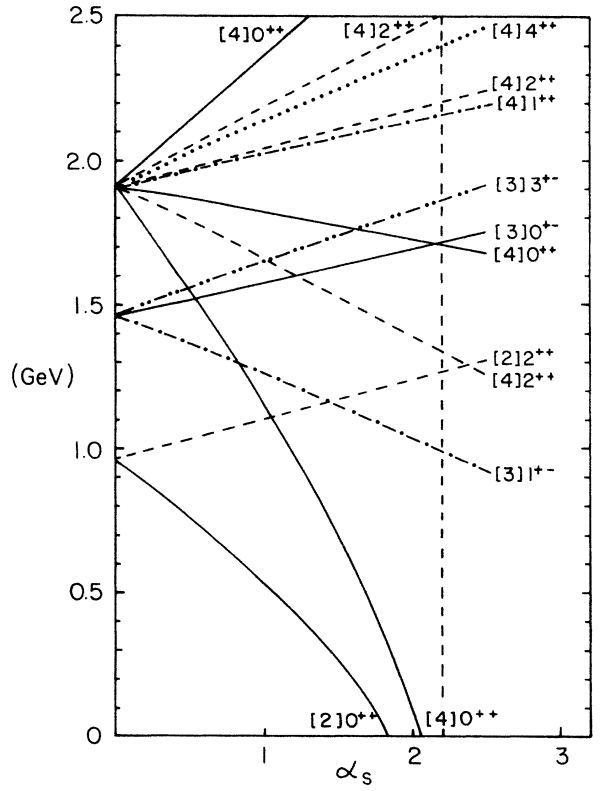

FIG. 6. The spectrum of $N=2$, 3, and 4 interacting $M1$ gluons with $J^{PC}=1^{+-}$ in lowest cavity mode for a color $\{1\}$ state. The energy levels are shown as a function of the strong coupling constant α_s and are labeled with the quantum numbers $[N]J^{PC}$.

break down. Of course, one should be careful in drawing such conclusions too early, since this degeneracy property depends critically on the model parameters and the perturbative treatment of the gluon-gluon interaction. The single-gluon energy, for instance, will be modified by the self-energy correction which may turn out as large.[21,22] Moreover, the fine-structure constant α_s could very well be much smaller than the value assigned in the original MIT bag model.[23] For this reason the spectrum of interacting $M1$ gluons is shown in Fig. 6 as a function of the coupling constant α_s, as well. In fact, the glueball spectrum, as calculated in lattice gauge theories, is in marked contrast to our expectation, that some of the 0^{++} states may be degenerate with the perturbative vacuum. Ishikawa, Sato, Schierholz, and Teper,[24] e.g., find the first 0^{++} state of gluonium at 0.67 GeV and the first 2^{++} at 1.47 GeV. Thus the predicting power of our calculations is limited by the uncertainty in the parameters used.

It is interesting to note that the four-gluon state $|[4]\{1\}0^{++}\rangle$ which is degenerate with the vacuum is not just a product of two two-gluon states $|[2]\{1\}0^{++}\rangle\times|[2]\{1\}0^{++}\rangle$ as suggested by Hansson, Johnson, and Peterson.[21] In fact the $|[2]\{27\}2^{++}\rangle \times|[2]\{27\}2^{++}\rangle$ state contributes with 33%, almost as much as the $|[2]\{1\}0^{++}\rangle\times|[2]\{1\}0^{++}\rangle$ state with 35%. Finally, also the states

$$|[2]\{1\}0^{++}\rangle\times|[2]\{8\}1^{++}\rangle$$

and

$$|[2]\{10\}1^{++}\rangle\times|[2]\{\overline{10}\}1^{++}\rangle$$

with 11% and 9%, respectively, make an important contribution to the lowest $|[4]\{1\}0^{++}\rangle$ state.

Our results in Tables I and III are in disagreement with the paper by Hansson, Johnson, and Peterson,[21] who obtain four instead of three color $\{1\}$ spin-0 states. We could not find the reason why these authors get four such states, since we were not able to prove the validity of their method. On the other hand, our technique is a standard method which has already found the way into the textbook literature.[10] Moreover, there is a completely independent (non-group-theoretical) proof of the method, since we have evaluated the four-gluon wave functions explicitly for the color $\{1\}$ case. The components of the wave functions are listed in Table III in terms of the coefficients of fractional parentage and the resulting states agree with Table I.

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APPENDIX A: THE PLETHYSM OF $U(8)\supset SU(3)_{color}$

Let us consider the group chain $U(8)\supset U(3)_{color}\supset SU(3)_{color}$, where the irreps of U(8) are mapped on the irreps of the subgroups $U(3)_{color}$ and $SU(3)_{color}$. The fundamental irrep of U(8), for instance, is mapped on the $\{8\}$ representation of $SU(3)_{color}$ according to the chain

$$\begin{aligned}&[1\dot{0}]\rightarrow[210]\rightarrow\{8\}=(1,1)\,,\\&U(8)\supset U(3)_{color}\supset SU(3)_{color}\,.\end{aligned}\tag{A1}$$

Following Wybourne[10] we denote this mapping by

$$[210]\otimes_P[1\dot{0}]\ , \tag{A2}$$

where the symbol $\otimes_P$ denotes plethysm.

Our aim is to determine the mappings of a general irrep of U(8) on the irreps of $U(3)_{color}$ and $SU(3)_{color}$

$$\begin{aligned}&[h_1h_2h_3\dot{0}]\rightarrow ?\rightarrow ?\ ,\\ &U(8)\supset U(3)_{color}\supset SU(3)_{color}\ ,\end{aligned} \tag{A3}$$

or, in symbolic notation, we want to find the plethysm

$$[210]\otimes_P[h_1h_2h_3\dot{0}]\ . \tag{A4}$$

A few formulas in Wybourne's book are of interest in this context: e.g.,

$$([\mu]-[\nu])\otimes_P[h] = \sum_{[\alpha][\beta]}(-1)^r\Gamma_{\alpha\beta h}([\mu]\otimes_P[\alpha]\times([\nu]\otimes_P[\tilde{\beta}])\ , \tag{A5}$$

with

$$[i]=[i_1i_2i_3\cdots],\quad i=\mu,\nu,\alpha,\beta,h$$

and

$$r=\beta_1+\beta_2+\beta_3+\cdots\ .$$

Here the $\Gamma_{\alpha\beta\gamma}$'s denote the expansion coefficients of the direct product of $[\alpha]$ and $[\beta]$, i.e.,

$$[\alpha]\times[\beta]=\sum_{[\gamma]}\Gamma_{\alpha\beta\gamma}[\gamma] \tag{A6}$$

and $[\tilde{\beta}]$ stands for the transposed partition of $[\beta]$. In the same book we also find the relation

$$([\mu]\times[\nu])\otimes_P[h] = \sum_{[\alpha][\beta]} g_{\alpha\beta h}([\mu]\otimes_P[\alpha])\times([\nu]\otimes_P[\beta])\ , \tag{A7}$$

where the $g_{\alpha\beta h}$'s are the expansion coefficients of the inner product (Clebsch-Gordan series) in the symmetric group, i.e.,

$$[\alpha]\circ[\beta]=\sum_{[\gamma]}g_{\alpha\beta\gamma}[\gamma]\ . \tag{A8}$$

In order to solve our problem, let us now introduce the identities

$$\{8\}=\{3\}\times\{\bar{3}\}-\{1\} \tag{A9}$$

or

$$[210]=[1\dot{0}]\times[1^20]-[1^3]\ . \tag{A10}$$

Substituting Eq. (A10) in Eq. (A5) we immediately arrive at

$$[210]\otimes_P[h]=([1\dot{0}]\times[1^20]-[1^3])[h]=\sum_{[\alpha][\beta]}(-1)^r\Gamma_{\alpha\beta h}([1\dot{0}]\times[1^20])\otimes_P[\alpha])\times([1^3]\otimes_P[\tilde{\beta}])\ . \tag{A11}$$

The last factor in Eq. (A11) can be readily evaluated by noting that the general irreps of $[\tilde{\beta}]$ and $[\beta]$ are restricted to the particular irreps $[\tilde{\beta}]=[r00]$ or $[\beta]=[1^r]$, where r is a non-negative integer number. We thus obtain

$$[210]\otimes_P[h]=\sum_{[\alpha]r}(-1)^r\Gamma_{\alpha 1rh}([1\dot{0}]\times[1^20])\otimes_P[\alpha])\ . \tag{A12}$$

Using Eq. (A7) we arrive at

$$[210]\otimes_P[h]=\sum_{[\alpha]r}\sum_{[\beta][\gamma]}(-1)^r\Gamma_{\alpha 1rh}g_{\beta\gamma\alpha}([1\dot{0}]\otimes_P[\beta])\times([1^20]\otimes_P[\gamma])\ . \tag{A13}$$

The first plethysm on the right-hand side is the $U(3)_{color}$ irrep $[\beta]$ itself, i.e., we have

$$[1\dot{0}]\otimes_P[\beta]=[\beta]\ . \tag{A14}$$

Similarly, we obtain, for the second plethysm,

$$[1^20]\otimes_P[\gamma]=[\gamma]^*\ , \tag{A15}$$

where for a general irrep $[\gamma]=[\gamma_1\gamma_2\gamma_3]$ the complex conjugate irrep is defined as

$$[\gamma]^*=[\gamma_1-\gamma_3,\gamma_1-\gamma_2,0]\ . \tag{A16}$$

Finally defining

$$[210]\otimes_P[h]=[h_1h_2h_3\dot{0}]_{U(8)} \tag{A17}$$

we arrive at

$$[h_1h_2h_3\dot{0}]_{U(8)}=\sum_{r[\alpha][\beta][\gamma]}(-1)^r\Gamma_{\alpha 1rh}g_{\beta\gamma\alpha}[\beta]_{U(3)_{color}}\times[\gamma]^*_{U(3)_{color}} \tag{A18}$$

which is precisely the equation mentioned in Sec. II. Of course, the direct product in Eq. (A18) remains to be evaluated using the rules given in Refs. 6 and 10.

APPENDIX B: INTERACTION MATRIX ELEMENTS FOR PARTICULAR MANY-GLUON STATES

If the color (or spin) part of the many-gluon wave function has total symmetry or antisymmetry with respect to the interchange of any two gluons, we can determine the expectation value of the interaction (4.3) directly without knowing the coefficients of fractional parentage. In this context, it is interesting to note that a totally antisymmetric color wave function can be constructed only for $N\leq 3$, since we restrict our study to spin-1 gluons. In the case of total symmetry or antisymmetry in color, the many-gluon wave function factorizes in a product of a color and spin part

$$|[N]a\alpha\rangle=|[h](\lambda,\mu)YTT_3\rangle\times|[h]SS_3\rangle\ , \tag{B1}$$

where $[h]=[N]$ or $[1^N]$ denotes the permutation symmetry of the color (or spin) wave function and the $(\lambda,\mu),Y,T,T_3$ and S,S_3 characterize the individual members of the multiplets of $SU(3)_{color}$ and $SU(2)_{spin}$,

respectively. Let us assume that the two-gluon interaction can be factorized in a color and spin part, as well,

$$V(1,2)=V_c(1,2)V_s(1,2)\ , \tag{B2}$$

which is certainly the case for the interaction given in Eq. (4.3). Thus also the expectation value of this interaction factorizes yielding

$$\langle [N]a\alpha \mid V(1,2) \mid [N]a\alpha\rangle = \langle [h]YTT_3 \mid V_c(1,2) \mid [h]YTT_3\rangle \times \langle [h]SS_3 \mid V_s(1,2) \mid [h]SS_3\rangle\ , \tag{B3}$$

and we are left with the problem of calculating the matrix elements of the color and spin part of the interaction separately. We now introduce the explicit form of the interaction (4.3):

$$V_c(1,2)=\frac{\hbar c\alpha_s}{R}\mathbf{F}_1\cdot\mathbf{F}_2\ , \qquad V_s(1,2)=A+B\mathbf{S}_1\cdot\mathbf{S}_2+C(\mathbf{S}\cdot\mathbf{S}_2)^2\ . \tag{B4}$$

The matrix element of $\mathbf{F}_1\cdot\mathbf{F}_2$ can be readily expressed as

$$\langle [h](\lambda,\mu)YTT_3 \mid \mathbf{F}_1\cdot\mathbf{F}_2 \mid [h](\lambda,\mu)YTT_3\rangle = \frac{C_3(\lambda,\mu)-3N}{N(N-1)}\ , \tag{B5}$$

where the Casimir operator of $SU(3)_{color}$ is defined as

$$C_3(\lambda,\mu)=\tfrac{1}{3}(\lambda^2+\lambda\mu+\mu^2)+\lambda+\mu\ . \tag{B6}$$

Similarly, we arrive at the matrix element

$$\langle [h]SS_3 \mid \mathbf{S}_1\cdot\mathbf{S}_2 \mid [h]SS_3\rangle = \frac{S(S+1)-2N}{N(N-1)} \tag{B7}$$

in terms of the Casimir operator of $SU(2)_{spin}$.

In order to determine the remaining matrix element

$$\langle [h]SS_3 \mid (\mathbf{S}_1\cdot\mathbf{S}_2)^2 \mid [h]SS_3\rangle\ , \tag{B8}$$

we can make use of the fact that the permutation operator for the spin part of the wave function P^s_{12} can be written as

$$P^s_{12}=-1+\mathbf{S}_1\cdot\mathbf{S}_2+(\mathbf{S}_1\cdot\mathbf{S}_2)^2=\tfrac{1}{3}+2\mathbf{G}_1\cdot\mathbf{G}_2\ , \tag{B9}$$

where the $\mathbf{G}_i$ denote the generators of the symmetry group U(3), that describe the ith particle. We then readily arrive at

$$\langle [h]SS_3 \mid (\mathbf{S}_1\cdot\mathbf{S}_2)^2 \mid [h]SS_3\rangle = \frac{2C_3(p,q)-S(S+1)+4/3N^2-2N}{N(N-1)}\ , \tag{B10}$$

where p and q are defined as $p=h_1-h_2$ and $q=h_2-h_3$ with $[h]=[h_1h_2h_3]$. Thus we obtain, for the general matrix element (B3),

$$\langle [N]a\alpha \mid V(1,2) \mid [N]a\alpha\rangle = \frac{C_3(\lambda,\mu)-3N}{N(N-1)}\left[A+B\frac{S(S+1)-2N}{N(N-1)}+C\frac{2C_3(p,q)-S(S+1)+\frac{4}{3}N^2-2N}{N(N-1)}\right]\frac{\hbar c\alpha_s}{R}\ . \tag{B11}$$

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