Quantum field theories around a large-Z nucleus

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We analyze quantum electrodynamics around a hypothetical highly charged ($Z \gtrsim 137$) nucleus by treating it as an external source. In contrast with the foregoing analyses which rely on the one-particle theory we construct a framework which enables us to create the quantum-field-theoretic treatment of the system. To deal with such a nonperturbative question we develop novel truncation and approximation procedures. Keeping only the lowest partial wave of the electron and the photon fields we transcribe the system into the form of two-dimensional fermion theory. We further convert the theory into a two-dimensional boson theory by using a bosonization technique. We then argue that the semiclassical approximation in the resultant boson theory is reasonably good and in particular does take care of the quantum effects of the original fermion theory. We investigate the asymptotic particle state of the theory and find that electrons appear as topological solitons. By analyzing the boson theory with an external source classically we show that the ground state undergoes the phase transition at a certain value of Z ($Z \simeq 150$ for nucleus size $\simeq 20$ fm) from the normal QED vacuum to an "anomalous" one which is characterized by the occurrence of real pair creation of electrons and positrons. Our result is confronted with the one obtained by the one-particle-theoretic treatment. Some comments are made on the possibility of understanding the peak structure in positron spectrum observed in heavy-ion collisions.

I. INTRODUCTION

One of the old problems in quantum electrodynamics (QED) is the behavior of relativistic electrons around a point charge Ze with $Z > 137$. (For the time being we will refer generically to the system with $Z \gtrsim 137$ as a supercritical system. A more precise definition will be given in Sec. IV.) As is well known[1] the one-particle theory runs into trouble for $Z > 137$ by rendering the energy eigenvalues complex.

It has been recognized by many people that the remedy of this problem must involve, as an essential ingredient, the consideration of the finite spatial extension of the charged source.[2,3] In this paper we wish to demonstrate that the treatment of QED as the quantum field theory is also indispensable for a complete understanding of the problem.

What is the physical significance of the problem? We ascribe its importance to the fact that it tests the quantum-field-theoretic nature of QED in its nonperturbative aspects. Although well tested and verified to great accuracy in perturbative aspects,[4] QED so far has not been tested in such regions. For instance, Schwinger's calculation[5] indicates that in sufficiently strong constant electric field the electron-positron pair creation occurs at a finite rate per unit time. But the rate is too small to measure for presently available electric field strength in the laboratory experiments. Here we wish to call the reader's attention to the distinction in the role played by the quantum-field-theoretic effect in perturbative and nonperturbative QED. While it merely implies a tiny (albeit very important) correction to the one-particle-theory result in the former, it gives rise to the zeroth-order effect in the latter; the occurrence of the event itself is the consequence of the quantum-field-theoretic nature of the theory.

Now we do have the experimental apparatus which enables us to create a highly charged system, at least during certain times, using heavy-ion collisions. Moreover the recent measurements of the positron spectrum[6,7] show an anomalous peak which might be attributable to the vacuum pair creation in the strong electric field. Therefore our interest in the problem goes beyond the academic one. (We emphasize, contrary to the current belief,[8] the possibility that the peak structure in the positron energy spectrum can be explained by the pure QED effect. See also Sec. V.) We, however, restrict ourselves to the case of a static and spherically symmetric charged source in this paper. We feel that the study of this simpler system constitutes an important first step toward the understanding of the physics around highly charged collision systems.

The problem with static charge distribution has been analyzed by a number of authors[9–11] from the one-particle-theoretical point of view. They solve the Dirac equations with the Coulomb field of an external charge Ze and find that the $1S$ level "dives" to the negative-energy continuum at a certain value of Z around 170. They then argue that in the presence of the vacancy of this level the spontaneous creation of positrons occurs.

The treatment based on the one-particle theory is, however, without subtlety in its interpretation. One of the puzzling things concerns the interpretation of the unoccupied level dived to the negative energy but not to the continuum, namely, to $-m_e < E < 0$. The proposed reinterpretation[12] of the ground state results in regarding it as a positron-nucleus bound state. But it is hard to imagine that the positively charged objects bind together by electromagnetic force. Moreover, there is no reason to believe that the result of the one-particle-theoretic treatment is accurate quantitatively even though it might be correct qualitatively.

We present a quantum-field-theoretic treatment of the problem which is free from these subtleties. Since everyone agrees with the conclusion of positron emission, quantum field theory is the natural framework to describe it. We will also make a comment on the above-mentioned subtlety (Sec. II).

The quantum-field-theoretic treatment of the supercritical system is difficult because the problem is in essence nonperturbative and our machinery for such a problem is generally quite poor. We will overcome this difficulty by appealing the physical consideration leading to the truncation of the system keeping only the lowest partial wave of photons and electrons. As will be explained in detail in Sec. III the resultant system can be regarded as a two-dimensional theory of fermions interacting with an electromagnetic field. Since we do have a variety of techniques for dealing with two-dimensional field theories, it is not too difficult to analyze the system. More luckily the experience in other problems[13-17] helps us.

The organization of this paper is as follows. In Sec. II we will briefly summarize the conventional treatment of the supercritical system based on the one-particle theory. We show that the supercritical ground state is connected with the normal Fock vacuum by a Bogoliubov transformation. We also make a comment on the essential difference between the quantum-field-theoretic and the one-particle-theoretic treatments.

In Sec. III we explain the basic ideas behind our construction of the quantum field theory around a supercritical system. Based on the observations made above, we develop the lowest-partial-wave QED. Starting with the representation as a two-dimensional fermion theory we convert it into a two-dimensional boson theory using the bosonization technique. Considering that the readers of this paper may be nuclear and atomic physicists, we give a careful explanation of the bosonization procedure, although it might be familiar to particle physicists. We then present a detailed analysis of the resultant boson theory and in particular elucidate the asymptotic states of the theory. We finally argue in Sec. III that the classical analysis of the boson theory can give a good approximation to the quantum fermion theory.

In Sec. IV we discuss the physics around a highly charged source utilizing the formalism developed in Sec. III. We restrict ourselves to the static and spherically symmetric source in this paper. We therefore focus upon the structure of the ground state. It will be shown in Sec. IV that when Z increases the vacuum undergoes the phase transition from the normal QED vacuum to an "anomalous" one. The latter vacuum is characterized by the occurrence of the real pair creation of electrons and positrons. Section V is devoted to conclusion and outlook.

II. ONE-PARTICLE THEORY VERSUS QUANTUM FIELD THEORY

We intend to clarify in this section the relationship between the one-particle-theoretic and the quantum-field-theoretic treatments of the supercritical system. Our discussion will be confined to the case of static source.

The conventional one-particle-theoretic treatment deals with QED with background electromagnetic fields $A_{\rm bg}$,

$$\mathscr{L}=\bar{\psi}(i\partial\!\!\!/+eA\!\!\!/_{\rm bg}-m_e)\psi \ . \tag{1}$$

Since the Lagrangian is quadratic in the dynamical variables the system is exactly soluble.

We introduce the Fock space spanned by the basis constructed by the creation and the annihilation operators defined by

$$\psi(x)=\sum_n b_n u_n(x)+\sum_{\mathbf{k}}[b(\mathbf{k})u(\mathbf{k},x)+d(\mathbf{k})^\dagger v(\mathbf{k},x)] \ , \tag{2}$$

where the first term indicates the sum over the bound state levels and u_n, $u(\mathbf{k})$, and $v(\mathbf{k})$ form the complete orthonormal eigensolution of the Dirac equation with "smeared" Coulomb potential. The Hamiltonian of the system can now be diagonalized

$$H=\sum_n E_n b_n^\dagger b_n+\sum_{\mathbf{k}}[\epsilon(\mathbf{k})b(\mathbf{k})^\dagger b(\mathbf{k})+\epsilon'(\mathbf{k})d(\mathbf{k})^\dagger d(\mathbf{k})] \ . \tag{3}$$

At $Z<Z_{\rm cr}$, where $Z_{\rm cr}$ denotes the critical value of Z at which the $1S$ level dives into the negative-energy continuum, the ground state of the system is given by the Fock vacuum,

$$b_n\,|\,0\rangle=b(\mathbf{k})\,|\,0\rangle=d(\mathbf{k})\,|\,0\rangle=0 \ . \tag{4}$$

Namely, we are taking the Fermi surface at $E=-m_e$.

At $Z>Z_{\rm cr}$, however, the Fock vacuum ceases to be the ground state of the system. Under this circumstance we assume the existence of a unitary transformation from the old to the new vacuum:

$$|\,\theta\rangle=U\,|\,0\rangle \ . \tag{5}$$

Correspondingly we define the operators which annihilate the new vacuum (supercritical vacuum):

$$B_n=Ub_nU^\dagger,\ \ B(\mathbf{k})=Ub(\mathbf{k})U^\dagger \ , \qquad D(\mathbf{k})=Ud(\mathbf{k})U^\dagger \ . \tag{6}$$

Notice that the Hamiltonian should also be diagonal in terms of the new variables.

Suppose now that only the $n=0$ level dives into the negative-energy continuum, that is, $E_0<-m_e$. For the time being let us forget about the spin degeneracy, although we shall come back to this point later. After some trial it is easy to observe that the unitary transformation (6) is nothing but the Bogoliubov transformation

$$U=\exp\{\theta[b_0^\dagger d(\mathbf{k}=0)^\dagger-d(\mathbf{k}=0)b_0]\} \tag{7}$$

with special Bogoliubov angle $\theta=\pi/2$ (Ref. 18). That is,

$$B_0=-d(\mathbf{k}=0)^\dagger,\ \ D(\mathbf{k}=0)^\dagger=b_0 \tag{8}$$

while all the other modes remain unrotated, namely,

$B_n = b_n$ $(n \neq 0)$, etc. To single out the $k=0$ mode we have to work with a large but finite box but we will not elaborate this point here.

At $Z > Z_{cr}$ the Hamiltonian thus takes the form

$$H = -m_e B_0^\dagger B_0 - E_0 D(\mathbf{k}=0)^\dagger D(\mathbf{k}=0) + \sum_{n\neq 0} E_n B_n^\dagger B_n + \sum_{\mathbf{k}} \epsilon(\mathbf{k}) B(\mathbf{k})^\dagger B(\mathbf{k}) + \sum_{\mathbf{k}\neq 0} \epsilon'(\mathbf{k}) D(\mathbf{k})^\dagger D(\mathbf{k}) + (E_0 + m_e) \ . \tag{9}$$

The last term in (9) represents the vacuum energy and its sign is negative (positive) at $Z > Z_{cr}$ $(Z < Z_{cr})$, showing the instability of the Fock vacuum in the supercritical electric field.[19]

When the electron spin is taken into account there are two different ways to pair the b_0 and the $d(\mathbf{k}=0)$ operators depending upon the spin orientation. We need to include the electron-electron interaction to determine which is the right way to pair these operators. We will not enter into this problem in this paper.

Our treatment of the one-particle theory, which is originally intended to provide a pedagogical explanation, reveals an interesting aspect of the theory. Namely, the redefinition of the ground state done in Ref. 12 amounts to performing transformation (8) only on b_0 leaving $d(\mathbf{k}=0)$ unchanged. This transformation, however, is not acceptable because it is not unitarily implementable. Apparently this solves the puzzle mentioned in the Introduction.

In contrast with the background-field problem (1) the full quantum field theory of the supercritical system is a difficult unsolvable problem because of the nonlinearity of the electron fields. While we are planning to describe our proposal to deal with such a system in the next section, here we will make a brief comment on the essential difference between the quantum-field-theoretic and one-particle-theoretic approaches.

Consider the problem of calculating the induced charge around the external charged source. In Feynman-diagram language the one-particle theory deals with the diagram depicted in Fig. 1(a). This is the only diagram which one can draw without the electron-electron interactions. In quantum field theory, however, there exist many other diagrams, the simplest of which is exhibited in Fig. 1(b).

We note that the latter diagram is of order $\sim\alpha^1\times$(all orders in $Z\alpha$), while the former one is $\sim\alpha^0\times$(all orders in $Z\alpha$). In this sense the latter one is down by order α but it is far from obvious that its contribution is really small because we have to deal with the all order effects of $Z\alpha$ which is now larger than unity. In other words we do not know any evident reasons for believing that the perturbation theory is reliable for this problem.[20] Moreover there are many more diagrams besides the one depicted in Fig. 1(b).

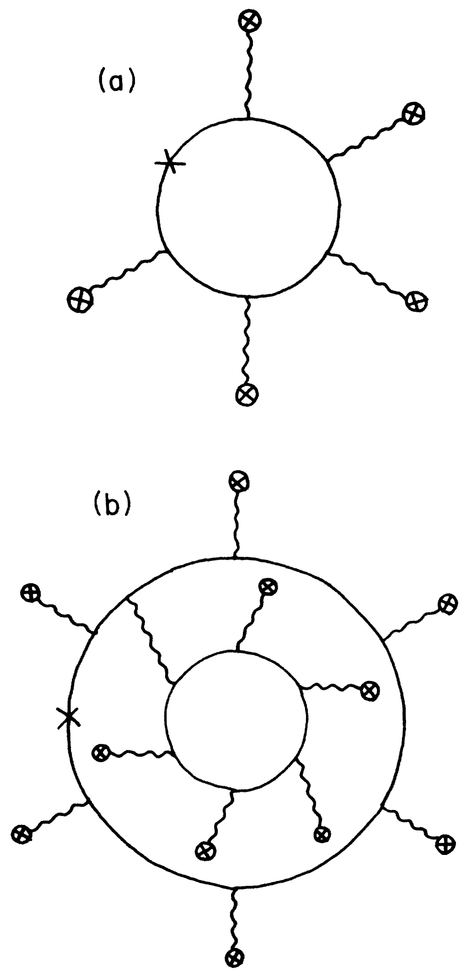

FIG. 1. The Feynman diagram for the induced electric charge in the background-field approach is drawn in (a), while in (b) one of the additional diagrams in quantum field theory is illustrated.

III. LOWEST-PARTIAL-WAVE QED

A. Quantum-field-theoretic approach to supercritical systems

We wish to develop the formalism which enables us to treat supercritical systems based on QED. Because of the nonperturbative nature of the problem it is inevitable to introduce the approximation and/or the truncation of the system.

To find a sensible way of truncation we go back to the original problem of the Dirac electron around the charge Ze. Let us ask the question "what is responsible for ruining the treatment of the bound-state problem?" The answer to this question is quite simple; as far as $Z < 274$ it is only the lowest partial wave, $j=\frac{1}{2}$, of the electrons that causes the trouble. This can be easily seen from the expression of the energy eigenvalue:[1]

$$E_{nj} = m_e \left[1 + \left[\frac{Z\alpha}{n-(j+\frac{1}{2})+[(j+\frac{1}{2})^2 - Z^2\alpha^2]^{1/2}} \right]^2 \right]^{-1/2} , \tag{10}$$

where m_e denotes the electron mass.

There is a simple explanation of the above fact. In the $j=\frac{1}{2}$ wave the Coulomb attraction is strong enough to dominate the centrifugal barrier and the "fall" of the electron to center occurs,[21] whereas in higher partial waves there is no such trouble because the centrifugal barrier wins the race. (When $Z>274$ the $J=\frac{3}{2}$ wave begins to fall but since this is not the presently attainable number of the electric charge we shall ignore this possibility.)

However, we are now considering not the point source but the one with finite spatial extension. In this case the bound-state problem is not ruined even when $Z>137$. As far as $Z \lesssim 300$, however, the only $j=\frac{1}{2}$ wave dives into the negative-energy continuum. (This estimate assumes the source size of the order of nuclear radius.) Therefore the dominance of the lowest partial wave seems to be quite good in one-particle theory provided that the source radius $\ll m_e{}^{-1}$.

We, however, wish to go beyond the one-particle theory. Are there any compelling reasons for believing that the validity of the lowest-partial-wave approximation continues to hold in the second-quantized theory? Although there is no *a priori* reason for this we can *a posteriori* justify this approximation; the electron-positron pair creation is largely suppressed by the centrifugal barrier and therefore the dominance of the lowest partial wave holds in a good approximation. Of course a complete justification of our approximation must await an explicit computation of the effects of higher partial waves, which is beyond the scope of this paper.[22]

We are thus led to analyze the lowest-partial-wave field theory around a supercritical source. The lowest-partial-wave approximation alone, however, does not guarantee the complete solubility of the theory at the quantum level. Therefore we have to invent a reliable approximation method for computation. Our proposal is the following. We first write the theory in the form of two-dimensional (one time and one-half space) fermion theory. Second, we convert the fermion theory into a two-dimensional boson theory via the bosonization technique. We then argue that the classical treatment of the resultant boson theory gives a reasonably good approximation to the original quantum fermion theory. In particular we claim that the classical analysis of the boson theory does take care of the quantum effects at the fermionic level. We will offer several arguments to justify our claim after developing the bosonized theory of the lowest-partial-wave QED.

B. Spherical harmonic expansion and the effective two-dimensional fermion theory

We consider QED with an external charge Ze which is described by the Lagrangian density

$$\mathscr{L}=-\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu}+\bar{\psi}(i\partial\!\!\!/+eA\!\!\!/-m_0)\psi-Ze\rho(x)A_0 \; . \tag{11}$$

In this paper we confine ourselves to a spherically symmetric source, $\rho(x)=\rho(r,t)$. The normalization is taken such that $\int d^3x\,\rho(x)=1$.

We shall keep only the lowest-partial-wave ($j=\frac{1}{2}$) fermion fields discarding all the "inessential" higher partial waves. We also retain only the s wave of the electromagnetic fields since our external source is spherically symmetric. This approximation implies throwing away the transverse photon while keeping the Coulomb interaction. By this we lose the opportunity to discuss the photon emission processes. Nonetheless, it does not mean any serious limitation for our present purpose. Furthermore the spherical source approximation seems to be quite appropriate also for the collision system because the "anomalous" positron emission is dominated[6] by large-angle scattering of heavy ions.

For definiteness we use the standard Dirac representation for the γ matrix

$$\gamma^0=\begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix},\quad \gamma^i=\begin{bmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{bmatrix},\quad \gamma_5=\begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix} \tag{12}$$

and introduce the two-component spinor X_δ ($\delta=\pm$) by which the fermion field with definite chirality, $\psi_{R,L}=\frac{1}{2}(1\pm\gamma_5)\psi$, can be expressed as

$$\psi_R=\begin{bmatrix} X_+ \\ X_+ \end{bmatrix},\quad \psi_L=\begin{bmatrix} X_- \\ -X_- \end{bmatrix}. \tag{13}$$

Using these chiral spinors (which are nothing but the Weyl spinors), the fermionic part of the Lagrangian can be written as

$$\mathscr{L}=\sum_{\partial=\pm} X_\delta^\dagger[(i\partial_0+A_0)+\delta\sigma_i(i\partial_i+A_i)]X_\delta - m_0 X_\delta^\dagger X_{-\delta}\; . \tag{14}$$

The s-wave electromagnetic fields have forms

$$A_0(x)=a_0(r,t)\;,\qquad A_i(x)=\hat{r}_i a_1(r,t)\;, \tag{15}$$

where $\hat{r}_i=r_i/r$ denotes the ith component of the radial unit vector. The fermion fields can be expanded as

$$X_\delta(x)=\frac{1}{r}\sum_{jm\sigma} v_{jm\sigma}^{\delta}(r,t)\Psi_{jm\sigma}(\Omega) \tag{16}$$

using the spherical harmonic basis $\Psi_{jm\sigma}$ which are the simultaneous eigenfunctions of J^2, J_3, and $\hat{\mathbf{r}}\cdot\boldsymbol{\sigma}$, where $\mathbf{J}=\mathbf{L}+\boldsymbol{\sigma}/2$, with eigenvalues $j(j+1)$, m, and σ, respectively. They are linear combinations of the parity-diagonal basis commonly used[1] in the hydrogen atom problem, and have the explicit form

$$\Psi_{jm\sigma=\pm}=\frac{1}{\sqrt{2}}\begin{bmatrix}\left[\frac{j+m}{2j}\right]^{1/2}Y_{j-1/2}^{m-1/2}\pm\left[\frac{j-m+1}{2(j+1)}\right]^{1/2}Y_{j+1/2}^{m-1/2}\\ \left[\frac{j-m}{2j}\right]^{1/2}Y_{j-1/2}^{m+1/2}\mp\left[\frac{j+m+1}{2(j+1)}\right]^{1/2}Y_{j+1/2}^{m+1/2}\end{bmatrix}. \tag{17}$$

Using the formula

$$\sigma_i\partial_i f(r,t)\Psi_{jm\sigma}(\Omega)=\frac{\sigma}{r}\left[\left[r\frac{\partial}{\partial r}+1\right]f(r,t)\Psi_{jm\sigma}(\Omega)+(j+\tfrac{1}{2})f(r,t)\Psi_{jm-\sigma}(\Omega)\right] \tag{18}$$

and the orthonormality of the basis we readily obtain the fermionic action written in terms of the partial waves:

$$S_f=\int dr\,dt\sum_{jm\sigma}\sum_{\delta}v_{jm\sigma}^{\delta *}[i\partial_0+ea_0+\delta\sigma(i\partial_r+ea_1)]v_{jm\sigma}^{\delta}-i\delta\sigma\frac{j+\frac{1}{2}}{r}v_{jm\sigma}^{\delta *}v_{jm-\sigma}^{\delta}-m_0v_{jm\sigma}^{\delta *}v_{jm\sigma}^{-\delta}\ . \tag{19}$$

From now on we shall keep only the lowest partial wave ($j=\frac{1}{2}$) and rewrite (19) in the form of two-dimensional fermion theory. We define the two-dimensional spinor

$$u_m^{\delta}=\left[\frac{1+i}{2}+\frac{1-i}{2}\tau_3\right]\begin{bmatrix}v_{(1/2)m\delta}^{\delta}\\ v_{(1/2)m-\delta}^{\delta}\end{bmatrix}, \tag{20}$$

where the extra rotation in (20) is for our later convenience. Using the two-dimensional γ matrix

$$\gamma^0=\tau_2,\ \ \gamma^1=i\tau_1,\ \ \gamma_5=\gamma^0\gamma^1=\tau_3\ , \tag{21}$$

the fermionic action (19) can readily be transcribed into the form

$$S_f=\int dr\,dt\sum_{m\delta}\bar{u}_m^{\delta}[\gamma^0(i\partial_0+ea_0)+\gamma^1(i\partial_r+ea_1)]u_m^{\delta}+\frac{i}{r}\bar{u}_m^{\delta}\gamma_5u_m^{\delta}-m_0\bar{u}_m^{\delta}u_m^{-\delta}\ . \tag{22}$$

Adding to this the gauge field part including the interaction with the external source,

$$S_g=\int dr\,dt\,2\pi r^2(\partial_0a_1-\partial_ra_0)^2-4\pi Zer^2\rho(r,t)a_0\ , \tag{23}$$

we have obtained the total action of the lowest-partial-wave QED on which we concentrate in this paper. From (22) and (23) we recognize that our system is precisely the two-dimensional QED with r-dependent charge $e/2\sqrt{\pi}r$. The last two-terms in (22) represent the effect of centrifugal barrier and the fermion mass, respectively.

Before closing this subsection we mention that transcribing the field theory with definite partial wave into the effective two-dimensional field theory can be done in all partial waves beyond the lowest one. In the case of the magnetic-monopole—fermion system this has been worked out in Ref. 23, and it is straightforward also in our case.

C. Boundary condition

When we rewrite the four-dimensional field theory into the set of two-dimensional field theories careful attention must be paid to the boundary condition at the origin. For higher partial waves ($j\geq\frac{3}{2}$) we have to impose the boundary conditions such that all $v_{jm\sigma}^{\delta}$ vanish at the origin. It should be noticed that the fermion fields would be singular at the origin were the boundary condition not obeyed; if one calculates the value of the fermion field at the origin by taking the limit to $r=0$, the answer depends on the angle used to approach the origin.

The boundary condition for the lowest partial wave, on the other hand, depends upon the basis one uses. If one works with an interaction representation where only the gauge interaction and the mass terms in (22) are regarded as perturbations the boundary condition for the lowest partial wave is the same as the one for the higher partial wave.[24] This is due to the presence of the centrifugal-barrier term in the unperturbed Hamiltonian.

On the other hand, we want to work with a different interaction representation where all the interactions including the centrifugal-barrier term are taken as perturbations. That is, we are to work with the massless free-fermion basis in two dimensions. This choice is required in order that the conventional bosonization procedure carries through[25] as will be seen later. Being the free-fermion basis it is only necessary to avoid the singularity (angular dependence) at the origin. Thus the proper boundary condition in our case is

$$v_{(1/2)m+}^{\delta}(0,t)-v_{(1/2)m-}^{\delta}(0,t)=0\ , \tag{24}$$

which implies that the coefficient of $\Psi_{(1/2)m+}-\Psi_{(1/2)m-}$ vanishes. Notice that the coefficients of $\Psi_{(1/2)m+}+\Psi_{(1/2)m-}$ need not vanish because this combination does not have angular dependence as one can easily check from (17).

The condition (24) does allow a clear physical interpretation for massless electrons. For this we recall that $\hat{\mathbf{r}}\cdot\sigma$ is equal to the helicity for outgoing waves and to the minus helicity for incoming waves. Therefore (24) simply states that the helicity conserves at the origin.

The boundary condition can be written, in terms of the two-dimensional spinor defined in (20), as

$$(1-\gamma^0)u_m^{\delta}(0,t)=0\ . \tag{25}$$

One may note the similarity with the Rubakov-Callan[13,14] boundary condition in the monopole-fermion system.

D. Bosonization

Our next step is to rewrite the resultant two-dimensional fermion theory into the equivalent two-dimensional boson theory. Since the discovery due to Coleman[26] and Mandelstam,[27] the fact that the fermion theory can be mapped into the corresponding boson theory (and vice versa) in two dimensions is widely accepted and is referred to under the name of bosonization.

In the sine-Gordon and massive Thirring models one can find the correspondence between the Heisenberg operators in both models.[27] In most cases, however, this is not possible and the bosonization proceeds via the following path:[28] we take the free massless fermion basis regarding the mass term and other terms as interactions. For free massless fermions one can find the operator correspondence between fermions and bosons. [See Eq. (26).] Working in the interaction representation one can rewrite the whole theory using the boson variable.

Let us start by introducing the boson-fermion correspondence for massless free fields:

$$u_m^\delta(r,t)=\left[\frac{c\mu}{2\pi}\right]^{1/2}K_m^\delta\begin{bmatrix}-iN_\mu\exp\{i\sqrt{\pi}[\phi_m^\delta(r,t)+\tilde\phi_m^\delta(r,t)]\}\\ N_\mu\exp\{i\sqrt{\pi}[-\phi_m^\delta(r,t)+\tilde\phi_m^\delta(r,t)]\}\end{bmatrix}, \tag{26}$$

where

$$\tilde\phi_m^\delta(r,t)=\lim_{\epsilon\to 0}\int_r^\infty ds\, e^{-\epsilon s}\dot\phi_m^\delta(s,t)\,. \tag{27}$$

Hereafter we frequently use the overdot and the prime for temporal and spatial derivatives, respectively. In (26) we have introduced four boson fields ϕ_m^δ ($m=\pm\frac{1}{2}$, $\delta=\pm$) corresponding to the four fermion fields u_m^δ. The symbol N_μ indicates the normal ordering with respect to mass μ (Ref. 26), c is a constant whose value is irrelevant in what follows, and K_m^δ stands for the Klein factor which guarantees the proper anticommutation relation between different Fermi fields.

The fact that the right-hand side of (26) does really have the property of a free massless Fermi field can be demonstrated by showing that all the n-point functions calculated in terms of the Bose variable agree with that of the free massless Fermi field. In the usual two-dimensional theories this can readily be done but in our case we have to be careful about the boundary condition (25). In our case it can properly be dealt with simply by imposing the boundary condition for the boson fields

$$\phi_m^\delta(0,t)=0\,. \tag{28}$$

Because of this boundary condition the commutation relations between boson fields contain extra terms in addition to the unusual one,

$$[\tilde\phi^{(+)}(r,t)+\eta\phi^{(+)}(r,t),\tilde\phi^{(-)}(r',t')+\eta'\phi^{(-)}(r',t')]$$
$$=-\frac{1}{4\pi}[(1-\eta)(1-\eta')A_+ +(1+\eta)(1+\eta')A_- +(1-\eta)(1+\eta')B_+ +(1+\eta)(1-\eta')B_-]\,, \tag{29}$$

where

$$A_\pm(r,t;r',t')=\ln ic\mu[(t-t')\pm(r-r')-i\epsilon]\,,$$
$$B_\pm(r,t;r',t')=\ln ic\mu[(t-t')\pm(r+r')-i\epsilon]\,, \tag{30}$$

where $\phi^{(+)}$ and $\phi^{(-)}$ indicate the positive- and the negative-frequency parts, respectively. The extra pieces (B's) in (29) serve to make the fermion's boundary condition (25) hold.

The Klein factors may be taken as

$$K_{1/2}^+=1,\quad K_{-1/2}^+=K_{1/2}^-=e^{i\pi F_+}\,,$$
$$K_{-1/2}^-=e^{i\pi F_+}e^{i\pi F_-}\,, \tag{31}$$

where

$$F_m=\int_0^\infty dr\sum_\delta \bar u_m^\delta\gamma^0u_m^\delta \tag{32}$$

denotes the fermion number of spin-up (-down) fermions for $m=\frac{1}{2}$ $(-\frac{1}{2})$.

In this paper, however, we work exclusively in the zero-fermion-number sector of the theory. This enables us to disregard the Klein factors since they are expressed only by F_m. It should be noticed that this restriction does not prevent us from discussing the charged state such as the one-electron state. Because of the cluster property we can talk about the behavior of the electron quite independently of the associated positron located at spatial infinity. We therefore disregard the Klein factor in our following discussions.

Upon accepting the correspondence (26) we can readily express our fermion operators in terms of the boson fields. We obtain the formulas

$$i\bar u_m^\delta\gamma^\mu\partial_\mu u_m^\delta=\tfrac{1}{2}\partial_\mu\phi_m^\delta\partial^\mu\phi_m^\delta\,, \tag{33a}$$

$$\bar u_m^\delta\gamma^\mu u_m^\delta=-\frac{1}{\sqrt{\pi}}\epsilon^{\mu\nu}\partial_\nu\phi_m^\delta\,, \tag{33b}$$

$$\bar u_m^\delta\gamma_5u_m^\delta=-\frac{i}{2\pi r}\cos(2\sqrt{\pi}\phi_m^\delta)\,, \tag{33c}$$

$$\sum_\delta \bar{u}^{\delta}_m u^{-\delta}_m = \frac{2c\mu}{\pi}\cos[\sqrt{\pi}(\phi^+_m+\phi^-_m)] \times \cos[\sqrt{\pi}(\tilde{\phi}^+_m-\tilde{\phi}^-_m)] \;. \tag{33d}$$

It should be noticed that these formulas can be obtained only after the careful point-splitting procedure. In (33c) the expected μ dependence is replaced by r^{-1} which comes from the additional terms in the commutator given in (29). We have used the covariant notation $x^\mu=(x^0,x^1)=(t,r)$ and $\epsilon^{\mu\nu}$ denotes the antisymmetric symbol with $\epsilon^{10}=1$.

Using the formula (33) we bosonize our system (22) and (23); for definiteness we choose the $a_1=0$ gauge. We have, after integration by parts,[29]

$$S=\int dr\,dt\,2\pi r^2 a_0'^2+e\Phi(r)a_0'+\sum_{m,\delta}\left[\tfrac{1}{2}(\partial_\mu\phi^\delta_m)^2+\frac{e}{\sqrt{\pi}}a_0'\phi^\delta_m+\frac{1}{2\pi r^2}\cos(2\sqrt{\pi}\phi^\delta_m)\right]$$
$$+\sum_m\frac{2c\mu m_0}{\pi}\cos[\sqrt{\pi}(\phi^+_m+\phi^-_m)]\cos[\sqrt{\pi}(\tilde{\phi}^+_m-\tilde{\phi}^-_m)]\;, \tag{34}$$

where

$$\Phi'(r,t)=4\pi Z r^2\rho(r,t)\;. \tag{35}$$

Here we have fixed the arbitrariness of the chiral angle so that the vacuum values of the Bose fields vanish in the absence of the external source.

From (34) we observe that the electromagnetic field a_0 has no dynamical degrees of freedom as usual in the Coulomb gauge. The absence of the transverse photon is, of course, due to our restriction to the s wave. We eliminate a_0' by using the equation of motion and construct the Hamiltonian as a functional of ϕ^δ_m and their canonical conjugates π^δ_m (Ref. 30):

$$H=\int dr\sum_{m,\delta}\left[\frac{1}{2}(\pi^{\delta\,2}_m+\phi^{\delta\prime\,2}_m)-\frac{1}{2\pi r^2}\cos(2\sqrt{\pi}\phi^\delta_m)\right]$$
$$-\frac{2c\mu m_0}{\pi}\sum_m\cos[\sqrt{\pi}(\phi^+_m+\phi^-_m)]\cos\left[\sqrt{\pi}\int_r^\infty ds\,[\pi^-_m(s)-\pi^+_m(s)]\right]+\frac{e^2}{8\pi r^2}\left[\Phi(r,t)-\frac{1}{\sqrt{\pi}}\sum_{m,\delta}\phi^\delta_m\right]^2. \tag{36}$$

The particle content of the theory is far from obvious from (36) because of the complexity of the mass terms. Therefore we perform a canonical transformation to the "physical" Bose variables:

$$\Phi_m=\tfrac{1}{2}(\phi^+_m+\phi^-_m)-\tfrac{1}{2}\int_r^\infty ds[\pi^+_m(s)-\pi^-_m(s)],\quad Q_m=\tfrac{1}{2}(\phi^+_m+\phi^-_m)+\tfrac{1}{2}\int_r^\infty ds\,[\pi^+_m(s)-\pi^-_m(s)]\;,$$
$$\Pi_m=\tfrac{1}{2}(\pi^+_m+\pi^-_m)+\tfrac{1}{2}(\phi^{+\prime}_m-\phi^{-\prime}_m),\quad P_m=\tfrac{1}{2}(\pi^+_m+\pi^-_m)-\tfrac{1}{2}(\phi^{+\prime}_m-\phi^{-\prime}_m)\;. \tag{37}$$

We thus have the final form of the Hamiltonian

$$H=\int dr\sum_m\tfrac{1}{2}(\Pi_m{}^2+P_m{}^2+\Phi_m'{}^2+Q_m'{}^2)+\sum_{m,\delta}\frac{1}{2\pi r^2}\left[1-\cos\sqrt{\pi}\left[\Phi_m+Q_m-\delta\int_r^\infty ds[\Pi_m(s)-P_m(s)]\right]\right]$$
$$+\sum_m\frac{M^2}{\pi}[2-\cos(2\sqrt{\pi}\Phi_m)-\cos(2\sqrt{\pi}Q_m)]+\frac{e^2}{8\pi r^2}\left[\left[\Phi(r,t)-\frac{1}{\sqrt{\pi}}\sum_m(\Phi_m+Q_m)\right]^2-\Phi(r,t)^2\right]\;, \tag{38}$$

where we have renormal ordered the Hamiltonian at $(\pi/4)^2c^{-1}m_0$ and $M=(\pi/4)m_0$. In (38) we have made the subtraction of the c-number terms so that the energy of the temporary vacuum with $\Phi_m=Q_m=0$ vanishes. The term "temporary" implies that this configuration does not give the true ground state of the theory with external source although it does for the sourceless theory. This fact will be verified in our numerical computation in Sec. IV.

The boundary condition for (25) or (28) has the form

$$\Phi_m(0,t)+Q_m(0,t)=0 \tag{39}$$

in terms of the physical boson variables.

Apparently there exists another boundary condition

$$\Phi_m(\infty,t)-Q_m(\infty,t)=0 \tag{40}$$

due to the definition of these variables. However, it should not be regarded as the boundary condition to be imposed in doing quantum computation. Since π_m^δ in the integrand is a quantum-mechanical operator,

$$\lim_{r\to\infty}\int_r^\infty ds\,\pi_m^\delta(s)$$

does not necessarily vanish. It should not come as a surprise if one remembers that Mandelstam's bosonization formula (26) does not imply the fermionic boundary condition $u_1^*(\infty)=iu_2(\infty)$ (u_1 and u_2 being the up and the down components) in performing fermionic path integral.

E. Electron excitation

We wish to demonstrate the existence of electron excitation in our bosonized lowest-partial-wave QED. From our experience in the monopole-fermion system[14] it is natural to expect that the electron exists as a soliton in our theory. To identify the asymptotic particle state we hunt the soliton excitations far from the location of the external source. For this purpose we are allowed to keep only the kinetic and the mass term in (38).

We look for the lowest energy excitation in our system. The obvious candidate for such a soliton solution is

$$\Phi_m^{\rm cl}(r)=\frac{2}{\sqrt{\pi}}\arctan(e^{-2M(r-r_0)})\ , \tag{41}$$

where $m=+\frac{1}{2}$ or $-\frac{1}{2}$ (not both). It should be noticed that (41) give an accurate solution which is consistent with the boundary condition (39) only if $r_0 \gg M^{-1}$. The energy of this configuration can easily be calculated as $(4/\pi)M=m_0$ which may be interpreted as the mass of the soliton.

Can the soliton (41) be interpreted as an electron? To answer this question we calculate the quantum numbers of the soliton. The electric charge and the third component of the angular momentum allow the following expressions in terms of the Bose variables:

$$Q_{\rm EM}\equiv e\int d^3x\,\psi^\dagger\psi$$

$$=e\sum_m F_m=-\frac{e}{\sqrt{\pi}}\sum_m[\Phi_m(r)+Q_m(r)]\Bigg|_0^\infty\ , \tag{42a}$$

$$J_3\equiv\int d^3x\,\psi^\dagger\left[\mathbf{r}\times\frac{1}{i}\nabla+\frac{\sigma}{2}\right]_3\psi$$

$$=\frac{1}{\sqrt{\pi}}\sum_m m\,[\Phi_m(r)+Q_m(r)]\Bigg|_0^\infty\ . \tag{42b}$$

It should be noticed that the angular momentum of the soliton at rest may be identified as the soliton's intrinsic spin.

Using (42) we can easily calculate the quantum number of the soliton and we have $(Q_{\rm EM},J_3)=(-e,\pm\frac{1}{2})$ for $m=\pm\frac{1}{2}$. It is the electron with spin up and down for $m=+\frac{1}{2}$ and $-\frac{1}{2}$, respectively. It is, therefore, tempting to conclude that the soliton (41) is the electron.

A problem, however, arises in this conclusion. The solution (41) exists also for $Q_m(r)$ which has exactly the same mass and the same quantum numbers. Therefore we are left with an "embarrassment of riches," having the two kinds of electrons.

The origin of this degeneracy can be traced back to the parity invariance of the theory. To understand this, and also for completeness, we work out the transformation properties of the physical Bose fields under the parity (P), the charge-conjugation (C), and the time-reversal (T) transformations. By knowing the transformation properties of the original fermion field it can readily be done with the results

$$\mathscr{P}\Phi_m\mathscr{P}^{-1}=Q_m\ ,\quad \mathscr{P}Q_m\mathscr{P}^{-1}=\Phi_m\ ; \tag{43a}$$

$$\mathscr{C}\Phi_m\mathscr{C}^{-1}=-Q_{-m}\ ,\quad \mathscr{C}Q_m\mathscr{C}^{-1}=-\Phi_{-m}\ ; \tag{43b}$$

$$\mathscr{T}\Phi_m(r,t)\mathscr{T}^{-1}=\Phi_{-m}(r,-t)\ ,\quad \mathscr{T}Q_m(r,t)\mathscr{T}^{-1}=Q_{-m}(r,-t)\ . \tag{43c}$$

We have not presented the transformation properties of the canonical momenta in (43) since they are exactly the same as that of the corresponding fields. Here we are taking the phase convention such that an electron state has even parity.

From (43a) it is evident that the degeneracy of the soliton solution reflects the parity invariance of the theory. Our problem now has a natural solution; we have to form the parity eigenstate so that the physical states respect the symmetry. However, we cannot do the job in a purely classical context and therefore we have to proceed to quantum mechanics.

Since we wish to remain in the semiclassical approach we employ the coherent state formalism.[31] Following Cahill[32] we construct the quantum-soliton states $|\Phi\rangle$ and $|Q\rangle$ corresponding, respectively, to the classical solutions $\Phi_m^{\rm cl}$ and $Q_m^{\rm cl}$ as

$$|\Phi\rangle=\exp\left[-i\int dr\,\Phi_m^{\rm cl}(r)\Pi_m(r,0)\right]|0\rangle\ , \tag{44a}$$

$$|Q\rangle=\exp\left[-i\int dr\,Q_m^{\rm cl}(r)P_m(r,0)\right]|0\rangle\ , \tag{44b}$$

where the sum over m is not implied. We note that

$$\delta\langle\Phi|H|\Phi\rangle/\delta\Phi=0$$

is solved by $\Phi_m^{\rm cl}$. It should also be noticed that these states have a number of desirable properties:

$$\langle 0|\Phi\rangle=\langle 0|Q\rangle=\langle\Phi|Q\rangle=0\ , \tag{45a}$$

$$\langle\Phi|Q_{\rm EM}|\Phi\rangle=\langle Q|Q_{\rm EM}|Q\rangle=-e\ , \tag{45b}$$

$$\langle\Phi|Q_{\rm EM}|Q\rangle=\langle Q|Q_{\rm EM}|\Phi\rangle=0\ . \tag{45c}$$

The equations in (45a) hold because of the property of the soliton configuration which differs by a constant from the vacuum value in an infinite range of r.

Now the parity eigenstate can easily be constructed:

$$|E(\pm)\rangle = \frac{1}{\sqrt{2}}(|\Phi\rangle \pm |Q\rangle) . \quad (46)$$

Owing to (45c) the electric charge and the spin of the new states $(Q_{\rm EM}, J_3)$ are still given by $(-e, \pm\frac{1}{2})$ for $m=\pm\frac{1}{2}$. The parity-even state $|E(+)\rangle$ is nothing but the s-wave electron state, while the parity-odd one $|E(-)\rangle$ represents the p-wave state.

Both of them are degenerate within our approximation of ignoring $1/r^2$ terms. If these terms were included the parity eigenstates (46) would naturally come out through diagonalizing the Hamiltonian. In the problem of hydrogenlike atom such a procedure will bring us the $1S_{1/2}$-$2P_{1/2}$ splitting, which will be the subject of a forthcoming publication.

F. Validity of semiclassical analysis of the boson theory

In Sec. IV we will give a detailed analysis of the resultant boson theory (38) to investigate the ground state of the supercritical system. Before doing this we wish to explain the reasons for our claim that the semiclassical analysis of the boson theory does take into account the quantum effects of the original fermion theory.

It is well known that the semiclassical treatment of the sine-Gordon theory comes close (at least qualitatively) to the quantum massive Thirring model, the equivalent fermionic theory of the former. Since our system essentially behaves like a sine-Gordon theory far away from the source it is plausible that the same thing happens. The fact that we have obtained the sensible particle spectra in the previous subsection does support our viewpoint.

Moreover if we turn our eyes to wider physical contexts there are good examples which further support our belief. One of the best examples concerns the fermion fractionization,[33] the phenomenon that the fermions induce their charge on solitons in the topologically nontrivial solitonic background. It is known[15,16] that in this case the classical analysis of the boson theory exactly reproduces the results of the fermion one-loop calculations.[34]

Our foregoing remarks, however, do not guarantee the validity of the semiclassical approximation at $r \lesssim M^{-1}$ or more importantly near the external source. (In the fermion fractionization the result depends only on the behavior of Bose fields at spatial infinity because of the topological nature of the problem.) One of the most serious objections would be that the classical treatment cannot properly deal with the uncertainty principle which is certainly one of the most important features in quantum theories.

It is, nevertheless, true that our classical analysis of the boson theory does take into account the uncertainty principle. To show this we consider a simplified system by dropping the mass term and the centrifugal force term in (38). Notice that by doing this we lose no essential ingredients of the model for our purpose. Since the uncertainty principle is effective at $r \lesssim m_e^{-1}$ and the centrifugal barrier prevents electrons from falling to the center, our simplified system would suffer an even more severe problem of ignoring the uncertainty principle.

The simplified system is exactly soluble quantum mechanically at the fermionic level. In fact it is essentially the dyon-fermion system which has been discussed in Ref. 17 apart from the number of the fermion species. Using the path-integral method one can easily calculate the induced fermionic charge in the vacuum. We obtain

$$\begin{aligned} \langle \bar{\psi}\gamma^0\psi \rangle &= \frac{1}{4\pi\sqrt{\pi} r^2}\partial_r b(r,t) , \\ b(r,t) &= \sqrt{2\pi}K \int dr'\, dt' \frac{1}{r'^2}\mathscr{R}_K(r,t;r',t')\Phi(r',t') , \end{aligned} \quad (47)$$

where $\mathscr{R}_K$ is defined by

$$\left[\partial_t{}^2 - \partial_r{}^2 + \frac{K}{r^2}\right]\mathscr{R}_K(r,t;r',t') = \delta(r-r')\delta(t-t') \quad (48)$$

and $K = e^2/4\pi$ $(e^2/8\pi)$ for our simplified system (the dyon-fermion system).

On the other hand, (47) is nothing but the result of the classical treatment of the boson theory.[17] [Notice that $b(r,t)$ in (47) is the solution of the classical equation of motion.] Thus the classical analysis gives the quantum-mechanically exact answer in our simplified system.[35]

Even more dramatic evidence that the classical analysis contains the quantum effects can be provided by showing that the hydrogenlike atom constructed by the classical Bose theory has a characteristic size of the order of the Bohr radius. Recall that the size of the ground-state atom is determined by the balance between the repulsion due to the uncertainty principle and the Coulomb attraction is nonrelativistic quantum mechanics. The result of our preliminary analysis of the problem, which is dealing not with the hydrogen but with the uranium atom seems to support our viewpoint; the size of our bosonic atom is much closer, on the order of magnitude, to the Bohr radius rather than the nuclear charge radius.

This completes the arguments to justify our claim that the classical analysis of the boson theory takes care of the fermionic quantum effects.

IV. INDUCED FERMIONIC CHARGE AROUND A SUPERCRITICAL SOURCE AND POSITRON EMISSION

We now address our original question of the physics around a supercritical system. We restrict ourselves into the static and the spherically symmetric source in this paper although our formalism can accommodate the time-dependent source. Because of the restriction we are able to directly talk about neither the positron emission in heavy-ion collisions nor the time development of the processes. Nevertheless, it is interesting to examine the structure of the ground state of our system. We do this by computing the vacuum energy and the induced charge around the supercritical source. Furthermore the problem with the static source is of physical significance provided that the nuclear collisions occur with very long reaction times.[36,37] (We argue, however, in Sec. V that this possibility is unlikely in view of the data on positron emission.)

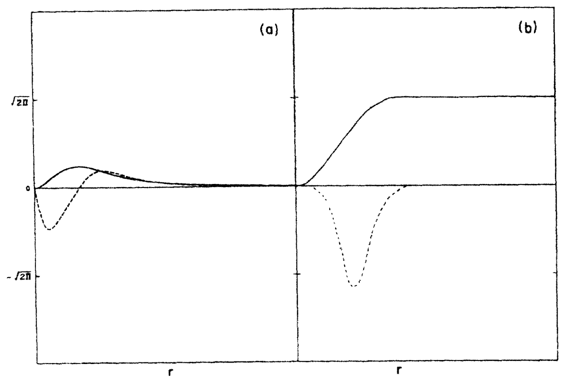

FIG. 2. Schematic illustration of the induced charge density (dashed line) and the corresponding boson wave function (solid line) for (a) undercritical and (b) supercritical cases, respectively. For definition of these terms see the text.

To appeal to the reader's intuition and to make it easier to understand what is going on we first give a sketch of the method for analysis and a schematic description of our results. Taking the validity of the classical analysis for granted it is easy to calculate the induced fermionic charge in our framework. That is, we look for the configuration which minimizes our Hamiltonian by solving the classical equations of motion.

For each value of Z we find two local minima in the parameter space of the solutions. They correspond to the charge distributions depicted in Figs. 2(a) and 2(b), respectively. The corresponding bosonic fields are also shown in Figs. 2(a) and 2(b), respectively.

For a small enough value of Z the type-(a) minimum gives the lowest-energy state of the system whereas for large enough Z the type-(b) one is the true ground state. When we increase Z the vacuum changes from type-(a) to type-(b) at a certain critical value of Z. Notice that we are comparing the vacuum energies of the type-(a) and -(b) ground states as we also did in one-particle theory (Sec. II). From here on a source is called supercritical (undercritical) if it has an electric charge greater (smaller) than the critical value.

The type-(a) minimum allows an immediate interpretation as the normal QED vacuum with the vacuum-polarization effects through radiative corrections (this again supports our claim that the classical analysis of the boson theory contains the quantum effects), whereas the type-(b) one is much more difficult to interpret. Apparently the ground state becomes charged. But this should not be the case as will be explained below.

Since we are dealing with the one particle state, a superheavy nucleus state, the charge conservation alone does not generally prohibit that our ground state is electrically charged. This situation in fact occurs in a certain magnetic monopole-fermion system where the monopole ground state carries fermion number[33,34] as well as electric charges.[15,16]

Let us consider, however, to adiabatically increase the external charge Z starting from the type-(a) phase just below the critical point. At $Z=Z_c$ the ground state undergoes the phase transition from the type-(a) to the type-(b) states. Since the former is charge neutral (apart from the external charge) the latter should also be charge neutral unless the charge conservation is broken. Notice that this argument does not apply to the case of magnetic monopoles since the magnetic charge cannot be turned off adiabatically.

Another important difference between our and the monopole-fermion system is that while we are working with the external source, one is talking about the fermionic charge induced around a fixed background field in the latter. Since the fermion comes into the problem only through the coupling with the dynamical gauge fields in the former, there is no way to have a charged ground state, unless the charge conservation is spontaneously broken. This last possibility is, however, highly unlikely and in fact contradictory to the general argument due to Vafa and Witten.[38]

To interpret correctly the type-(b) ground state we calculate the charge induced around the external source. Using (42) and noticing that Φ_m and Q_m take $\sqrt{\pi}$ times the integer we realize that the value of the induced charge has to be an integral multiple of the electron charge. Therefore we were observing the trace of the real (opposed to virtual) pair creation of electrons and positrons. We emphasize that this interpretation is possible only if the integral valuedness of the induced charge is demonstrable.[39] It is the topological nature of the electron excitation that makes it possible in our classical analysis of the boson theory.

Thus we have arrived at a natural interpretation of the type-(b) ground state. It accompanies at spatial infinity positrons whose electric charge exactly cancel the one induced around the external source. In comparing the energy of this ground state with that of type-(a) we have to add the rest mass of these positrons to the energy computed before. Hereafter we denote the type-(a) and -(b) ground states as the no-emission and the n-positron-emission states, respectively.

Now we enter into the details of our actual analysis. We take, as spherically symmetric sources, the solid-sphere and the spherical-shell forms both with radius R:

$$\begin{aligned} \rho(r,t) &= \frac{3}{4\pi R^3}\theta(R-r) \quad \text{(solid sphere)}\,, \\ \rho(r,t) &= \frac{1}{4\pi R^2}\delta(r-R) \quad \text{(spherical shell)}\,, \end{aligned} \tag{49}$$

where $\theta(x)$ denotes the step function. The corresponding Φ's defined by (35) are given, respectively, as

$$\begin{aligned} \Phi(r,t) &= Z\left[\frac{r}{R}\right]^3\theta(R-r)+Z\theta(r-R) \quad \text{(solid sphere)}\,, \\ \Phi(r,t) &= Z\theta(r-R) \quad \text{(spherical shell)}\,. \end{aligned} \tag{50}$$

Once giving the source it is in principle straightforward to numerically obtain the lowest-energy state. We note that all the canonical momenta can be set equal to zero because of the time independence of the system. The systems are still coupled ones and we need further simplifications. We confine ourselves in this paper to the case

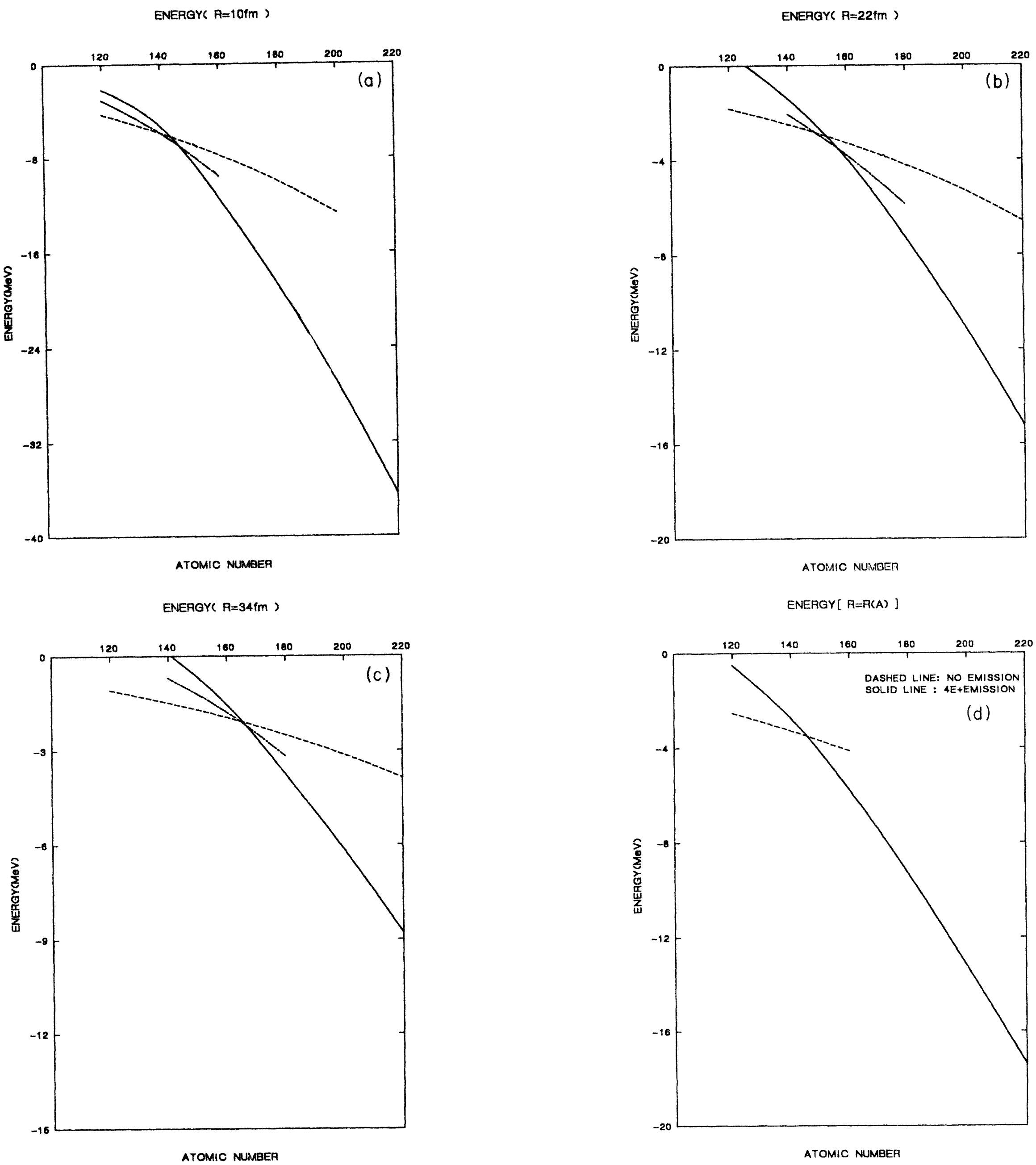

FIG. 3. The energies of the no-emission and the n-positron-emission states are plotted as functions of the atomic number Z for the solid-sphere source with radii (a) 10 fm, (b) 22 fm, (c) 34 fm, and (d) $R=2m_\pi{}^{-1}(2.5\times Z/2)^{1/3}$ cases, respectively. In each figure the dashed, dash-dotted, and solid lines show the energies of the no-emission, the two-, and the four-positron-emission states, respectively.

$\Phi_m = Q_m$ which simplifies the analysis considerably. The condition $\Phi_m = Q_m$, however, implies an important restriction that the single electron (or positron) cannot be emitted.

We employ the variational method to obtain the lowest-energy configuration of the bosonized Hamiltonian (38). We display mainly the results for the solid-sphere source and occasionally make comments on the spherical-shell case, considering the fact that the latter results are quite similar to the former's. In Figs. 3(a)—3(c), the energies of the three minima, the no-emission, the two-positron-emission, and the four-positron-emission states, are depicted as functions of the atomic number Z for $R=10$, 22, and 34 fm. The energy of the n-positron-emission state includes the rest mass of the n-positrons at spatial infinity. In Fig. 3(d), the case of source with mass number dependent radius,

$$R(A)=2m_\pi{}^{-1}(2.5\times Z/2)^{1/3} ,$$

is shown.

From these figures we observe that the vacuum under-

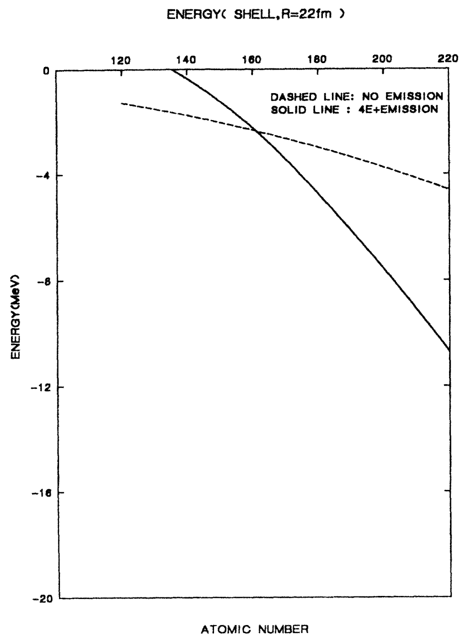

FIG. 4. The energies of the no-emission (dashed line) and the four-positron-emission states (solid line) are depicted as functions of Z for the spherical shell case with $R=22$ fm.

goes the phase transition from the normal to the "anomalous" one at the critical values of Z, $Z_c \simeq 140$, 150, and 160, for $R=10$, 22, and 34 fm, respectively.

It should be noticed that we are measuring the energy with respect to the temporary vacuum in which all the boson fields vanish. Therefore the absolute magnitude of the energies is of no meaning but the difference between the normal and the anomalous vacua does make sense. We note that these values of R agree, on the order of magnitude, with the distance to the peripheries in Coulomb scattering with incident energies of several MeV/nucleon.

In Fig. 4 the energies of the two types of ground states are shown for the case of spherical-shell source with $R=22$ fm. The critical value of Z tends to be higher than that of the solid-sphere case and $Z_c \simeq 160$.

We next show how the bosonic wave function and the induced charge density around a highly charged nucleus behave. We restrict ourselves to the cases with two typical values of Z; $Z=120$ for undercritical and $Z=180$ for supercritical sources. The wave functions for other values of Z behave qualitatively similar with either the $Z=120$ or 180 cases.

In Fig. 5 we show the boson wave function and the induced charge density around the solid-sphere source with $R=22$ fm. Only in this figure we are presenting the result of the Runge-Kutta method since it gives a slightly better solution (see below). A notable feature of the induced charge distribution is that it is sharply peaked at $r \simeq R$. This is true for both the supercritical and the undercritical case and also true for other values of R. This feature persists in the spherical shell case.[40] We do not present the wave functions for other values of R since

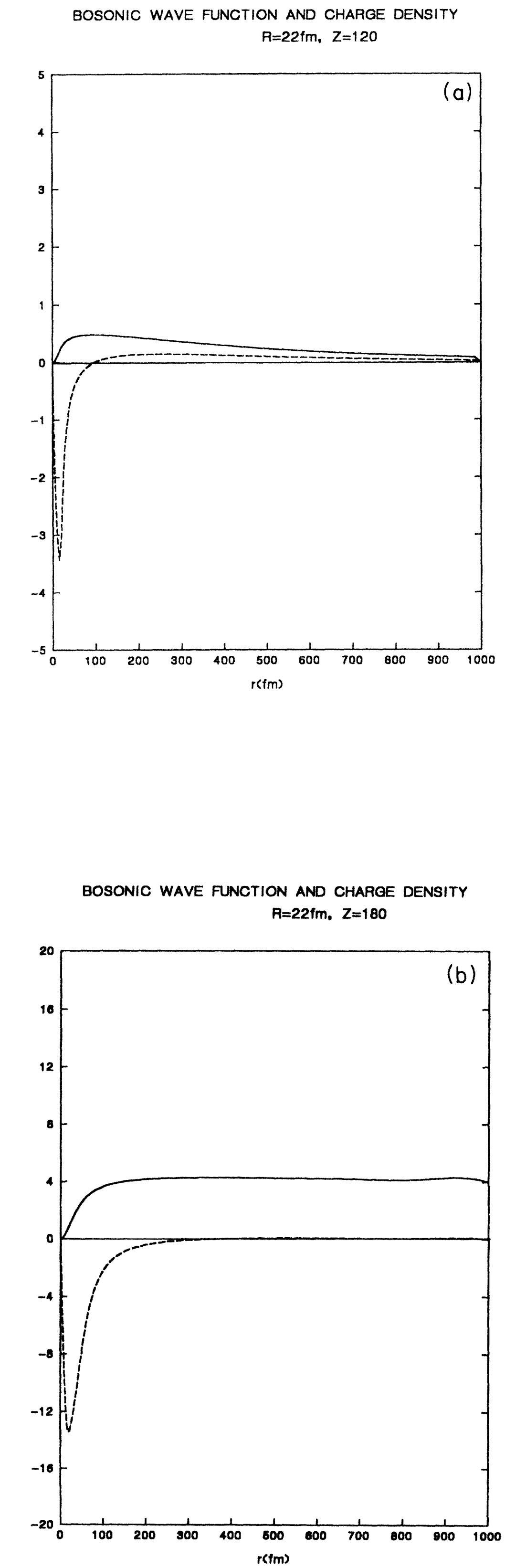

FIG. 5. The induced charge density (dashed line) and the corresponding boson wave function (solid line) are represented as functions of r for (a) $Z=120$ and (b) $Z=180$ cases, respectively.

they are very similar apart from the shift of the peak at $r \simeq R$.

We make a comment on the accuracy of our variational calculation. Of course it depends on how well the trial function was chosen in this method. In order to check this point we have performed independent calculations by numerically solving the differential equations via the Runge-Kutta algorithm. This has been done for the no-emission and the four-positron-emission states. By comparing the results of two methods the accuracy of our variational calculation is estimated to be about $\sim 5\%$ on the ground-state energy.

We note, in Fig. 3, that the energies of the no-emission, the two- and the four-positron-emission states approximately degenerate at the critical point. Even when the electron-electron interactions are ignored one expects that the two-electron state is preferred over the four-electron state there because the $2P_{1/2}$ state has higher energy than the $1S_{1/2}$ state. The semiclassically constructed state with the Φ and the Q excitation, however, correspond to the two-electron state one of which is in $1S_{1/2}$ and the other in $2P_{1/2}$ levels, respectively. (Since it carries spin 1 there is no other choice under the assumption that we are dealing with radially unexcited states.) Therefore, in our calculation, the two- and four-electron states should degenerate at the critical point apart from the effect of electron-electron interaction. The fact that the two-electron state is slightly favored over the four-electron state around the critical point can naturally be explained by its effect. Clearly the above-mentioned approximate degeneracy is the artifact of our restriction to the $\Phi_m = Q_m$ case due to technical reasons. This and the related problem concerning the mixed-state nature of our semiclassical state should be resolved by an improved treatment in the future.

Our result on the phase transition from the undercritical to the supercritical ground states qualitatively agrees with the one expected from the one-particle theory. At the quantitative level, however, our results differ from the predictions of the one-particle theory.

One of the most significant differences appears in how many positrons are emitted for a given number of Z. The one-particle theory predicts[9,10] that it is equal to the number of the unoccupied dived level. At $Z = 180$ two positrons are predicted to be emitted. Whereas in our quantum-field-theoretic treatment the number of induced electrons, which is equal to the number of emitted positrons, is determined by the energetics of the system, we gain energy by creating an electron-positron pair since they screen the electric field and thereby decrease the Coulomb energy. Of course we lose the rest mass, the kinetic energy, and the centrifugal barrier energy by pair creation. By balancing the gain and the loss of the energies the number of the electron-positron pair is determined. At $Z = 180$ we have four positrons as shown in Fig. 3.

Similarly Z_c, the critical value of Z for the positron emission, can differ for differing treatments. Our quantum-field-theoretic treatment predicts that, assuming the source radius $\simeq 20$ fm, $Z_c \simeq 150$ which is smaller by about 20 than the one deduced from the usual one-particle-theory treatment. This smaller value of Z_c is favored experimentally in view of the fact that the positrons due to the strong electric field seem to be observed already at $Z = 163$ (Ref. 41).

To conclude that the difference between our treatments and the one-particle-theoretic treatments is due to the quantum-field-theoretic effects such as represented in Fig. 1(b), we have to guarantee the quantitative accuracy of our semiclassical analysis. Unfortunately it is not beyond doubt because in our system the semiclassical expansion is not a systematic expansion with respect to a small expansion parameter. Of course our machinery is not meant to describe whole aspects of QED in a quantitatively accurate manner but designed to mainly deal with the systems around a highly charged nucleus. In fact the features of the ground state of the uranium atom do allow a reasonable description by our classical Bose theory. Therefore we believe that our results suggest that the quantum-field-theoretic effect enhances the pair creation around a supercritical source.

V. CONCLUSION AND OUTLOOK

In this paper we have described a quantum-field-theoretic formulation of the supercritical system. By discarding all the "inessential" higher partial waves we have constructed the effective two-dimensional fermion theory. The resultant theory has been further transcribed into the two-dimensional boson theory via the bosonization technique. We have presented a thorough analysis of the boson theory and in particular revealed the particle spectrum of the theory. Several arguments have been offered to justify our claim that the classical treatment of the boson theory does contain the quantum effects at the fermionic level. Finally we have performed a detailed numerical analysis of the theory in the classical approximation to elucidate the structure of the ground state around the supercritical source.

The result of our analysis indicates that as Z increases the vacuum undergoes the phase transition from the normal QED vacuum to the "anomalous" one which is characterized by the occurrence of the real pair creation of electrons and positrons. While this feature of the vacuum around the highly charged source has been anticipated from the one-particle-theoretic point of view, our analysis provides a demonstration based on quantum field theory.

Moreover our results differ quantitatively from those of the one-particle-theoretic treatment. The critical value of Z at which the phase transition occurs tends to be smaller. In our case $Z_c \simeq 150$ ($R = 22$ fm), whereas the one-particle theory predicts that $Z_c \simeq 170$. The number of emitted positrons is also different; at $Z = 180$ it is four in our case while it is two in one-particle theory. Unfortunately our restriction to the classical approximation prevents us from making the definite statement that the above difference does reflect the quantum-field-theoretic effect. Nevertheless we believe that our treatment constitutes an important first step toward the quantum field theory of the supercritical system.

What does our result imply for the hypothetical superheavy nucleus with supercritical charge if it exists as a real object? It seems to indicate that the superheavy nucleus wears the electron cloud and thereby carries a smaller number of electric charge than the number of protons inside.

Finally we wish to make some comments on the supercritical collision system. It should be noticed that from the viewpoint of quantum field theory the collision systems are entirely different physical systems from the ones with stable supercritical nuclei. This is because there is no asymptotic state with supercritical charge in the former. Therefore it is only for the very limited situation that the static source problem is a good guide for the collision system. The suitable condition would be

$$\tau_{\text{creation}} \ll \tau_{\text{collision}} , \tag{51}$$

where τ_{creation} and $\tau_{\text{collision}}$ denote the characteristic time scales for the pair creation and the heavy-ion collisions, respectively.

Condition (51) does not seem to be satisfied in the actual situation. We note that the e^+e^- pair creation in heavy-ion collisions is the strong-interaction process because $Z\alpha \simeq 1$. Then it is reasonable to guess that $\tau_{\text{creation}} \simeq m_e^{-1}$ by recalling the standard argument on the space-time structure of particle creation in hadronic collisions,[42] especially with the classical external sources.[43] On the other hand, $\tau_{\text{collision}}$ may be estimated, by assuming the Coulomb scattering, as the time during which the two ions come close to each other within $r_A \sim 10$ fm. This gives on the order of magnitude $\tau_{\text{collision}} \sim (m_A r_A{}^3 / Z^2\alpha)^{1/2}$, where m_A is the reduced mass. Numerically τ_{creation} and $\tau_{\text{collision}}$ estimated as above happen to coincide:

$$\tau_{\text{creation}} \simeq \tau_{\text{collision}} \sim 10^{-21} \text{ sec} .$$

Therefore the extreme situation (51) cannot be realized apart from the possibility[36] of the formation of molecular-type intermediate states.

Furthermore the observed peak structure in the positron spectrum[6,7] seems to indicate that τ_{creation} is even longer. Since the intrinsic width of the peak should be less than the observed value (about 80 keV), $\tau_{\text{creation}} > 10^{-20}$ sec. Therefore the physically interesting situation is, instead of (51),

$$\tau_{\text{creation}} \gg \tau_{\text{collision}} . \tag{52}$$

It should be noticed that under the circumstance (52) the emitted electron and the positron behave more or less symmetrically.[44] Moreover the too rapid Z dependence of the positron kinetic energy[6] may tend to flatten when the large space-time region is relevant to the pair creation.

One may doubt the occurrence of the pair creation itself under condition (52). We, however, argue that it may be possible thanks to the cooperative effects of the strong Coulomb field and their time dependence. We feel that our formalism is best suited to examine such a possibility. Work toward this direction is in progress.

In conclusion this work implies only the first step toward the complete understanding of the physics around the supercritical systems. Many things remain to be done in the future. First, our claim that the classical treatment of the bosonized theory takes proper account of the fermionic quantum effects should be confirmed. The accuracy of the classical approximation should be quantitatively estimated. Second, the problem of a hydrogenlike atom in our bosonized QED has to be worked out. Third, the effects of higher partial waves of the electromagnetic field should be examined. This may be important if one works on the collision system since in this case the spherical symmetry of the source, in general, does not exist. Fourth, the effects of higher partial waves of the fermions have to be evaluated. We, however, emphasize that our formalism developed in this paper provides a firm basis for the investigation of the above subjects. We hope that we will return to these questions in the future.

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