One-loop vertex function in Coulomb-gauge QED

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The one-loop vertex function in Coulomb-gauge QED has been evaluated. The result is expressed in a parametric integral form. The ultraviolet divergence has been isolated, and the matrix structure of the result has been optimized for applications. Examples of the use of the vertex function are discussed.

I. INTRODUCTION

Techniques for calculating multiloop Feynman diagrams in covariant-gauge QED are well known.[1] Corresponding techniques in the Coulomb gauge are not so well developed. Because of the practical importance of the Coulomb gauge in work on bound states in QED, the study of diagrams involving loops in the Coulomb gauge is of great interest.

There is a short history of research on loop diagrams in the Coulomb gauge. Johnson[2] studied the spectral form of the Coulomb-gauge electron propagator, and performed an explicit calculation of the spectral functions to one loop. This propagator was also considered by Hagen,[3] who studied its behavior near the mass shell. Heckathorn[4] dealt with multiloop diagrams, and obtained the important result that the pole part of the (dimensionally regularized) vertex function is proportional to γ_μ. A convenient explicit expression for the one-loop electron self-energy function was obtained by the present author.[5] Loop diagrams in Coulomb gauge have also been used in calculations of physical processes. The Lamb shift was obtained to order $\alpha(Z\alpha)^6$ by Sapirstein,[6] and the order-α correction to the decay rate of parapositronium was computed by the present author.[7]

In this article I give an expression for the one-loop vertex function in Coulomb gauge. The momentum integral has been done, the ultraviolet divergence isolated, and the result presented in a form useful for applications. In Sec. II I describe the calculation and write out the result. In Sec. III I apply the vertex function in evaluations of the one-loop renormalization constant Z_1 and the one-loop vertex correction to the decay rate of parapositronium.

II. AN EXPRESSION FOR THE VERTEX FUNCTION

The Feynman diagram for the one-loop vertex function is shown in Fig. 1. The corresponding analytic expression is[8,9]

$$-ie_0(\omega)\Lambda^\lambda(p',p)=\int (dl)'_\omega[-ie_0(\omega)\gamma_\mu]\frac{i}{\gamma(p'-l)-m}[-ie_0(\omega)\gamma^\lambda]\frac{i}{\gamma(p-l)-m}[-ie_0(\omega)\gamma_\nu]iD_C^{\mu\nu}(l)\ . \tag{1}$$

Regularization of the ultraviolet divergence in (1) is accomplished via dimensional regularization, with $d=2\omega$ (one time, $2\omega-1$ space) the dimension of spacetime. The momentum-space measure is $(dl)'_\omega=(d^{2\omega}l)/(2\pi)^{2\omega}$. The coupling $e_0(\omega)$ that appears in the Feynman rules is the product $(m)^{2-\omega}e_0$ of a power of the physical electron mass m with the (dimensionless) bare coupling. The Coulomb-gauge photon propagator has the form

$$D_C^{\mu\nu}(l)=\begin{bmatrix} 1/l^2 & 0 \\ 0 & \delta_{ij}^T/l^2 \end{bmatrix}, \tag{2}$$

where $\delta_{ij}^T=\delta_{ij}-l_il_j/l^2$. The momentum-space integration in (1) is performed by way of the integration formulas tabulated in the Appendix of Ref. 5.

The result for the one-loop vertex function can be expressed as

$$\Lambda^\mu(p',p)=\frac{\alpha}{4\pi}\gamma^\mu\left[D-\int_0^1 du\ln\left[\frac{\Delta}{m^2}\right]\right]+\frac{\alpha}{4\pi}\int_0^1 dx\int_0^1 du\left[\frac{1}{\sqrt{x}}\frac{R_X^\mu}{\Delta_X}+\frac{2R_Y^\mu}{\Delta_Y}\right]$$
$$+\frac{\alpha}{4\pi}\int_0^1\frac{dx}{\sqrt{x}}\int_0^1 ds\int_0^1 du\left[\frac{xR_Z^\mu}{\Delta_Z}+\frac{2x^2sR_Z'^\mu}{(\Delta_Z)^2}\right]. \tag{3}$$

Here $\alpha=e^2/4\pi=e_0{}^2/4\pi+O(\alpha^2)$ is the fine-structure constant and D is the divergent quantity

$$D=\frac{1}{2-\omega}-\gamma_E+\ln(4\pi)\ , \tag{4}$$

where γ_E is Euler's constant. The Δ factors are given by

$$\Delta = m^2 - u(1-u)k^2 \,, \tag{5a}$$

$$\Delta_X = m^2 - u(1-u)(k_0{}^2 - x\mathbf{k}^2) + (1-x)[u\mathbf{p}'^2 + (1-u)\mathbf{p}^2] \,, \tag{5b}$$

$$\Delta_Y = xm^2 - xu(1-u)k^2 + (1-x)u(m^2 - p'^2) + (1-x)(1-u)(m^2 - p^2) \,, \tag{5c}$$

$$\Delta_Z = sm^2 - su(1-u)(k_0{}^2 - x\mathbf{k}^2) + (1-s)u(m^2 - p_0'^2) + (1-s)(1-u)(m^2 - p_0{}^2) + (1-xs)[u\mathbf{p}'^2 + (1-u)\mathbf{p}^2] \,. \tag{5d}$$

The $\mu=0$ gamma-matrix factors are

$$\begin{aligned} R_X^0 = {} & -\boldsymbol{\gamma}\cdot\mathbf{p}'\gamma^0\boldsymbol{\gamma}\cdot\mathbf{p}(1-x) - \boldsymbol{\gamma}\cdot\mathbf{p}'\gamma^0 m - \gamma^0\boldsymbol{\gamma}\cdot\mathbf{p}m + \boldsymbol{\gamma}\cdot\mathbf{p}'k_0 u(-1-x+2xu) \\ & + \boldsymbol{\gamma}\cdot\mathbf{p}k_0(1-u)[1+x-2x(1-u)] + \gamma^0\{2xu(1-u)\mathbf{k}^2 + (1-x)[u\mathbf{p}'^2 + (1-u)\mathbf{p}^2]\} + (1-2u)mk_0 \,, \end{aligned} \tag{6a}$$

$$\begin{aligned} R_Y^0 = {} & -\boldsymbol{\gamma}\cdot\mathbf{p}'\gamma^0\boldsymbol{\gamma}\cdot\mathbf{p} - \boldsymbol{\gamma}\cdot\mathbf{q}x(2xq - p' - p)_0 \\ & + \gamma^0 \Bigg[(1-x^2)m^2 + (xq - p')_0(xq - p)_0 + x^2u(1-u)k_0{}^2 \\ & \qquad - \tfrac{1}{2}x(3-2x)[u(m^2 - p_0'^2) + (1-u)(m^2 - p_0{}^2)] + \left[-1 + \frac{x}{2} - 2x^2u(1-u) \right]\mathbf{k}^2 \\ & \qquad + \left[1 - \frac{x}{2} - \frac{x}{2}(5-4x)u \right]\mathbf{p}'^2 + \left[1 - \frac{x}{2} - \frac{x}{2}(5-4x)(1-u) \right]\mathbf{p}^2 \Bigg] + m(2xq - p' - p)_0 \,, \end{aligned} \tag{6b}$$

$$R_Z^0 = 2\boldsymbol{\gamma}\cdot\mathbf{p}'\gamma^0\boldsymbol{\gamma}\cdot\mathbf{p} + \boldsymbol{\gamma}\cdot\mathbf{p}'(sq - p)_0 + \boldsymbol{\gamma}\cdot\mathbf{p}(sq - p')_0 - \boldsymbol{\gamma}\cdot\mathbf{k}\gamma^0 m - 2\gamma^0\mathbf{p}'\cdot\mathbf{p} \,, \tag{6c}$$

$$R_Z'^0 = \boldsymbol{\gamma}\cdot\mathbf{p}'\gamma^0\boldsymbol{\gamma}\cdot\mathbf{p}\mathbf{q}^2 + \boldsymbol{\gamma}\cdot\mathbf{q}[\mathbf{p}'\cdot\mathbf{q}(sq - p)_0 + (sq - p')_0\mathbf{q}\cdot\mathbf{p}] - \boldsymbol{\gamma}\cdot\mathbf{q}\gamma^0 m\mathbf{q}\cdot\mathbf{k} - \gamma^0\mathbf{p}'\cdot\mathbf{p}\mathbf{q}^2 \,, \tag{6d}$$

where $q = up' + (1-u)p$ and $k = p - p'$. The corresponding $\mu = n$ gamma-matrix factors are

$$\begin{aligned} R_X^n = {} & \boldsymbol{\gamma}\cdot\mathbf{p}'\gamma^n\boldsymbol{\gamma}\cdot\mathbf{p}(1-x) + \boldsymbol{\gamma}\cdot\mathbf{p}'\gamma^n\gamma^0[(1-x)uk_0] + \gamma^0\gamma^n\boldsymbol{\gamma}\cdot\mathbf{p}[-(1-x)(1-u)k_0] \\ & + \boldsymbol{\gamma}\cdot\mathbf{p}'\gamma^n m + \gamma^n\boldsymbol{\gamma}\cdot\mathbf{p}m + \boldsymbol{\gamma}\cdot\mathbf{p}'(2xup'^n) + \boldsymbol{\gamma}\cdot\mathbf{p}[2x(1-u)p^n] \\ & + \gamma^n\{m^2 + u(1-u)(k_0{}^2 + x\mathbf{k}^2) - 2x[u\mathbf{p}'^2 + (1-u)\mathbf{p}^2]\} + \gamma^0[2xu(1-u)k_0k^n] + \gamma^n\gamma^0 mk_0 + 2xmq^n \,, \end{aligned} \tag{7a}$$

$$\begin{aligned} R_Y^n = {} & -\boldsymbol{\gamma}\cdot\mathbf{p}'\gamma^n\boldsymbol{\gamma}\cdot\mathbf{p}(2-x) - \boldsymbol{\gamma}\cdot\mathbf{p}'\gamma^n\gamma^0(xq - p)_0 - \gamma^0\gamma^n\boldsymbol{\gamma}\cdot\mathbf{p}(xq - p')_0 - \boldsymbol{\gamma}\cdot\mathbf{p}'\gamma^n m - \gamma^n\boldsymbol{\gamma}\cdot\mathbf{p}m \\ & + \boldsymbol{\gamma}\cdot\mathbf{p}'[x(1-2xu)q^n - 2(1-xu)p^n] + \boldsymbol{\gamma}\cdot\mathbf{p}\{x[1-2x(1-u)]q^n - 2[1-x(1-u)]p'^n\} \\ & + \gamma^n\{\tfrac{1}{2}x[u(m^2 - p'^2) + (1-u)(m^2 - p^2)] + (2-x)\mathbf{p}'\cdot\mathbf{p}\} \\ & + \gamma^0[x(2xq - p' - p)_0q^n - 2(xq - p')_0p^n - 2(xq - p)_0p'^n] + 2m(xq - p' - p)^n \,, \end{aligned} \tag{7b}$$

$$\begin{aligned} R_Z^n = {} & 3\boldsymbol{\gamma}\cdot\mathbf{p}'\gamma^n\boldsymbol{\gamma}\cdot\mathbf{p} + 2\boldsymbol{\gamma}\cdot\mathbf{p}'\gamma^n\gamma^0(sq - p)_0 + 2\gamma^0\gamma^n\boldsymbol{\gamma}\cdot\mathbf{p}(sq - p')_0 + 2\boldsymbol{\gamma}\cdot\mathbf{p}'\gamma^n m + 2\gamma^n\boldsymbol{\gamma}\cdot\mathbf{p}m \\ & + 2\boldsymbol{\gamma}\cdot\mathbf{p}'p^n + 2\boldsymbol{\gamma}\cdot\mathbf{p}p'^n + \gamma^n[m^2 + sq_0(p' + p)_0 - p_0'p_0 - 2\mathbf{p}'\cdot\mathbf{p}] + \gamma^n\gamma^0(-mk_0) \\ & + \gamma^0[2(sq - p')_0p^n + 2(sq - p)_0p'^n] + 2m(p' + p)^n \,, \end{aligned} \tag{7c}$$

$$\begin{aligned} R_Z'^n = {} & \boldsymbol{\gamma}\cdot\mathbf{p}'\gamma^n\boldsymbol{\gamma}\cdot\mathbf{p}\mathbf{q}^2 + \boldsymbol{\gamma}\cdot\mathbf{p}'\gamma^0\boldsymbol{\gamma}\cdot\mathbf{p}(1-s)q_0q^n + \boldsymbol{\gamma}\cdot\mathbf{p}'\boldsymbol{\gamma}\cdot\mathbf{p}mq^n \\ & + \boldsymbol{\gamma}\cdot\mathbf{p}'\gamma^n\gamma^0u[-(sq - p')_0\mathbf{q}\cdot\mathbf{p} + (sq - p)_0\mathbf{p}'\cdot\mathbf{q}] + \gamma^0\gamma^n\boldsymbol{\gamma}\cdot\mathbf{p}(1-u)[(sq - p')_0\mathbf{q}\cdot\mathbf{p} - (sq - p)_0\mathbf{p}'\cdot\mathbf{q}] \\ & + \boldsymbol{\gamma}\cdot\mathbf{p}'\gamma^n(-um\mathbf{k}\cdot\mathbf{q}) + \gamma^n\boldsymbol{\gamma}\cdot\mathbf{p}[(1-u)m\mathbf{k}\cdot\mathbf{q}] + \boldsymbol{\gamma}\cdot\mathbf{p}'\gamma^0(-umk_0q^n) + \gamma^0\boldsymbol{\gamma}\cdot\mathbf{p}[(1-u)mk_0q^n] \\ & + \boldsymbol{\gamma}\cdot\mathbf{p}'\{[(um^2 + usq_0)(p' + p)_0 - up_0'p_0 - (1-u)\mathbf{p}^2]q^n + 2(1-u)\mathbf{q}\cdot\mathbf{p}p^n\} \\ & + \boldsymbol{\gamma}\cdot\mathbf{p}\{[(1-u)m^2 + (1-u)sq_0(p' + p)_0 - (1-u)p_0'p_0 - u\mathbf{p}'^2]q^n + 2u\mathbf{q}\cdot\mathbf{p}'p'^n\} + \gamma^n(-\mathbf{q}^2\mathbf{p}'\cdot\mathbf{p}) \\ & + \gamma^0\{-(1-u)(sq - p')_0(\mathbf{p}^2q^n - 2\mathbf{q}\cdot\mathbf{p}p^n) - u(sq - p)_0(\mathbf{p}'^2q^n - 2\mathbf{p}'\cdot\mathbf{q}p'^n)\} \\ & + \{u[\mathbf{p}'^2 - (1-u)\mathbf{k}^2]mp'^n + (1-u)(\mathbf{p}^2 - u\mathbf{k}^2)mp^n\} \,. \end{aligned} \tag{7d}$$

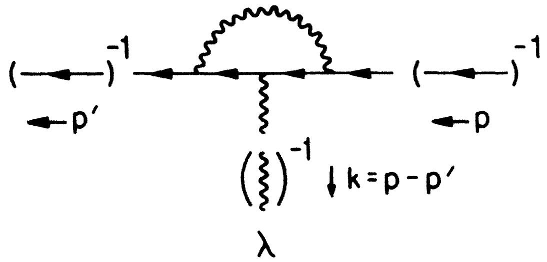

FIG. 1. Graphic expression for the one-loop vertex function $-ie_0(\omega)\Lambda_\mu(p',p)$.

The gamma-matrix factors are written so that $\gamma\cdot\mathbf{p}'$ appears on the left and $\gamma\cdot\mathbf{p}$ appears on the right. This convention facilitates the use of (3) in applications.

III. APPLICATIONS

In this concluding section I discuss the use of expression (3) for the one-loop vertex function in two examples.

A. The renormalization constant Z_1

The full vertex function $\Gamma'_\mu(p',p)=\gamma_\mu+\Lambda_\mu(p',p)$ can be related to a finite renormalized vertex function by way of a renormalization constant Z_1:

$$\Gamma'_\mu(p',p)=Z_1{}^{-1}\Gamma^R_\mu(p',p) . \qquad (8)$$

I define Z_1 so that the renormalized vertex function is effectively γ_μ for electrons at rest:

$$(\gamma p'+m)\Gamma^R_\mu(p',p)(\gamma p+m)\rightarrow(\gamma mn+m)\gamma_\mu(\gamma mn+m) \qquad (9)$$

as $p'\rightarrow mn$, $p\rightarrow mn$ where $n=(1,\mathbf{0})$. The corresponding relation for Z_1 is

$$\tfrac{1}{2}(\gamma n+1)n^\mu\Gamma'_\mu(mn,mn)\tfrac{1}{2}(\gamma n+1)=\tfrac{1}{2}(\gamma n+1)Z_1{}^{-1} . \qquad (10)$$

When $p'=p=mn$, expression (3) reduces to

$$n^\mu\Lambda_\mu(mn,mn)=\frac{\alpha}{4\pi}\gamma nD+\text{const}\times(\gamma n-1) . \qquad (11)$$

Consequently one has for the renormalization constant

$$Z_1=1-\frac{\alpha}{4\pi}D$$

through terms of $O(\alpha)$. This agrees with the result of Ref. 5.

B. The vertex correction in parapositronium decay

The one-loop vertex contribution to the decay rate of parapositronium has been calculated in the Coulomb gauge without the use of expression (3) for the vertex function.[7] The use of expression (3) simplifies the calculation tremendously. Consider the expressions

$$\text{tr}\left[\gamma\cdot\hat{\epsilon}_2(\gamma p'+m)\gamma\cdot\hat{\epsilon}_1\begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix}\right] \qquad (12a)$$

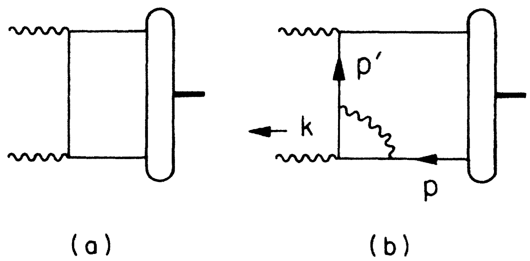

FIG. 2. The lowest-order (a) and vertex correction (b) diagrams for the parapositronium decay amplitude. The two-photon final state is shown to the left and the initial bound state to the right.

and

$$\text{tr}\left[\gamma\cdot\hat{\epsilon}_2(\gamma p'+m)\mathbf{\Lambda}(p',p)\cdot\hat{\epsilon}_1\begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix}\right] \qquad (12b)$$

which enter into the lowest-order [Fig. 2(a)] and vertex-correction [Fig. 2(b)] terms in the decay rate, respectively.[10] The matrix on the right in each trace represents the 4×4 Dirac spin structure of the bound-state wave function in the limit of slowly moving particles. The momentum assignments are indicated in Fig. 2, which shows an electron emerging from the bound state with momentum $p=(m,\mathbf{0})$, a real photon carrying away momentum $k=(m,\mathbf{k})$, and a virtual electron with momentum $p'=(0,-\mathbf{k})$. The photons split the available energy between them equally. In the context of (12b) the one-loop vertex function is effectively

$$\Lambda^n(p',p)\rightarrow\xi\gamma^n+\zeta\gamma\cdot\mathbf{k}\gamma^n , \qquad (13)$$

where use has been made of the transversality of real photons ($\mathbf{k}\cdot\hat{\epsilon}=0$) and the projective nature of the spin matrix (factors of γ^0 on the right in Λ^n are effectively equal to one). Using (13) in (12b) one finds for the vertex-correction term

$$2im\,(\xi+m\zeta)\hat{\mathbf{k}}\cdot(\hat{\epsilon}_1\times\hat{\epsilon}_2) , \qquad (14)$$

compared to

$$2im\,\hat{\mathbf{k}}\cdot(\hat{\epsilon}_1\times\hat{\epsilon}_2) \qquad (15)$$

for the lowest-order term (12a). Evidently the vertex-correction contribution to the decay amplitude is $(\xi+m\zeta)$ times the lowest-order amplitude. It only remains to evaluate $(\xi+m\zeta)$. In the kinematic situation of interest here one has

$$\Delta=m^2 \qquad (16a)$$

$$\Delta_X=m^2H_X,\quad H_X=1+u^2(1-x) , \qquad (16b)$$

$$\Delta_Y=m^2H_Y,\quad H_Y=x+2u(1-x) , \qquad (16c)$$

$$\Delta_Z=m^2H_Z,\quad H_Z=s[1+u^2(1-x)]+2u(1-s) . \qquad (16d)$$

The R factors are easily evaluated, and one obtains

$$(\xi+m\zeta)=\frac{\alpha}{4\pi}\left[D+\int_0^1 dx\int_0^1 du\left[\frac{1-u^2(1+x)}{\sqrt{x}H_X}+\frac{2x}{H_Y}\right]+\int_0^1\frac{dx}{\sqrt{x}}\int_0^1 ds\int_0^1 du\left[\frac{-xs(1-u)}{H_Z}-\frac{2x^2s^2u^2(1-u)}{(H_Z)^2}\right]\right]. \quad (17)$$

The integrations here have been done, with the result

$$(\xi+m\zeta)=\frac{\alpha}{4\pi}\left[D+\frac{\pi^2}{4}-4\ln 2+2A+2B\right], \quad (18)$$

where $A=\pi^2/8-\frac{1}{2}\ln^2(1+\sqrt{2})$ and $B=-1+\sqrt{2}\ln(1+\sqrt{2})$. Multiplied by 2 (to account for the mirror image vertex correction diagram) this is exactly the result of Ref. 7.

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[1]Standard textbook treatments include those of J. D. Bjorken and S. D. Drell, *Relativistic Quantum Mechanics* (McGraw-Hill, New York, 1964), and C. Itzykson and J.-B. Zuber, *Quantum Field Theory* (McGraw-Hill, New York, 1980). The Feynman-gauge one-loop vertex function was considered by R. Karplus and N. M. Kroll, Phys. Rev. **77**, 536 (1950), and was corrected by J. M. Jauch and F. Rohrlich, *The Theory of Photons and Electrons,* 2nd expanded ed. (Springer, New York, 1976), pp. 197—202. The parametrization and evaluation of multiloop diagrams are discussed for example in P. Cvitanović and T. Kinoshita, Phys. Rev. D **10**, 3978 (1974); **10**, 3991 (1974).

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[8]For a more complete discussion of Feynman rules in Coulomb-gauge QED see Ref. 5.

[9]The gamma-matrix conventions and natural units [$\hbar=c=1$, $\alpha=e^2/4\pi\simeq(137)^{-1}$] of Bjorken and Drell, *Relativistic Quantum Mechanics* (Ref. 1) are used throughout. The symbol m represents the electron mass, $m\simeq0.511$ MeV.

[10]This formalism is developed in Ref. 7. Expressions (12) are correct only to lowest order. To higher orders one must take into account the momentum dependence of the bound-state wave function.