Large-n behavior of the deep-inelastic structure functions and the renormalization- and factorization-scheme dependences

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The problem of singular perturbation coefficients for $n \to \infty$ that appear in the second-order QCD calculation of the moment of the deep-inelastic structure function F_2^n is investigated by applying the optimized perturbation theory (OPT). Special emphasis is paid to the connection with the kinematical analysis based on the resummation of singular terms. The main results of the present paper are as follows. (i) The simultaneous application of the generalized two-scale OPT to the factorization- and renormalization-scheme dependences of F_2^n not only absorbs the most singular $\ln^2 n$ terms in the perturbation coefficients into the effective mass scale with which the coupling constant runs, but also sweeps up the less singular $\ln n$ terms to end up with a suppressing "form factor." (ii) Applying the single-scale OPT to the renormalization-scheme dependence of the Q^2 dependence of the structure function $R_2^n \sim d \ln F_2^n / d \ln Q$, all the singular terms are completely absorbed into the effective mass scale, leaving the nonsingular perturbation coefficients. The R_2^n thus optimized and the R_2^n obtained from the F_2^n optimized as in (i) are shown to agree remarkably well more than expected formally and also to agree numerically very well. (iii) Results (i) and (ii) agree up to the most singular terms with those obtained through the kinematical analyses of Brodsky and Lepage and of Amati *et al.*

I. INTRODUCTION

The quantum-chromodynamic (QCD) perturbation expansion in terms of the strong coupling constant $a = g_s^2/4\pi^2$ has been observed to break down near phase-space boundaries. For example, in the conventional calculations based on the minimal-subtraction (MS) scheme or on its variations the next-to-leading order term for deep-inelastic structure functions behaves as $a \ln^2(1-x)$ near the boundary of phase space $x \sim 1$. Or, equivalently, in the QCD series for the Mellin-transformed moments

$$F_2^n(Q^2) \equiv \int_0^1 dx\, x^{n-2} F_2(x, Q^2) = A_n a^{d_n} (1 + r_{1,n} a + r_{2,n} a^2 + \cdots) \quad (1)$$

higher-order coefficients $r_{i,n}$ $(i \geq 1)$ contain terms that become logarithmically divergent as $n \to \infty$ (e.g., $r_{1,n} \sim \ln^2 n$). These terms being singular as $x \to 1$ (or $n \to \infty$) have been pointed out[1-3] to be of kinematical origin, i.e., to come from the phase-space boundary effect in the phase-space integral of relevant Feynman diagrams. In fact the origin of, at least, the most singular terms has been understood with such kinematical analyses.

Let us briefly see the consequences of the kinematical analyses[1-3] in actual calculations up to the next-to-leading order. In the case of hadronic structure functions, it is well known that conventional calculations in the MS-oriented schemes, the second-order coefficient $r_{1,n}$ in Eq. (1) behaves in the large-n regions as[3]

$$r_{1,n} = r_{1,n}^{\langle 2 \rangle} \ln^2 n + r_{1,n}^{\langle 1 \rangle} \ln n + O(1) . \quad (2)$$

The first term proportional to $\ln^2 n$ has been shown to be[2,4] beautifully resummed to all orders of a within the leading-double-logarithmic approximation, and has been absorbed into the effective coupling constant running with kinematically rescaled effective mass. Thus after the resummation of $a \ln^2 n$ terms, the second-order correction term becomes free of the most singular $\ln^2 n$ term. As for the second term in Eq. (2) proportional to $\ln n$ we have not yet succeeded to deal with it along the line of kinematical consideration. In the case of the photon structure function, the second-order coefficient behaves,[3] as $n \to \infty$, as

$$r_{1,n} = r_{1,n}^{\langle 1 \rangle} \ln n + O(1) , \quad (3)$$

and the leading-logarithmic resummation of the kinematically singular terms can absorb[3] only a portion of the $\ln n$ term into the modified leading-order expression. Thus kinematical consideration concerning the resummation of singular terms as $n \to \infty$ successfully works for the most singular $a \ln^2 n$ terms, but not for less singular terms.

Here let us turn to the other problem inherent in the QCD perturbation theory. It is well known that any *finite-order* QCD perturbation calculation depends essentially[5] on artificial choices of two calculational schemes: the renormalization scheme[6-9] (RS) and the factorization scheme[10-12] (FS). The problem of the scheme dependences of QCD perturbation theory should be analyzed in principle, as we have already shown,[12] by taking simultaneously the RS and FS dependences into account. In fact the resolution of scheme dependences of the photon structure function can be carried out only when both (RS

and FS) dependences are considered simultaneously.[9,13] There are, however, many physical quantities that are believed to be independent of FS, e.g., the Q^2 dependence of the hadronic structure function and the ratio of the gluonic to electromagnetic decay widths of the heavy quarkonium, for which the RS dependences *can be studied* by itself.

If we restrict, for brevity, our attention to the problem of RS dependences in the QCD calculations, the effective coupling constant a can be defined, for example, as[9]

$$\int_0^{a(\mu)} \frac{dx}{\beta(x)} - \int_0^{\infty} \frac{dx}{bx^2(1+cx)} = \ln \frac{\mu}{\tilde{\Lambda}} , \tag{4}$$

$$\beta(x) = bx^2(1+cx+c_2x^2+c_3x^3+\cdots) , \tag{5}$$

where b and c are RS-independent constants, while $c_2, c_3, \ldots$ as well as $\ln\mu$ vary depending on the RS chosen. As was proven by Stevenson[9] the renormalization scale $\ln\mu$ and the RS-dependent perturbation coefficients c_i ($i \geq 2$) can be chosen as parameters labeling the RS dependence. Returning to the QCD series for the structure function (1), the coupling constant a and the coefficients $r_{i,n}$ ($i \geq 1$) depend on the RS chosen as functions of $\ln\mu$ and c_i. Thus any finite-order calculations for $F_2^n(Q^2)$ depend on the RS. Because of the equivalence of all RS's, it seems apparent that, without further information in addition to the RS dependences, we have no definite criterion to choose any specific RS, except that the result neatly reflecting the renormalization-group (RG) invariance of the all-order expression might be favorable.[14] It is also obvious, if we take account of the aforementioned singular behaviors as $n \to \infty$ of the perturbation coefficients, that the desirable perturbation series is as follows. All the perturbation coefficients $r_{i,n}$ are free of singular terms as $n \to \infty$; thus, the QCD series makes sense as a perturbation expansion in terms of the effective coupling constant a.

Now the question we should ask is whether we can find such a calculational scheme that satisfies the above-mentioned two requirements: (i) The perturbation calculation *in the scheme* neatly reflects the RG invariance of the full-order expression; (ii) the perturbation coefficients calculated in the scheme are free of singular terms as $n \to \infty$, that might be of kinematical origin. This problem has been partly investigated in the analysis of the photon structure function,[13] where we have shown that the optimized perturbation theory (OPT) based on the principle of minimal sensitivity[9] (PMS) may have dealt with the large second-order corrections at least in the most singular terms in the large-n limit. In the following we make a detailed investigation of the above problem and show explicitly that the OPT based on the PMS actually realizes such a desirable scheme in the calculations of deep-inelastic structure functions. It is to be noted that the present analysis is concerned not only with the most singular terms but also with the less singular terms in the limit $n \to \infty$.

In Sec. II the large-n behavior of the optimized QCD series for deep-inelastic structure functions is studied. Analysis of the hadronic structure function is done in two stages. First, in Sec. II A the perturbative QCD result for the Q^2 dependence of the structure function, which is free of the FS dependence, is investigated. The optimization of the RS dependence is carried out through the original single-scale OPT, studying consequences about the structure of the optimized QCD series. Second, in Sec. II B the same analysis is done for the optimized series for the structure function itself after the simultaneous optimization of the RS and FS dependences by employing the generalized two-scale OPT. In Sec. II C we compare the results from the above two analyses and show that they agree remarkably well more than expected formally. A brief survey of the analysis of the photon structure function in the large-n limit and the comparison of it with the hadronic structure function case are presented in Sec. II D. Finally in Sec. III conclusions and several discussions are given.

II. LARGE-n BEHAVIOR OF THE DEEP-INELASTIC STRUCTURE FUNCTION

A. Single-scale OPT analysis of the RS dependence alone

Original single-scale OPT (Ref. 9) can be applied to physical quantities being independent of the FS. This is because the FS dependence is closely connected with the RS dependence and each of them (FS and RS) cannot be treated separately. Thus we here consider the Q^2 dependence of the structure function, which is free from the FS dependence.[9,11,15] For later convenience, however, we start from the perturbative expression for the structure function F_2^n. Up to the next-to-leading (second) order, the structure function is given by[16]

$$F_2^n(Q^2) = A_n a^{d_n}(1+r_{1,n}a) , \tag{6}$$

where the coupling constant a is defined by Eq. (4) with the second-order β function

$$\beta(a) = ba^2(1+ca) , \tag{7}$$

namely,

$$|b| \ln \frac{\mu}{\tilde{\Lambda}} = \frac{1}{a} + c \ln \left[\frac{ca}{1+ca} \right] . \tag{8}$$

The scheme-labeling parameter to this order is the renormalization scale $\ln\mu$ alone. Then the Q^2 dependence of the structure function can be calculated in the second-order approximation as

$$R_2^n(Q^2) \equiv \frac{1}{bd_n} \frac{d \ln F_2^n(Q^2)}{d \ln Q} = a(1+h_{1,n}a) , \tag{9}$$

where

$$h_{1,n} = c + r_{1,n}/d_n . \tag{10}$$

Applying the single-scale OPT to $R_2^n(Q^2)$ we get the consistency equation

$$\frac{d h_{1,n}}{d \ln\mu} = -b , \tag{11}$$

which can be integrated to give

$$h_{1,n} = b \ln \frac{Q}{\mu} + \rho_n , \tag{12}$$

where ρ_n is the scheme invariant that can be calculated in any scheme. The optimization equation becomes

$$c+2(1+ca)h_{1,n}=0\ . \tag{13}$$

Thus we get the optimized solutions as

$$(h_{1,n})_{\rm opt}=-\frac{c}{2(1+ca_{\rm opt})}\ , \tag{14}$$

$$[R_2^n(Q^2)]_{\rm opt}=a_{\rm opt}[1+(h_{1,n})_{\rm opt}a_{\rm opt}]$$
$$=a_{\rm opt}\frac{2+ca_{\rm opt}}{2(1+ca_{\rm opt})}\ , \tag{15}$$

the optimal renormalization scale $\mu_{\rm opt}$ and the corresponding coupling constant $a_{\rm opt}$ are the solutions to the coupled equations

$$|b|\ln\frac{Q}{\mu_{\rm opt}}=\rho_n+\frac{c}{2(1+ca_{\rm opt})}\ , \tag{16}$$

$$|b|\ln\frac{\mu_{\rm opt}}{\tilde{\Lambda}}=\frac{1}{a_{\rm opt}}+c\ln\left[\frac{ca_{\rm opt}}{1+ca_{\rm opt}}\right]\ . \tag{17}$$

In order to study the large-n behavior of the optimized series we consider the small-coupling limit[17] $a\simeq 0$. Then the optimized solutions (14)—(17) become

$$(h_{1,n})_{\rm opt}=-\frac{c}{2}+O(a)\ , \tag{18}$$

$$[R_2^n(Q^2)]_{\rm opt}=a_{\rm opt}\left[1-\frac{c}{2}a_{\rm opt}+O(a^2)\right]\ , \tag{19}$$

$$\ln\frac{Q}{\mu_{\rm opt}}=\frac{1}{|b|}\left[\rho_n+\frac{c}{2}\right]+O(a)\ , \tag{20}$$

and the optimal coupling constant $a_{\rm opt}$ becomes, in the limit $n\to\infty$,

$$a_{\rm opt}=\frac{1}{|b|\ln(\mu_{\rm opt}/\tilde{\Lambda})}+O\left[\frac{\ln\ln(\mu_{\rm opt}/\tilde{\Lambda})}{\ln^2(\mu_{\rm opt}/\tilde{\Lambda})}\right]\ , \tag{21}$$

where

$$\mu_{\rm opt}=Qn^{-1/4}g_2(n)[1+O(a)]\ , \tag{22a}$$

$$g_2(n)=e^{3c/2b}f_2(n)\ , \tag{22b}$$

$$f_2(n)\underset{n\to\infty}{\to}0.1734\quad(n_f=4)\ , \tag{22c}$$

and n_f is the number of quark flavors.

The perturbative second-order coefficient $r_{1,n}$ [or $h_{1,n}$ through Eq. (10)] has been already calculated in the MS or $\overline{\rm MS}$ scheme, where, for large n,

$$(r_{1,n})_{\rm MS\ or\ \overline{MS}}=r_{1,n}^{\langle 2\rangle}\ln^2 n+r_{1,n}^{\langle 1\rangle}\ln n+O(1)\ , \tag{23a}$$

or

$$(h_{1,n})_{\rm MS\ or\ \overline{MS}}=h_{1,n}^{\langle 1\rangle}\ln n+O(1)\ , \tag{23b}$$

which should be compared with the optimized coefficients that behave in the same limit $n\to\infty$ as

$$(r_{1,n})_{\rm opt}\equiv d_n[(h_{1,n})_{\rm opt}-c]=-\frac{4c}{|b|}\ln n+O(1)\ , \tag{24a}$$

or

$$(h_{1,n})_{\rm opt}=-\frac{c}{2}\ . \tag{24b}$$

Namely, through the optimization those terms giving the most singular contributions as $n\to\infty$ are absorbed into the optimized coupling constant $a_{\rm opt}$ defined at the rescaled mass-scale $\mu_{\rm opt}$. Thus in the optimized QCD series for $R_2^n(Q^2)$, Eq. (15), all the singular terms are completely absorbed into the rescaling of the effective mass scale, leaving the nonsingular perturbation coefficient $(h_{1,n})_{\rm opt}$. Note, however, if we look at the "optimized series for $F_2^n(Q^2)$,"

$$[F_2^n(Q^2)]_{\rm opt}\equiv A_n(a_{\rm opt})^{d_n}[1+(r_{1,n})_{\rm opt}a_{\rm opt}]\ , \tag{25}$$

the perturbation coefficient $(r_{1,n})_{\rm opt}$ still contains terms being singular as $n\to\infty$, i.e., $(r_{1,n})_{\rm opt}\sim\ln n$, and the "optimized series (25)" still formally loses its sense as $n\to\infty$. We have been careful so that we have *not* applied here the single-scale OPT to F_2^n itself. The "optimized series for F_2^n" means the F_2^n evaluated by substituting various optimized quantities for R_2^n.

In the general ith-order approximation, application of the single-scale OPT to R_2^n gives in the limit $n\to\infty$ the optimized results as [up to $O(a)$]

$$(h_{1,n})_{\rm opt}=\frac{c(i-3)}{2(i-1)} \tag{26a}$$

or

$$(r_{1,n})_{\rm opt}=-\frac{4c(i+1)}{3|b|(i-1)}\ln n+O(1)\ . \tag{26b}$$

In this case again the most singular terms as $n\to\infty$ are absorbed into the optimized coupling constant $a_{\rm opt}^{(i)}\equiv a(\mu_{\rm opt}^{(i)})$, which shows the same large-n behavior as the second-order case, Eqs. (21) and (22), namely

$$\mu_{\rm opt}^{(i)}=Qn^{-1/4}g_i(n)[1+O(a)]\ , \tag{27a}$$

$$g_i(n)=\exp\left[-\frac{c}{2|b|}\frac{i+1}{i-1}\right]f_2(n)\ , \tag{27b}$$

where $f_2(n)$ is given in Eqs. (22b) and (22c). As for the higher-order coefficients $(h_{i,n})_{\rm opt}$ or $(r_{i,n})_{\rm opt}$ $(i\geq 2)$ we cannot say much at present without information about the higher-order calculations for the β function, anomalous dimensions, and coefficient functions in some calculational schemes.

We can carry out the same analysis even for the photon structure function if we consider its Q^2 dependence and formally obtain the similar result.

B. Two-scale OPT analysis of the RS and FS dependences simultaneously

Optimization analysis of the structure function $F_2^n(Q^2)$ itself that depends both on the FS and RS simultaneously can be carried out through the generalized two-scale OPT. Such analysis has been already completed for the nonsinglet (NS) component of the structure function in the general ith-order calculations.[12] In this case besides the parameters labeling the RS, the renormalization scale $\ln\mu$ and the β-function coefficients[18] c_i and $\tilde{c}_i$ $(i\geq 2)$, there

appear parameters labeling the FS: the factorization scale $\ln M$, and the perturbation coefficients of anomalous dimension for spin-n operator $d_{i,n} \equiv \gamma_n^i / b$ ($i \geq 1$ because the leading-order coefficient $d_{0,n} \equiv d_n$ is a scheme invariant). Factorization of the moment can be carried out in terms of the Wilson operator-product expansion (OPE) as

$$F_2^n(Q^2) = \mathcal{O}_n(a(M)) \mathcal{C}_n(\tilde{a}(\mu)) , \tag{28}$$

where $\mathcal{O}_n$ is the matrix element of the operator with spin n renormalized at M and $\mathcal{C}_n$ is the OPE coefficient function corresponding to the subprocess hard-scattering cross section.

Following the analysis given in Ref. 12, we first study in detail the calculation up to the second order:

$$F_2^n(Q^2) = A_n a^{d_n} (1+ca)^{-d_n + d_{1,n}/c} (1+\tilde{r}_{1,n}\tilde{a}) , \tag{29}$$

where A_n is the scheme-independent constant and the two coupling constants a and $\tilde{a}$ are defined as[9,12,18] [see Eqs. (4) and (5)]

$$|b| \ln \frac{M}{\tilde{\Lambda}} = \frac{1}{a} + c \ln \left[\frac{ca}{1+ca} \right] , \tag{30a}$$

$$|b| \ln \frac{\mu}{\tilde{\Lambda}} = \frac{1}{\tilde{a}} + c \ln \left[\frac{c\tilde{a}}{1+c\tilde{a}} \right] . \tag{30b}$$

Applying the two-scale OPT to $F_2^n(Q^2)$ we first get, by integrating the consistency equations, $\tilde{r}_{1,n}$ as a function of scheme-labeling parameters:[12]

$$\tilde{r}_{1,n} = b d_n \ln \frac{Q}{M} - d_{1,n} + d_n \rho_n , \tag{31}$$

where the scheme invariant ρ_n is defined in Eq. (12). The optimization equations become, after simple manipulations, a set of three equations:

$$\tilde{r}_{1,n} = 0 , \tag{32a}$$

$$a - \tilde{a} + \frac{d_{1,n}}{d_n} a^2 = 0 , \tag{32b}$$

$$\ln(1+ca) - c\tilde{a} = 0 . \tag{32c}$$

By solving a set of six equations (30a)–(32c) simultaneously we can get the optimized solutions for M, μ, $d_{1,n}$, a, $\tilde{a}$, and $\tilde{r}_{1,n}$, thus determining the optimized structure function completely.

In order to study the large-n behavior of the optimized series, we again consider the small-coupling limit $a \simeq \tilde{a} \simeq 0$. Then we get the optimized solutions as

$$(\tilde{r}_{1,n})_{\text{opt}} = 0 \quad \text{or} \quad (\mathcal{C}_n)_{\text{opt}} = 1 , \tag{33a}$$

$$(d_{1,n})_{\text{opt}} = -\frac{c d_n}{2} + O(a) , \tag{33b}$$

$$\ln \frac{Q}{M_{\text{opt}}} = \frac{1}{|b|} \left[\rho_n + \frac{c}{2} \right] + O(a) , \tag{33c}$$

and

$$(F_2^n)_{\text{opt}} = A_n (a_{\text{opt}})^{d_n} (1+ca_{\text{opt}})^{-3d_n/2 + O(a)} . \tag{34}$$

The optimized coupling constant $a_{\text{opt}} \equiv a(M_{\text{opt}})$ is the solution to Eq. (30a) with M replaced by M_{opt}, and becomes, in the limit $n \to \infty$,

$$a_{\text{opt}} = \frac{1}{|b| \ln(M_{\text{opt}}/\tilde{\Lambda})} + O\left[\frac{\ln\ln(M_{\text{opt}}/\tilde{\Lambda})}{\ln^2(M_{\text{opt}}/\tilde{\Lambda})} \right] , \tag{35}$$

where

$$M_{\text{opt}} = Q n^{-1/4} g_2(n) [1 + O(a)] , \tag{36}$$

and $g_2(n)$ is defined in Eqs. (22a)–(22c). The following fact is worth mentioning. In comparing Eq. (33c) with Eq. (20), i.e., the two coupling constants $a_{\text{opt}}^{(B)} \equiv a(M_{\text{opt}})$ and $a_{\text{opt}}^{(A)} \equiv a(\mu_{\text{opt}})$ (Ref. 19), we recognize that the transmutation of the large-n singular terms in the perturbation coefficient into the effective mass scale agrees up to $O(a)$ between the two cases. Namely, the two optimal coupling constants defined at such rescaled masses agree well [see Eq. (54) below].

In order to see how the singular terms are dealt with in the present case, let us expand the second factor in Eq. (34). Then we get the perturbative expression of $(F_2^n)_{\text{opt}}$ similar to Eq. (1),

$$[F_2^n(Q^2)]_{\text{opt}} = A_n (a_{\text{opt}})^{d_n} [1 + \hat{r}_{1,n} a_{\text{opt}} + \cdots] , \tag{37}$$

where the second-order coefficient $\hat{r}_{1,n}$ is given by

$$\hat{r}_{1,n} = -\tfrac{3}{2} c d_n , \quad \underset{n \to \infty}{\sim} -\frac{4c}{|b|} \ln n + O(1) . \tag{38}$$

Namely, in the present case, though the most singular terms proportional to $\ln^2 n$ are completely absorbed into the mass scale, those terms proportional to $\ln n$ are still remaining. Thus the perturbation series (37) may lose its validity as n becomes large and $a_{\text{opt}} \ln n$ becomes of order 1. This situation is quite reminiscent of the results in Sec. II A and those found through the kinematical analyses.[2,3] There, although terms of order $a \ln^2 n$ can be resummed up to be absorbed into the effective mass scale with which the coupling constant runs, less singular terms of order $a \ln n$ are still remaining without any "prescription" to deal with such terms.

Note, however, that the optimal solution in the present case is not Eq. (37) but the compact expression (34). This fact indicates that through the simultaneous optimization procedure for both RS and FS dependences *double* resummation of potentially dangerous terms are carried out; namely, the most singular terms of order $a \ln^2 n$ are correctly resummed up together with its higher order terms and absorbed into the effective mass scale, and *at the same time remaining less singular terms of order $a \ln n$ are also resummed up into the compact expression* (34). It is worth stressing that the physical effects coming from the two types of resummations of singular terms are dif-

ferent. By taking notice of the fact that both c and d_n are positive-definite quantities, the factor $(1+ca_{\rm opt})^{-3d_n/2}$ arising from the second resummation is a suppression factor, i.e., works as a kind of "form factor." We know many examples[20] where potentially dangerous terms are resummed up to arrive at a suppressing form factor.

Now let us consider briefly the general ith-order approximation. In Ref. 12, it is shown that the application of two-scale OPT gives, in general,

$$(\mathscr{C}_n)_{\rm opt}=1\ ; \tag{39}$$

namely, that the optimized structure function takes the form

$$[F_2^n(Q^2)]_{\rm opt}=A_n(\bar{a})^{d_n}(1+c\bar{a})^{-d_n}\exp\left\{\int_0^{\bar{a}}dx\left[\left[\sum_{l=0}^{i-1}\bar{d}_{l,n}x^l\right]\Big/\left[x\sum_{l=0}^{i-1}\bar{c}_{l,n}x^l\right]-\frac{d_n}{x(1+cx)}\right]\right\}, \tag{40}$$

where the quantities with overbars represent optimal values of corresponding quantities, e.g., $\bar{a}\equiv a_{\rm opt}$, etc. The optimal coupling constant $\bar{a}$ again shows the same leading large-n behavior as in the previous cases,[21] Eqs. (21) and (35). How the optimized formula (40) improves the naive perturbation series (1), we cannot fully answer without information for higher-order coefficients. However, we may say that the most singular terms of order $a\ln^2 n$ are resummed up into the effective mass scale, and the next-singular terms of order $a\ln n$ might also be resummed up to give rise to the optimal two-loop anomalous dimension $\bar{d}_{1,n}$. These resummations leave the singularity-free optimal coefficient $\bar{r}_{1,n}$.

Concerning less singular terms that may appear in the higher-order coefficients $r_{i,n}$ $(i\geq 2)$ we can say nothing conclusive, but it is suggestive that the last factor in the optimized formula (40) works as a kind of suppressing form factor. This observation may indicate the following possibility: Remaining less singular but potentially dangerous terms, say, of order $a^2\ln n$, etc., in the higher-order coefficients are successively resummed up to be absorbed in order into the optimal scheme-labeling parameters $\bar{c}_i$ and/or $\bar{d}_{i,n}$ $(i\geq 2)$, depending on its hierarchies, i.e., depending on the order of the coupling constant, in which the physical structure function may show responses to their variations. This successive resummation may sweep and garnish all singular terms in the perturbation coefficients $r_{i,n}$ to end up with the suppressing form factor. In order to see whether such a possibility is in fact realized or not, calculations of higher-order terms are awaited.

At this point one might wonder, although the singular terms in the perturbation coefficients $r_{i,n}$ are swept up and, as a result, convergence of the perturbation series is *formally* improved, whether the perturbation theory as a whole is really improved or not. This is in fact the case at least for the hadronic structure functions. In the original MS calculations, perturbation series (1) becomes, in the large-n limit,

$$1+r_{1,n}a\sim 1+\delta\ln^2 n/\ln(Q/\Lambda)\ , \tag{41}$$

where δ is a factor of order 1. Thus the perturbation expansion may be valid for n satisfying

$$\ln n\lesssim[\ln(Q/\Lambda)]^{1/2}\ . \tag{42}$$

On the contrary, the optimized perturbation series for R_2^n in the single-scale OPT [see Eq. (15)] becomes

$$1+\bar{h}_{1,n}a(\bar{M})\sim 1+\delta'/\ln(Qn^{-1/4}/\tilde{\Lambda})\ , \tag{43}$$

where δ' is also of $O(1)$, and thus may be valid for n satisfying

$$\ln n\lesssim\ln(Q/\tilde{\Lambda})\ . \tag{44}$$

In the two-scale OPT analysis of F_2^n, the perturbation series for the OPE coefficient function $\mathscr{C}_n$ [see Eq. (33a)] becomes trivial,

$$1+(\tilde{r}_{1,n})_{\rm opt}\tilde{a}_{\rm opt}=1\ ,$$

and the anomalous dimension becomes[12]

$$\gamma_n(\bar{a})=d_n\bar{a}\left[1+\frac{\bar{d}_{1,n}}{d_n}\bar{a}\right], \tag{45}$$

$$\frac{\bar{d}_{1,n}}{d_n}=-\frac{c}{2}\ . \tag{46}$$

Namely, the second-order coefficients $(\tilde{r}_{1,n})_{\rm opt}$ and $\bar{d}_{1,n}/d_n$ become independent of n. Thus we can say that the OPT based on the PMS greatly enlarges the applicable region in the perturbation expansion. Above observation is based on the second-order analysis, and we expect that the situation is quite the same as for the higher-order calculations and for the higher-order coefficients.

C. Comparison of the two OPT analyses

We have already seen that the two OPT analyses, i.e., the two-scale OPT applied to the structure function F_2^n itself and the single-scale OPT to its Q^2-dependence R_2^n, have given almost the same "optimized" solutions when we consider the same quantity, i.e., the structure function F_2^n expanded in the same form of the perturbation series (1). In both cases the optimized second-order coefficient $(r_{1,n})_{\rm opt}$ is given by

$$(r_{1,n})_{\rm opt}=-\tfrac{3}{2}cd_n+O(a)\ , \tag{47a}$$

thus showing the same large-n behavior:

$$(r_{1,n})_{\rm opt}=-\frac{4c}{|b|}\ln n+O(1)\ . \tag{47b}$$

It is therefore interesting to study to what extent does this "equivalence" hold between the two OPT analyses. For this purpose we compare here the optimized second-order results for the Q^2 dependence of the structure function.

Through the single-scale OPT analysis of R_2^n we have obtained

$$(R_2^n)_{\text{opt},1}=\alpha\frac{2+c\alpha}{2(1+c\alpha)}\ , \tag{48}$$

where the optimized coupling constant $\alpha\equiv a(\mu_{\text{opt}})$ is the solution to the equation coming from Eqs. (16) and (17):

$$|b|\ln\frac{Q}{\tilde{\Lambda}}-\rho_n=\frac{1}{\alpha}+c\ln\left[\frac{c\alpha}{1+c\alpha}\right]+\frac{c}{2(1+c\alpha)}\ . \tag{49}$$

The same quantity can be calculated by using the various optimized quantities determined through the two-scale OPT analysis of F_2^n. Substituting the solutions to Eqs. (32a)–(32c) into the expression

$$R_2^n=a\left[1+\frac{d_{1,n}}{d_n}a\right]+\frac{\tilde{r}_{1,n}}{1+\tilde{r}_{1,n}\tilde{a}}\frac{1}{d_n}\tilde{a}^2(1+c\tilde{a})\ , \tag{50}$$

we get

$$(R_2^n)_{\text{opt},2}=\frac{1}{c}\ln(1+ca) \tag{51a}$$

$$=\tilde{a}\ , \tag{51b}$$

where in Eqs. (51a) and (51b) we have written, for brevity, the optimized coupling constants $a_{\text{opt}}\equiv a(M_{\text{opt}})$ and $\tilde{a}_{\text{opt}}\equiv\tilde{a}(\mu_{\text{opt}})$ as a and $\tilde{a}$, respectively, and a is given by the solution to the equation coming from Eqs. (30a), (31), and (32a)–(32c):

$$|b|\ln\frac{Q}{\tilde{\Lambda}}-\rho_n=\frac{2}{a}+c\ln\left[\frac{ca}{1+ca}\right]-\frac{\ln(1+ca)}{ca^2}\ . \tag{52}$$

Note that we have *not* applied here the two-scale OPT to R_2^n, but to the structure function F_2^n itself. The "optimized" result for R_2^n, Eqs. (51a) and (51b), is calculated by using the optimal quantities determined through the two-scale OPT analysis of F_2^n as mentioned above.

Now let us see the difference between the two results Eqs. (48) and (51a):

$$\Delta R_2^n\equiv(R_2^n)_{\text{opt},2}-(R_2^n)_{\text{opt},1}\ . \tag{53}$$

From Eqs. (49) and (52) we get the relation satisfied by the two optimal coupling constants α and a, which becomes, in the small coupling regions,

$$\alpha=a\left[1-\tfrac{1}{6}c^2a^2+\tfrac{1}{12}c^3a^3+\tfrac{11}{180}c^4a^4+O(a^5)\right]\ . \tag{54}$$

By using this expression we get the very nontrivial result

$$\Delta R_2^n=-\tfrac{1}{36}c^4a^5+O(a^6)\ , \tag{55}$$

which should be compared with the ordinary expectation that ΔR_2^n is to be $O(a^3)$, coming from the formal analysis. Thus we can expect that the difference between the two OPT analyses, Eq. (53), may be very small. This expectation is in fact realized through the numerical comparison. Taking account of the purpose of the present paper we present here only sample data for convenience to get the idea of the order of magnitude of ΔR_2^n. For example, at $Q^2=20$ GeV^2 the difference (53) is less than 0.02% even at $n=50$, where we take $\Lambda_{\overline{\text{MS}}}=0.2$ GeV, or, $\tilde{\Lambda}_{\overline{\text{MS}}}=0.224$ GeV, and $n_f=4$.

D. Photon structure function

We have already carried out the optimization of the photon structure function $F_{2,\gamma}^n$ and studied its consequences in Ref. 13, where it has been stressed that the second-order QCD formula for $F_{2,\gamma}^n$ can be optimized only through the generalized two-scale OPT. In this subsection several remarks and discussions are given by concentrating our focus on the main subject of this paper, the large-n behavior of the structure function, and by comparing the photon case with the hadronic structure function case. Because we are interested in the large-n behavior of $F_{2,\gamma}^n$ we shall consider only the asymptotically dominant pointlike contributions that can be calculated purely from QCD without suffering from any unknown matrix elements of local operators.[22,23]

Working within a class of schemes[24] where the hadronic two-loop anomalous dimensions vanish, we get the general second-order expression for the photon structure function[13] (NS denotes the nonsinglet contribution)

$$\begin{aligned}\hat{F}_n^{\gamma}&\equiv-bF_{2,n}^{\gamma}/e^4\\
&=\delta_{\psi}K_{n,\psi}^0\left[\frac{1+d_{n,GG}}{(1+d_{n+})(1+d_{n-})}(1+B_{n,\psi}\tilde{a})-\frac{d_{n,G\psi}}{(1+d_{n+})(1+d_{n-})}B_{n,G}\tilde{a}\right]\frac{1+ca}{a}\\
&+\delta_{\psi}(K_{n,\psi}^1-cK_{n,\psi}^0)\left[\frac{d_{n,GG}}{d_{n+}d_{n-}}(1+B_{n,\psi}\tilde{a})-\frac{d_{n,G\psi}}{d_{n+}d_{n-}}B_{n,G}\tilde{a}\right]+\delta_{\psi}K_{n,G}^1\left[-\frac{d_{n,\psi G}}{d_{n+}d_{n-}}(1+B_{n,\psi}\tilde{a})+\frac{d_{n,\psi\psi}}{d_{n+}d_{n-}}B_{n,G}\tilde{a}\right]\\
&+\delta_{\text{NS}}K_{n,\text{NS}}^0\frac{1}{1+d_{n,\text{NS}}}(1+B_{n,\text{NS}}\tilde{a})\frac{1+ca}{a}+\delta_{\text{NS}}(K_{n,\text{NS}}^1-cK_{n,\text{NS}}^0)\frac{1}{d_{n,\text{NS}}}(1+B_{n,\text{NS}}\tilde{a})-b\delta_{\gamma}B_{n,\gamma}\ ,\end{aligned} \tag{56}$$

where $d_{n,i} \equiv \gamma_{n,i}/b$ ($i = \mathrm{NS}, \psi, G$) are the hadronic one-loop anomalous dimensions, and $K^0_{n,i}$ and $K^1_{n,i}$ ($i = \mathrm{NS}, \psi, G$) are the photonic one- and two-loop anomalous dimensions, respectively. The two coupling constants a and $\tilde{a}$ are defined by Eqs. (30a) and (30b). Parameters denoting the FS and RS dependences are M (the factorization scale), μ (the renormalization scale), and $K^1_{n,i}$ ($i = \mathrm{NS}, \psi, G$). By studying the structure of the optimization equations we find[13] that among these five parameters only two of them can be chosen as independent optimization variables: the first one is M, and the second one is any linear combination of μ, $K^1_{n,\mathrm{NS}}$, and $K^1_{n,\psi}$. Thus there are only three choices of the optimization variables: (A) (M,μ); (B) $(M,K^1_{n,\mathrm{NS}})$; (C) $(M,K^1_{n,\psi})$. In the large-n limit choice (C) gives the same result as (B), and we have essentially only two choices: (A) and (B). Parameters not listed are fixed at the outset, which do not affect the final optimized result for the physical quantity $\hat{F}^\gamma_n$. Thus in the following we take two choices (A) and (B) with the remaining parameters fixed as

$$\text{(A) } (M,\mu);\ K^1_{n,\mathrm{NS}} = K^1_{n,\psi} = K^1_{n,G} = 0\ , \tag{57a}$$

$$\text{(B) } (M,K^1_{n,\mathrm{NS}});\ \mu = M,\ K^1_{n,\psi} = K^1_{n,G} = 0\ . \tag{57b}$$

In keeping appropriate large-n terms for various quantities we can carry out the optimization procedure in the large-n limit. Studying in the small coupling limit we get, in both cases (A) and (B), the same optimal scale M_{opt} [or the optimal coupling constant $a_{\mathrm{opt}} \equiv a(M_{\mathrm{opt}})$]:

$$a_{\mathrm{opt}} = \frac{1}{|b|\ln(M_{\mathrm{opt}}/\tilde{\Lambda})} + O\left[\frac{\ln\ln(M_{\mathrm{opt}}/\tilde{\Lambda})}{\ln^2(M_{\mathrm{opt}}/\tilde{\Lambda})}\right], \tag{58}$$

where

$$M_{\mathrm{opt}} = Qn^{-1/4}h_2(n)[1+O(a)]\ , \tag{59a}$$

$$h_2(n) \underset{n\to\infty}{\longrightarrow} 0.1148\ \ (n_f = 4)\ . \tag{59b}$$

In order to compare the optimal result with that of the conventional calculations,[22] e.g., in the $\overline{\mathrm{MS}}$ scheme, we present the optimized structure function in the small coupling limit:

$$\hat{F}^\gamma_{n,\mathrm{opt}} = \frac{a_n}{a_{\mathrm{opt}}} + b_{n,\mathrm{opt}} + O(a)\ . \tag{60}$$

Then we get in the limit $n \to \infty$ the optimized second-order coefficient as

$$b_{n,\mathrm{opt}}/a_n = \tfrac{2}{3}\ln n + O(1)\ , \tag{61}$$

which should be compared with the result in the $\overline{\mathrm{MS}}$ calculation[3,22] in the same limit:

$$b_{n,\overline{\mathrm{MS}}}/a_n = (b/4 + \tfrac{2}{3})\ln n + O(1)\ . \tag{62}$$

We can easily recognize that through the optimization part of the singular second-order coefficient, i.e., the first term in Eq. (62), has been absorbed into the rescaling of the effective mass scale

$$Q \to M_{\mathrm{opt}} = Qn^{-1/4}h_2(n)\ , \tag{63}$$

at which the effective coupling constant $a_{\mathrm{opt}} = a(M_{\mathrm{opt}})$ is defined.

Results (61) and (63) and their x-space counterparts are *exactly the same* as those obtained through the analysis of phase-space boundary effects.[2,3] In both (OPT and kinematical) analyses, only a part of leading large-n terms in the second-order coefficients, which is $O(\ln n)$ in the photon case, can be swept up with the rescaling of the coupling constant, in clear contrast with the previous OPT analysis of the hadronic structure function case (Sec. II B) where, not only the leading singular terms of $O(\ln^2 n)$ are completely absorbed into the coupling constant, but also the remaining less singular terms of $O(\ln n)$ are swept up giving the suppressing form factor. Namely, while in the case of the hadronic structure function, the two-scale OPT analysis sweeps and garnishes all the singular terms in the perturbation coefficients, thus resolving completely the problem of large perturbation corrections in the limit $n \to \infty$; in the photon case it can give only the same results as the kinematical method, leaving part of the large-n singular terms even in the optimized perturbation coefficient. Then we naturally pose the following question: Has the optimized solution given above in fact not solved the problem of large corrections in the large-n (or large-x) limit?

At this point it is worth noticing the fact that the remaining singular term of $O(\ln n)$ in the optimized coefficient, Eq. (61), comes entirely from the optimized counterpart of the last term in the original second-order expression (56), i.e., $-b\delta_\gamma(B_{n,\gamma})_{\mathrm{opt}}$, and that this is the very term that comes from the contributions coupled to the local composite operators constructed from the photon field operator (contribution P). All other terms in (56), whose singular perturbation terms are completely swept up through the optimization, come from contributions coupled to those constructed from the quark and gluon field operators (contribution H). Namely, those singular terms that can or cannot be swept up through the optimization have their proper origin of contribution P and H, which *can* be calculated separately but is, needless to say, *not* separately invariant under changes of schemes. Within the second-order QCD calculations, the contribution H consists of the leading- and second-order terms, while the contribution P consists of only the second-order term that is nothing but the *leading* term for the contribution P. Thus if we separate the total series $\hat{F}^\gamma_n$, (56), into two series $\hat{F}^P_n$ and $\hat{F}^H_n$ as[25,26]

$$\hat{F}^\gamma_n = \hat{F}^P_n + \hat{F}^H_n\ , \tag{64a}$$

$$\hat{F}^P_n = -b\delta_\gamma B_{n,\gamma}\ , \tag{64b}$$

$$\hat{F}^H_n = (\text{all other terms}) = \frac{a_n}{a} + b^H_n + O(a)\ \ (a \simeq \tilde{a} \simeq 0)\ , \tag{64c}$$

then the result of the optimization carried out in Ref. 13 says that

$$b^H_{n,\mathrm{opt}}/a_n = O(1)\ , \tag{65a}$$

$$\hat{F}^P_{n,\mathrm{opt}}/a_n = \tfrac{2}{3}\ln n + O(1)\ , \tag{65b}$$

namely, that the two-scale OPT (in this case, also the kinematical method) can solve the large-n problem of the QCD perturbation series for the photon structure function.

III. CONCLUSIONS AND DISCUSSION

In this paper we applied the optimized perturbation theory (OPT) to the deep-inelastic structure functions, or their Q^2 dependences, and studied the consequences mainly bringing into focus how the large-n (or large-x) behavior of the perturbation coefficients changed through the optimization and as a result how applicable region in the QCD perturbation expansion was enlarged. Here we briefly summarize our main results and give some discussions. First, the application of the OPT to hadronic structure functions solves not only the problem of scheme dependences but also the problem of singular perturbation coefficients as $n \to \infty$ as follows.

(i) Applying the original single-scale OPT to the perturbative QCD calculation of the Q^2 dependence of the structure function R_2^n, Eq. (9), that depends only on the RS, we find that the singular part of the perturbation coefficient $h_{1,n}$, behaving $\ln n$, are completely absorbed into the effective mass scale (22a) or (27a) at which the optimal coupling constant is defined, leaving only the nonsingular term $(h_{1,n})_{\rm opt}$, Eq. (24b) or Eq. (26a). If we look at the "optimal" structure function (25) that is calculated by using various optimal quantities for R_2^n, however, the "optimal" perturbation coefficient $(r_{1,n})_{\rm opt}$ still contains terms of $O(\ln n)$, Eqs. (24a) or (26b); thus, the "optimal series" (25) still loses its sense as $n \to \infty$.

(ii) Applying the generalized two-scale OPT to the perturbative calculation of the structure function F_2^n itself, Eq. (29), that depends both on the RS and FS simultaneously, we find that the most singular $\ln^2 n$ terms present in the conventional second-order coefficient $r_{1,n}$ [see, Eq. (1)] together with a part of less singular $\ln n$ terms is absorbed into the effective mass scale (36), and in this case the remaining less singular $\ln n$ terms are also swept and garnished to end up with the suppressing form factor [see Eqs. (34) and (40)]. Thus the two-scale OPT can completely solve the problem of large perturbative corrections in the limit $n \to \infty$ at least within the second-order calculations.

(iii) The fact that the optimized perturbation coefficients that appear in the optimized QCD series for R_2^n in case (i) and for F_2^n in case (ii) are both *completely free* from the $\ln n$ singularities, seems to be interesting. The point to be noted is that the physical quantity R_2^n or F_2^n is just the quantity to which the single- or two-scale OPT can be applied and is in fact applied.

(iv) The above results are quite similar to those obtained in the kinematical considerations.[1-3] In fact, the treatment of the most singular terms in the kinematical analysis completely agrees with the above OPT results. Because the kinematical analysis has not dealt with the remaining singular terms we cannot thoroughly study the relation between the kinematical approach and the OPT, but the treatment of the less singular terms in the two-scale OPT, (ii), is quite suggestive for the possibility of further resummation of $a \ln n$ terms.

(v) In order to compare the single-scale and the two-scale OPT we calculated the same physical quantity in the two OPT analyses, i.e., the "optimized" results for the Q^2 dependence of the structure function, Eqs. (48) and (51). Note that Eq. (48) is truly the optimized result for R_2^n, namely, it is obtained by applying the single-scale OPT to R_2^n itself, while Eqs. (51a) and (51b) are not, namely, they are calculated by using the optimal quantities obtained by applying the two-scale OPT to the structure function F_2^n. The two "optimized" results are shown to agree remarkably well up to the $O(a^5)$ accuracy level; namely, their difference is of $O(a^5)$ [see Eq. (55)]. This fact is quite nontrivial because the difference is formally of $O(a^3)$.

Second, the application of the OPT to the photon structure function $F_{2,\gamma}^n$ gives interesting results that differ slightly from the hadronic structure function case. As noted in Ref. 13 the photon structure function can be optimized only through the two-scale OPT, whose optimized results with respect to the effective mass scale or the coupling constant completely agree with those obtained through the kinematical analyses.[2,3] In this case only a part of leading singular terms that behave $\ln n$ as $n \to \infty$ is absorbed into the rescaling of the effective mass scale (59a), still leaving the singular $\ln n$ terms even in the optimized perturbation coefficient, Eq. (61). This fact is clearly contrasted with the hadronic structure function case where the two-scale OPT completely sweeps up all the singular terms. Thus in appearance the optimized perturbation series for the photon structure function seems to still suffer from the problem of large perturbative corrections as $n \to \infty$. The real situation, however, may not be as is worried about. This becomes clarified by examining the structure of the perturbation series for $F_{2,\gamma}^n$ in more detail, namely, by recognizing the fact that $F_{2,\gamma}^n$ consists of two terms F_n^H and F_n^P which can be calculated separately, Eqs. (64a)–(64c). While the first term F_n^H contains the leading contribution inversely proportional to the coupling constant, followed by the sub-leading-order correction terms, the second term F_n^P lacks the leading contribution, starting from the next-to-leading constant term, followed by the further higher-order terms. Thus within the second-order calculation, though F_n^H consists of the leading and the second-order terms, F_n^P consists of only the second-order term that is nothing but the "leading" contribution to F_n^P. The fact we have clarified in Sec. II D is that through the optimization of the total series $F_{2,\gamma}^n \equiv F_n^H + F_n^P$ based on the two-scale OPT the singular $\ln n$ terms in the second-order coefficient for F_n^H are completely absorbed into the rescaling of the effective mass scale, giving the nonsingular optimized coefficient (65a), and the would-be troublesome optimized coefficient of $O(\ln n)$ appears solely in the "leading" contribution to F_n^P, Eq. (65b). If we take this fact into account seriously, then we can say that even in the photon case the two-scale OPT may solve the problem of singular perturbation coefficient.

From the above observations we may conclude as follows. The two-scale OPT not only resolves the calculational scheme dependences in the perturbative QCD cal-

culations but also solves the problem of large perturbative corrections as $n \to \infty$ (or $x \to 1$). Or, at least we can say that the simultaneous resolution of the RS and FS dependences through the two-scale OPT deduces an important physical effect that may help us to understand the structure of the QCD perturbation theory.

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[17]In order that the perturbation expansion in terms of the coupling constant works, the external momentum scale Q is assumed to be large enough to satisfy $Qn^{-1/4} \gtrsim \tilde{\Lambda}$, see Eq. (21) below.

[18]Simultaneous analysis of the RS and FS dependences inevitably introduces two coupling constants a and $\tilde{a}$; see Eq. (28) below and Ref. 12. The coupling a satisfies [cf. Eqs. (4) and (5)]

$$\partial a / \partial \ln M = \beta(a) = b a^2 (1 + ca + c_2 a^2 + \cdots) ,$$

while $\tilde{a}$ satisfies

$$\partial \tilde{a} / \partial \ln \mu = \tilde{\beta}(\tilde{a}) = b \tilde{a}^2 (1 + c\tilde{a} + \tilde{c}_2 \tilde{a}^2 + \cdots) .$$

Note that two β-functions β and $\tilde{\beta}$ coincide within the second-order approximation.

[19]The coupling constants to be compared in its exact sense are those defined at the renormalization scale μ, namely, $a = a(\mu)$ in Sec. II A and $\tilde{a} = \tilde{a}(\mu)$ in the present case. In the small coupling limit the optimal renormalization scale $\mu_{\rm opt}$ and the corresponding coupling constant $\tilde{a}_{\rm opt} \equiv \tilde{a}(\mu_{\rm opt})$ are determined as

$$\tilde{a}_{\rm opt} = \frac{1}{|b| \ln(\mu_{\rm opt}/\tilde{\Lambda})} + O\left(\frac{\ln\ln(\mu_{\rm opt}/\tilde{\Lambda})}{\ln^2(\mu_{\rm opt}/\tilde{\Lambda})} \right)$$

and

$$\mu_{\rm opt} = M_{\rm opt} \exp\left(\frac{c}{2|b|} + O(\tilde{a}) \right) .$$

Therefore, to be exact, the transmutation of large-n singular terms in the perturbation coefficient into the effective mass scale slightly differs between the two cases. Although the *leading* large-n behaviors of the optimal coupling constants $a(\mu_{\rm opt})$ in Sec. II A and $\tilde{a}(\mu_{\rm opt})$ in Sec. II B are the same, the subleading behaviors become different. The reason why such difference emerges becomes clarified below.

[20]When x (or n) is sufficiently large, there appear effectively two large mass scales Q^2 and $s \simeq Q^2(1-x)$ that satisfy $Q^2 \gg s \gg \tilde{\Lambda}^2$. In such situations there usually appear in the perturbation calculation of the double-logarithmic terms of the form $a \ln^2 n$, which can be resummed up to all orders of $a \ln^2 n$ to arrive at the suppressing form factor, just as in the present case, see, e.g., Yu. L. Dokshitzer, D. I. Dyakonov, and S. I. Troyan, in Proceedings of the XIIIth Winter School of LNPI, Leningrad, 1978 (unpublished) [Report No. SLAC-TRANS-183, 1978 (unpublished)]; Phys. Rep. **58**, 269 (1980).

[21]The subleading behavior depends on the order of approximation as in Sec. II A. In the ith-order approximation we get the following expression:

$$a_{\rm opt}^{(i)} = \frac{1}{|b| \ln(M_{\rm opt}^{(i)}/\tilde{\Lambda})} + O\left(\frac{\ln\ln(M_{\rm opt}^{(i)}/\tilde{\Lambda})}{\ln^2(M_{\rm opt}^{(i)}/\tilde{\Lambda})} \right) ,$$

where

$$M_{\rm opt}^{(i)} = Q n^{-1/4} g_i(n) [1 + O(a)] ,$$

and $g_i(n)$ is given in Eqs. (27a) and (27b).

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large-n moments because the optimal scheme under full ten-parameter space is shown in fact to lie within a class of schemes where two-loop hadronic anomalous dimensions vanish, see Ref. 13.

[25]This separation applies not only to the second-order QCD calculations but also to the general ith-order QCD calculations as long as we confine our interest to the nontrivial lowest-order approximation with respect to the electromagnetic interactions.

[26]It should be noted that the OPT *can be,* and thus *is* in the present case, applied only to the total series $\hat{F}_n^{\gamma}$, *not* to the two series $\hat{F}_n^{p}$ and $\hat{F}_n^{H}$ separately.