Hamiltonian and Lagrangian constraints of the bosonic string

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We study the Hamiltonian and Lagrangian constraints of the Polyakov string. The gauge fixing at the Hamiltonian and Lagrangian level is also studied.

The interest in string theory has increased in the last two years because of its potential as a unified[1] theory of all interactions. A characteristic property of all these models is the double reparametrization invariance of the action; this means that the associated Lagrangians are singular. Recently the relations between the Hamiltonian and Lagrangian formalism for constrained systems was studied.[2] In particular it was shown how to obtain the Lagrangian constraints from the Hamiltonian constraints. In this work we will apply this method to obtain the Lagrangian constraints of the Polyakov string.[3]

The Lagrangian of the Polyakov string is $L=\int d\sigma\, \mathscr{L}(\sigma,\tau)$, where

$$\mathscr{L}=\frac{\sqrt{-g}}{2}g^{\alpha\beta}\partial_\alpha x^\mu \partial_\beta x_\mu\ ,\qquad \mu=0,1,\ldots,D-1,\ \ \alpha,\beta=1,2\ . \tag{1}$$

The canonical momenta are

$$P_\mu=\frac{\delta L}{\delta \dot{x}^\mu}=\frac{-1}{(-g)^{1/2}}(g_{11}\dot{x}_\mu-g_{01}x'_\mu)\ , \tag{2}$$

$$\Pi_{\alpha\beta}=\frac{\delta L}{\delta \dot{g}_{\alpha\beta}}=0 \tag{3}$$

which imply the primary constraints

$$\Pi_{00}=0,\ \ \Pi_{01}=0,\ \ \Pi_{11}=0\ . \tag{4}$$

The canonical Hamiltonian density, defined unambiguously on the surface of the primary constraints, is

$$\mathscr{H}_c=-\frac{1}{2}\frac{\sqrt{-g}}{g_{11}}(P^2+x'^2)+\frac{g_{01}}{g_{11}}(Px') \tag{5}$$

and, therefore, the Dirac Hamiltonian[4] is

$$H_D=\int d\sigma \left[\mathscr{H}_c+\sum_{\alpha<\beta}\lambda_{\alpha\beta}\Pi_{\alpha\beta}\right],$$

where $\lambda_{\alpha\beta}$ are arbitrary functions of the evolution parameter τ. The stability of the primary constraints (4) requires

$$\begin{aligned}
\dot{\Pi}_{00}&=\{\Pi_{00},H_D\}=-\frac{1}{4}\frac{P^2+x'^2}{(-g)^{1/2}}=0\ ,\\
\dot{\Pi}_{01}&=\{\Pi_{01},H_D\}=\frac{1}{2}\frac{g_{01}}{g_{11}}\frac{P^2+x'^2}{(-g)^{1/2}}-\frac{(Px')}{g_{11}}=0\ ,\\
\dot{\Pi}_{11}&=\{\Pi_{11},H_D\}\\
&=-\frac{1}{2}\frac{P^2+x'^2}{(-g)^{1/2}}\left[\frac{g_{00}}{2g_{11}}-\frac{g}{{g_{11}}^2}\right]+\frac{g_{01}}{{g_{11}}^2}(Px')\ .
\end{aligned} \tag{6}$$

In order to have a consistent Lagrangian density we need $\sqrt{-g}$ and g_{11} to be different from zero, and therefore the relations (6) give two independent secondary constraints:

$$\chi_1=\tfrac{1}{2}(P^2+x'^2),\ \ \chi_2=(Px')\ . \tag{7}$$

The stability conditions for the secondary constraints (7) are automatically satisfied due to the relations

$$\begin{aligned}
&\{\chi_i,\Pi_{\alpha\beta}\}=0\ ,\\
&\{\chi_1(\sigma),\chi_1(\sigma')\}=\chi_2(\sigma)\partial_\sigma\delta(\sigma-\sigma')-\chi_2(\sigma')\partial_{\sigma'}\delta(\sigma-\sigma')\ ,\\
&\{\chi_1(\sigma),\chi_2(\sigma')\}=\chi_1(\sigma)\partial_\sigma\delta(\sigma-\sigma')-\chi_1(\sigma')\partial_{\sigma'}\delta(\sigma-\sigma')\ ,\\
&\{\chi_2(\sigma),\chi_2(\sigma')\}=\chi_2(\sigma)\partial_\sigma\delta(\sigma-\sigma')-\chi_2(\sigma')\partial_{\sigma'}\delta(\sigma-\sigma')\ ;
\end{aligned} \tag{8}$$

thus the model contains five Hamiltonian constraints (6), (7) which are first class.

The Hamilton-Dirac equations of motion are

$$\begin{aligned}
\dot{x}_\mu&=-\frac{\sqrt{-g}}{g_{11}}P_\mu+\frac{g_{01}}{g_{11}}x'_\mu\ ,\\
\dot{g}_{\alpha\beta}&=\lambda_{\alpha\beta}\ ,\\
\dot{\Pi}_{\alpha\beta}&=0\ ,\\
\dot{P}_\mu&=-\partial\sigma\left[\frac{\sqrt{-g}}{g_{11}}x'_\mu-\frac{g_{01}}{g_{11}}P_\mu\right].
\end{aligned} \tag{9}$$

Note that these equations of motion contain three arbitrary functions associated with the gauge symmetries of the Lagrangian (1):

$$\begin{aligned}
\delta x^\mu&=\epsilon^\alpha\partial_\alpha x^\mu\ ,\\
\delta g_{\alpha\beta}&=\lambda g_{\alpha\beta}+\epsilon^\gamma\partial_\gamma g_{\alpha\beta}+\partial_\alpha\epsilon^\gamma g_{\gamma\beta}+\partial_\beta\epsilon^\gamma g_{\alpha\gamma}\ ,
\end{aligned} \tag{10}$$

where ϵ^0, ϵ^1, and λ are arbitrary infinitesimal functions. These transformations can be interpreted as two reparametrizations plus a Weyl dilatation, so that the number of gauge transformations is the same as the number of primary constraints.[5]

If we want to eliminate the arbitrary functions in the equation of motion, we need to introduce, by hand, three gauge-fixing constraints: for example,

$$\Phi_1=g_{00}-\lambda(x,P),\ \ \Phi_2=g_{01},\ \ \Phi_3=g_{11}+\lambda(x,P)\ , \tag{11}$$

where λ is any function of x and P. These constraints convert the primary constraints (4) into second-class constraints. Their stability requires

$$\lambda_{00}=\dot{\lambda},\ \ \lambda_{01}=0,\ \ \lambda_{11}=-\dot{\lambda}\ , \tag{12}$$

so that the $\lambda_{\alpha\beta}$ are determined and we have no arbitrary functions in the equations of motion. For the variables x and P we have

$$\dot{x}^{\mu}=P^{\mu},\ \ \dot{P}^{\mu}=x^{\mu''}\ . \tag{13}$$

These equations are known as the conformal gauge equations of motion. Note that still we have two first-class constraints: χ_1, χ_2; therefore if we want to work with the true degrees of freedom, we need to introduce two more gauge constraints such that χ_1 and χ_2 become second class. Furthermore their stability must be automatically satisfied because we have no arbitrary functions in the equation of motion (13). For example, let us consider

$$\begin{aligned}&\chi_3=x^{+}-P_{-}\tau=0\ ,\\ &\chi_4=P'_{-}=0\end{aligned} \tag{14}$$

which convert χ_1 and χ_2 into second-class constraints. The stability of χ_3, χ_4 requires using (13),

$$\begin{aligned}&0=\dot{\chi}_3=P^{+}-x''_{-}\tau-P_{-}=-x''_{-}\tau\ ,\\ &0=\dot{\chi}_4=\dot{P}'_{-}=x'''_{-}\end{aligned} \tag{15}$$

which are automatically satisfied because from (14) one can obtain $x'_{-}=0$.

At this point the analysis continues as in the Nambu-Goto[6] string on the light cone.[7] The number of physical degrees of freedom at the canonical level is therefore $3\times 2+D\times 2-5\times 2=2D-4$.

Let us now study the Lagrangian formalism; the Hessian matrix is

$$W_{AB}(\sigma,\sigma')=\frac{\delta^2 L}{\delta\dot{q}^{A}(\sigma)\delta\dot{q}^{B}(\sigma')}=-\frac{g_{11}}{(-g)^{1/2}}\begin{bmatrix}g_{\mu\nu} & 0\\ 0 & 0\end{bmatrix}, \tag{16}$$

where $q^{A}(\sigma,\tau)$ denotes any of the variables $x^{\mu}(\sigma,\tau), g_{\alpha\beta}(\sigma,\tau)$. The null vectors of the Hessian are obtained as in[2]

$$\gamma_a^{A}(\sigma,\sigma')=FL^{*}\frac{\delta\Phi_a(\sigma)}{\delta P_A(\sigma')}\ , \tag{17}$$

where Φ_a are the primary Hamiltonian constraints (4). Explicitly

$$\begin{aligned}&\gamma_1(\sigma,\sigma')=\delta(\sigma-\sigma')\begin{bmatrix}0\\1\\0\\0\end{bmatrix},\\ &\gamma_2(\sigma,\sigma')=\delta(\sigma-\sigma')\begin{bmatrix}0\\0\\1\\0\end{bmatrix},\\ &\gamma_3(\sigma,\sigma')=\delta(\sigma-\sigma')\begin{bmatrix}0\\0\\0\\1\end{bmatrix}.\end{aligned} \tag{18}$$

The Euler-Lagrangian equations of motion can be written as

$$\ddot{x}_{\mu}=\alpha_{\mu},\ \ \ddot{g}_{\alpha\beta}=\beta_{\alpha\beta}\ , \tag{19}$$

where

$$\begin{aligned}\alpha_{\mu}=&\ \partial_{\sigma}(-g)^{-1/2}(g_{00}x'_{\mu}-g_{01}\dot{x}_{\mu})\\ &+(-g)^{-1/2}(g_{00}x'_{\mu}-g_{01}\dot{x}_{\mu}+g_{00}x''_{\mu}-g_{01}\dot{x}'_{\mu})+\tfrac{1}{2}(-g)^{-3/2}g_{11}\dot{g}_{00}(g_{11}\dot{x}_{\mu}-g_{01}x'_{\mu})\\ &-(-g)^{-1/2}x'_{\mu}\dot{g}_{01}-(-g)^{-3/2}g_{01}\dot{g}_{01}(g_{11}\dot{x}_{\mu}-g_{01}x'_{\mu})+(-g)^{-1/2}\dot{x}_{\mu}\dot{g}_{11}\\ &+\tfrac{1}{2}(-g)^{3/2}g_{00}\dot{g}_{11}(g_{11}\dot{x}_{\mu}-g_{01}x'_{\mu})-g_{01}(-g)^{-1/2}\dot{x}'_{\mu}\end{aligned} \tag{20}$$

and $\beta_{\alpha\beta}$ are arbitrary functions of the evolution parameter. These equations of motion are only valid on the surface defined by the primary Lagrangian constraints. All the Lagrangian constraints are the images of the canonical constraints[2] under the operator

$$K=\frac{\partial}{\partial\tau}+\int d\tau\left[\dot{q}^{A}(\sigma)FL^{*}\frac{\delta}{\delta q^{A}(\sigma)}+\frac{\delta L}{\delta q^{A}(\sigma)}FL^{*}\frac{\delta}{\delta P_A(\sigma)}\right], \tag{21}$$

where FL^{*} is the pull back of the Legendre application FL: $TQ\rightarrow T^{*}Q$. The first generation of Lagrangian constraints is obtained by applying (21) to (4). In this we obtain only two independent Lagrangian constraints

$$\begin{aligned}&g_{11}\dot{x}^{2}-g_{00}x'^{2}=0\ ,\\ &g_{11}(\dot{x}x')-g_{01}x'^{2}=0\ .\end{aligned} \tag{22}$$

The application of the operator K to the secondary Hamiltonian constraints is automatically satisfied on the surface defined by (22). Therefore at the Lagrangian level we have only two constraints. This means that before the gauge fixing the number of degrees of freedom in the Hamiltonian and the Lagrangian formalisms are different.

Consider now the gauge fixing at the Lagrangian level. Given a canonical gauge fixing one can obtain the corresponding Lagrangian gauge-fixing constraints using the operators FL^* and K. Thus we obtain

$$\begin{aligned} FL^*(\Phi_1) &= g_{00} - FL^*\lambda \equiv \theta_1 , \\ FL^*(\Phi_2) &= g_{01} \equiv \theta_2 , \\ FL^*(\Phi_3) &= g_{11} + FL^*\lambda \equiv \theta_3 , \end{aligned} \tag{23}$$

and

$$\begin{aligned} K(\Phi_1) &= \dot{g}_{00} - K\lambda \equiv \bar{\theta}_1 , \\ K(\Phi_2) &= \dot{g}_{01} \equiv \bar{\theta}_2 , \\ K(\Phi_3) &= \dot{g}_{11} + K\lambda \equiv \bar{\theta}_3 . \end{aligned} \tag{24}$$

The stability of the θ_i gives the $\bar{\theta}_i$ and the stability of $\bar{\theta}_i$ determines the unknown accelerations by fixing the $\beta_{\alpha\beta}$ in Eq. (19). We still have a superfluous degree of freedom, which can be eliminated by projecting the canonical gauge constraints χ_3 and χ_4:

$$\begin{aligned} & FL^*(\chi_3) = x^+ - \dot{x}_-\tau, \quad FL^*(\chi_4) = \dot{x}'_- , \\ & K(\chi_3) = \dot{x}^+ - \dot{x}_-\tau K(P_-) = -\tau K(P_-) , \\ & K(\chi_4) = K(P'_-) . \end{aligned} \tag{25}$$

The stability of $FL^*(\chi_3)$ and $FL^*(\chi_4)$ gives $K(\chi_3)$ and $K(\chi_4)$. Since $K(\chi_3)$ and $K(\chi_4)$ are zero on the surface of constraints, they are not new constraints. Therefore, the number of physical degrees of freedom is

$$3\times 2 + D\times 2 - 2 - 3 - 3 - 2 = 2D - 4 , \tag{26}$$

where the $(2+3+3+2)$ corresponds to the two Lagrangian constraints plus the eight fixing constraints. It is important to note that this matching of the number of degrees of freedom at the Lagrangian and canonical levels can only be obtained after fixing the gauge.

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