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Covariant string field theory

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The covariant string field theory is presented in its full detail for the open-bosonic-string case. The previously reported gauge-fixed action and Becchi-Rouet-Stora (BRS) transformation are completed by adding the quartic string-interaction term constructed explicitly here. The properties of the 3-string and 4-string vertices are fully clarified. We thus establish the nilpotency of the full nonlinear BRS transformation and the BRS invariance of our gauge-fixed action. This, on the other hand, establishes also the gauge invariance of our gauge-unfixed action and the group law of the gauge transformations, which were also reported previously. The general N-string amplitudes are computed explicitly at the tree level and shown to reproduce correctly the usual dual amplitudes for the on-shell physical external states. In particular we prove that the on-shell physical amplitudes at the tree level are completely independent of the string length parameter α. The zero-slope limit of our covariant string action as well as of the BRS transformation is calculated completely off the mass shell. The resultant action has the same form as the usual covariant Yang-Mills action $-\frac{1}{4}\mathrm{tr}F_{\mu\nu}{}^2$ as for the gauge-invariant part, and the nontrivial α dependence appears only in the gauge-fixing terms.

I. INTRODUCTION

The remarkable progress made in the last two years in superstring theory[1,2] has revived physicist's interest in constructing string field theories in a manifestly covariant manner, either in gauge-invariant form or in gauge-fixed Becchi-Rouet-Stora- (BRS) invariant form. The construction of such string field theories is indeed very desirable for various purposes; for instance, to discuss compactification of extra dimensions, to reveal gauge symmetries and the geometry of string field theory, to develop efficient Feynman rules for higher-loop string diagrams, and so on.

The vital power of the covariant string field theory clearly resides in the point that it enables us to reveal nonperturbative aspects of the theory. For example, without field theory, it had never been suspected that the string theory could be formulated in a completely geometry-independent manner.[3,4] Further, it would be possible only in the framework of string field theory to confirm the exciting possibility suggested first by Freund[5] that all the consistent superstring theories come from the unique bosonic string theory in 26 dimensions.

This task of constructing a covariant string field theory was initiated by Siegel[6] for free cases of open and closed bosonic strings, based on the Kato-Ogawa[7,8] BRS formalism[9–12] of the first-quantized string. Subsequently, gauge-invariant actions have been proposed by many authors for a free bosonic string and for a free superstring.[13–15]

At the end of last year we reported a manifestly covariant field theory of an interacting bosonic string in two short papers[16,17] (hereafter referred to as I and II) for the open- and closed-string cases, respectively. There we constructed a gauge-fixed action and proved its invariance under a nonlinear nilpotent BRS transformation. It was also noted that it correctly reproduced the usual dual amplitudes for the string scatterings of physical modes.

Subsequently, in Ref. 18 (referred to as III), we presented a gauge-invariant action with local gauge symmetry for both open- and closed-string cases on the basis of the string vertex functions constructed in the above gauge-fixed action. [We became aware of a recent article by Aref'eva and Volovich[19] in which they also found the same gauge-invariant action as ours up to an $O(g^2)$ term for the open-string case.] It was recognized that the nilpotency of the previous BRS transformation almost simultaneously guaranteed the local gauge invariance of the new action. The group structure of local gauge transformations was also fully clarified there.

Our formulation is based on the string vertex functions which may be regarded as a natural covariant extension of those of light-cone gauge string field theory,[20–25] and the string field contains, in particular, the "length" parameter α as its argument. A similar vertex was adopted also by Neveu and West in their slightly different approach to gauge-invariant string theory.[26] A very different and more geometrical approach to gauge-invariant string field theory was proposed by Witten for the open-bosonic-string case[27] first and extended to the open-superstring case[28] recently. In his theory, the string field contains no string length parameter and the vertex has quite a new structure in which the string midpoint plays a special role. Other proposals for gauge-invariant interacting bosonic string field theories have been made in Refs. 29—33.

The purpose of the present paper is to give full details of our covariant string field theory in a complete manner and to show that it is actually a satisfactory theory. In particular, since the proofs for various statements given in I and II had to be very short by the nature of Letter articles, we fully spell out their details here. Further the quartic open-string interaction term, whose necessity was pointed out in I, is determined explicitly for the first time in this paper. For length reasons we are obliged to confine ourselves to the open-string case in this paper. However there should be no difficulty for the reader in convincing himself of the correctness of the closed-string field theory given in II after reading this paper. In any case we will present the full details for the closed-string case in a separate paper.

This paper is organized as follows. In Sec. II we explain some basic properties of our string field Φ and BRS operator Q_B and notations used in this paper. The full nonlinear BRS transformation of open string field Φ schematically takes the form

$$\begin{aligned}\delta_B\Phi &= Q_B\Phi+g\Phi^2V+g^2\Phi^3V^{(4)}\\ &\equiv \delta_B^0\Phi+g\delta_B^1\Phi+g^2\delta_B^2\Phi\ ,\end{aligned} \tag{1.1}$$

referring to the 3-string and 4-string vertex functions V and $V^{(4)}$, respectively. The 3-string vertex V is constructed in Sec. III in such a way that the reason becomes clear why the ghost factor should be multiplied to the naive overlapping δ functional. The $O(g^1)$ nilpotency $\{\delta_B^0,\delta_B^1\}=0$ is proved there to hold actually in $d=26$ with this 3-string vertex.

Section IV is devoted to the construction of the 4-string vertex, with which $\{\delta_B^0,\delta_B^2\}\Phi$ is proved to vanish again in $d=26$ leaving particular "surface terms" corresponding to end-point 4-string configurations. In Sec. V those "surface terms" are shown to be canceled exactly by the special terms from $(\delta_B^1)^2\Phi$ which were called "horn diagram" contributions in I. (In connection with this the authors must apologize for making an erroneous statement in I that the horn diagrams vanish by themselves on the basis of the observation that they contain two ghost factors at coincidental interaction points. They are, however, found to be multiplied by divergent determinant factors to give finite contributions after all, and are canceled by the contributions from the 4-string vertex as stated above.) For this cancellation to occur, the coupling strength of the 4-string vertex term (as well as its measure contained) is fixed relatively to the 3-string's one g; that is, the requirement of BRS nilpotency or the gauge invariance determines the relative weight of the 3-string and 4-string interaction terms.

The other terms in $(\delta_B^1)^2\Phi$ vanish by the mechanism explained in I, i.e., the cancellations between pairs of diagrams connected by the duality. This duality property is essential also for guaranteeing the $O(g^3)$ and $O(g^4)$ nilpotency. We complete the nilpotency proof of our BRS transformation in Sec. V. The useful and important properties of the vertex functions proved there are summarized in the last subsection by introducing a convenient notation of "string products."

We present the gauge-invariant action and the gauge-fixed BRS-invariant action in Sec. VI. Although they were identical with those reported already in III and I, respectively (except for the quartic interaction term not discussed in I), we cite them here as well as the group law of local string gauge transformations in the present notations for completeness. We omitted the discussion of the gauge-fixing procedure which can be found in III.

In Sec. VII we show that our theory with gauge-fixed BRS-invariant action actually reproduces the usual dual amplitudes for general N-string scattering at the tree level provided that the external string states are on-shell and physical modes. In particular the explicit calculations of 4-string tree amplitudes are presented in detail. Based on the general N-string amplitudes obtained there, we prove that the on-shell physical amplitudes at the tree level are completely *independent* of the length parameters α_r of external strings aside from the overall conservation factor $\delta(\sum_{r=1}^N\alpha_r)$. A discussion is given there that if such an α independence holds beyond the tree level one can consistently define physical states free from α parameters in such a way that the physical S matrix defined over them satisfies unitarity.

We calculate in Sec. VIII the zero-slope limit of our string theory and find a very encouraging result. The gauge-invariant part has no explicit α dependence and reproduces exactly the same form as the Yang-Mills theory, and all the explicitly α-dependent terms appear only in the gauge-fixing and the corresponding Faddeev-Popov terms summarized in the form[34] $S_{\mathrm{GF+FP}}=\delta_B(X)$. Although being zero-slope limit, this proves the desired α *independence* of on-shell physical amplitudes *at the full order level,* since the on-shell physical amplitudes are independent of the choice of gauge-fixing terms as is well known[35,36,11] in the usual Yang-Mills theory.

The final section is devoted to the discussion.

We add Appendixes A—I in the final part of this paper. Most of them deal with various technical details of the proofs for the statements given in the text. However, Appendix A is intended to give some basic definitions and properties of the Neumann functions associated with light-cone diagrams, which will probably be very helpful for the reader to understand the whole content of this paper. So the reader is recommended to read it first.

Before entering into the subject we give an important remark on notational changes from the previous papers I—III. There the string field $\Phi[X,c,\bar{c};\alpha]$ was a bosonic quantity carrying no ghost number, and was expanded with respect to the ghost zero mode c_0 into

$$\Phi=\phi+c_0\psi\ . \tag{1.2}$$

(This is indeed the usual convention adopted by Siegel[6] and many other authors.) However, these ghost variables $c(\sigma)$ and $\bar{c}(\sigma)$ are something like "momentum" variables rather than "coordinates" and it is much more convenient (and natural) to use their Fourier conjugate variables $i\pi_c(\sigma)$ and $i\pi_{\bar{c}}(\sigma)$ in constructing vertex functions, in particular, the 4-string vertex $V^{(4)}$. Therefore, we dared to change our conventions and to adopt the Fourier-transformed string field as our basic field Φ:

$$\left[\int [dc\, d\bar{c}]\exp\left[i\int d\sigma(\pi_c c+\pi_{\bar{c}}\bar{c})\right]\Phi[X,c,\bar{c};\alpha]=\tilde{\Phi}[X,i\pi_{\bar{c}},i\pi_c;\alpha]\right]^{\text{old}}\to\Phi[X,c,\bar{c};\alpha] \text{ in this paper .} \tag{1.3}$$

Here note that we have also changed the meaning of the variables $c(\sigma)$ and $\bar{c}(\sigma)$ which actually stand for $i\pi_{\bar{c}}(\sigma)$ and $i\pi_c(\sigma)$ in the old notations (and hence $i\pi_{\bar{c}}$ and $i\pi_c$ in our notations for c and $\bar{c}$ in the old ones). The definition of the ghost number N_{FP} is unchanged; c and $\bar{c}$ carry $N_{\text{FP}}=+1$ and -1, respectively, in both conventions (this N_{FP} is the *net* ghost number which may be carried by the coefficient fields as well as the coordinates c and $\bar{c}$; do not confuse it with the internal ghost number carried by ghost coordinates c and $\bar{c}$ alone; see Sec. VI). However, our string field Φ now becomes Grassmann odd and has $N_{\text{FP}}=-1$ since the integration measure $[dc\, d\bar{c}]^{\text{old}}$ in (1.3) carries the ghost number $N_{\text{FP}}=-1$ of dc_0^{old}. Fortunately, this notational change induces no change for nonzero-mode oscillators $c_n,\bar{c}_n$ $(n\neq 0)$ and thus the difference appears only in the ghost zero-mode part in the bra-ket representation which will be used extensively in the following. Equation (1.3) reads, in the ket representation,

$$\left[\int dc_0 e^{i\pi_c^0 c_0}(|\phi\rangle+c_0|\psi\rangle)=-i\pi_c^0|\phi\rangle+|\psi\rangle\right]^{\text{old}}\to(-\bar{c}_0|\phi\rangle+|\psi\rangle) \text{ in this paper .} \tag{1.4}$$

As another notational change we reverse the σ direction of the argument functions $X(\sigma)$, $c(\sigma)$, and $\bar{c}(\sigma)$ for the functional $\Phi[X,c,c;\alpha]$ with $\alpha<0$:

$$\Phi^{\text{old}}[X(\sigma),c(\sigma),\bar{c}(\sigma);\alpha]\to\begin{cases}\Phi[X(\sigma),c(\sigma),\bar{c}(\sigma);\alpha] & \text{for } \alpha\geq 0\ ,\\ \Phi[X(\pi-\sigma),-c(\pi-\sigma),\bar{c}(\pi-\sigma);\alpha] & \text{for } \alpha<0\ ,\end{cases} \tag{1.5}$$

with the understanding that Φ^{old} in the left-hand side (LHS) is the string field on which the previous change is already performed. This change (1.5) makes the overlapping δ functional of string vertices look like an antiparallel connection instead of the previous parallel one and simplifies the oscillator expressions of the vertices. It should be emphasized that these are mere changes of notations, of course, and the previous results reported in I—III are all correct and coincide with those in this paper although they have *apparently* different expressions.

II. STRING FIELD

The string field Φ in our manifestly covariant formulation is a functional of string coordinate $X_\mu(\sigma)$ $(\mu=1\text{—}d)$ and the FP ghost and antighost (Hermitian Grassmann) coordinates, $c(\sigma)$ and $\bar{c}(\sigma)$:

$$\Phi=\Phi[X_\mu(\sigma),c(\sigma),\bar{c}(\sigma);\alpha]\ . \tag{2.1}$$

The field Φ itself is a *Grassmann-odd* quantity with FP ghost number $N_{\text{FP}}=-1$. Besides $(X_\mu,c,\bar{c})$, Φ also depends on another (unphysical) variable α $(-\infty<\alpha<\infty)$ which we call the "string-length" parameter. The necessity of introducing α as an argument of Φ was pointed out in I and will be explained in more detail in Sec. V. It plays a role similar to p_+ in the light-cone gauge formulation.[20–24]

In the open-string case, to which we will confine ourselves in this paper, the string parameter σ runs from 0 to π, and the coordinates $(X_\mu,c,\bar{c})$ are subject to the following boundary conditions:

$$\frac{d}{d\sigma}X_\mu(\sigma)=c(\sigma)=\frac{d}{d\sigma}\bar{c}(\sigma)=0 \quad\text{at } \sigma=0,\pi\ . \tag{2.2}$$

It is convenient to introduce the oscillator representation

$$\begin{aligned}X^\mu(\sigma)&=\frac{1}{\sqrt{\pi}}\left[x^\mu+i\sum_{n\geq 1}\frac{1}{n}(\alpha_n^\mu-\alpha_{-n}^\mu)\cos(n\sigma)\right],\\ A_\pm^\mu(\sigma)&\equiv P^\mu(\sigma)\mp X'^\mu(\sigma)=\frac{1}{\sqrt{\pi}}\sum_{n=-\infty}^{\infty}\alpha_n^\mu e^{\pm in\sigma}\ ,\\ C_\pm(\sigma)&\equiv i\pi_{\bar{c}}(\sigma)\mp c(\sigma)=\frac{1}{\sqrt{\pi}}\sum_{n=-\infty}^{\infty}c_n e^{\pm in\sigma}\ ,\\ \bar{C}_\pm(\sigma)&\equiv\bar{c}(\sigma)\mp i\pi_c(\sigma)=\frac{1}{\sqrt{\pi}}\sum_{n=-\infty}^{\infty}\bar{c}_n e^{\pm in\sigma}\ ,\end{aligned} \tag{2.3}$$

where α_n^μ, c_n, and $\bar{c}_n$ satisfy the properties [we use the metric $\eta^{\mu\nu}=\text{diag}(-,+,+,\ldots,+)$]

$$\begin{aligned}&[\alpha_n^\mu,\alpha_m^\nu]=n\eta^{\mu\nu}\delta_{m+n,0},\quad \{c_n,\bar{c}_m\}=\delta_{n+m,0}\ ,\\ &\alpha_{-n}=\alpha_n^\dagger,\quad c_{-n}=c_n^\dagger,\quad \bar{c}_{-n}=\bar{c}_n^\dagger\ ,\\ &\alpha_0^\mu=p^\mu=-i\frac{\partial}{\partial x_\mu},\quad c_0=\frac{\partial}{\partial\bar{c}_0}\ .\end{aligned} \tag{2.4}$$

The vacuum state of the oscillators is denoted by $|0\rangle$:

$$(\alpha_n^\mu,c_n,\bar{c}_n)|0\rangle=0 \quad (n\geq 1)\ . \tag{2.5}$$

In (2.3), $P^\mu(\sigma)$, $\pi_c(\sigma)$, and $\pi_{\bar{c}}(\sigma)$ are the momenta conjugate to $X^\mu(\sigma)$, $c(\sigma)$, and $\bar{c}(\sigma)$, respectively:

$$\begin{aligned}P_\mu(\sigma)&=-i\frac{\delta}{\delta X^\mu(\sigma)}\ ,\\ \pi_c(\sigma)&=-i\frac{\delta}{\delta c(\sigma)},\quad \pi_{\bar{c}}(\sigma)=-i\frac{\delta}{\delta\bar{c}(\sigma)}\ .\end{aligned} \tag{2.6}$$

Then, we can represent the nonzero-mode dependence of the field functional $\Phi[X_\mu,c,\bar{c};\alpha]$ (2.1) equivalently by a ket vector

$$|\Phi(x,\bar{c}_0;\alpha)\rangle = -\bar{c}_0\,|\phi(x,\alpha)\rangle + |\psi(x,\alpha)\rangle\ , \tag{2.7}$$

while keeping the "coordinate representations" for α and the zero-mode variables x and $\bar{c}_0$. In particular, we have made explicit the dependence on the ghost zero mode $\bar{c}_0$. Similarly the Hermitian conjugate $\Phi^\dagger[X_\mu,c,\bar{c};\alpha]$ corresponds to a bra vector

$$\langle\Phi(x,\bar{c}_0;\alpha)| = -\bar{c}_0\langle\phi(x,\alpha)| + \langle\psi(x,\alpha)|\ . \tag{2.8}$$

Since Φ has $N_{\rm FP}=-1$, ϕ and ψ has $N_{\rm FP}=0$ and -1, respectively. In particular, physical modes of the string are contained in the bosonic field ϕ.

We can state the relation between the string field $\Phi[X,c,\bar{c};\alpha]$ in the functional representation and $|\Phi(x,\bar{c}_0;\alpha)\rangle$ and $\langle\Phi(x,\bar{c}_0;\alpha)|$ in the bra-ket representation, more explicitly as follows. We denote the set of coordinates $(X_\mu(\sigma),c(\sigma),\bar{c}(\sigma),\alpha)$ by Z, and by z if the "zero-mode" coordinates $(X_\mu,\bar{c}_0,\alpha)$ are omitted. Then the functional integration measure is defined by

$$\begin{aligned}[dZ] &\equiv \left[dx\,d\bar{c}_0\frac{d\alpha}{2\pi}\right][dz]\ ,\\ [dz] &\equiv \prod_{n=1}^{\infty}(dx_n\,i d\theta_n d\bar{\theta}_n)\ ,\end{aligned} \tag{2.9}$$

with Fourier coefficients x_n, θ_n, and $\bar{\theta}_n$ of X, c, and $\bar{c}$. The variables θ_n and $\bar{\theta}_n$ are related to the oscillators c_n and $\bar{c}_n$ in (2.3) as

$$c_{\pm n} = \frac{1}{\sqrt{2}}\left[\frac{\partial}{\partial\bar{\theta}_n} \pm i\theta_n\right],\quad \bar{c}_{\pm n} = \frac{1}{\sqrt{2}}\left[\bar{\theta}_n \pm i\frac{\partial}{\partial\theta_n}\right] \quad (n\geq 1)\ . \tag{2.10}$$

The measure $[dz]$ is Hermitian while $[dZ]$ is anti-Hermitian by our convention $(d\bar{c}_0)^\dagger=-d\bar{c}_0$. [We define the Grassmann integral as $\int d\bar{c}_0\bar{c}_0=1$. Then, the measure $d\bar{c}_0$ is anti-Hermitian: $(d\bar{c}_0)^\dagger=-d\bar{c}_0$, and $\delta(\bar{c}_0)=\bar{c}_0$.] In terms of the eigenstates of $\hat{z}$ operators, the completeness relation and orthonormality in the nonzero mode subspace are given by

$$\begin{aligned}&\int |z\rangle[dz]\langle z| = 1\ ,\\ &\langle z|z'\rangle = \delta(z-z')\\ &\qquad \equiv \prod_{n=1}^{\infty}[\delta(x_n-x_n')i\delta(\theta_n-\theta_n')\delta(\bar{\theta}_n-\bar{\theta}_n')]\ .\end{aligned} \tag{2.11}$$

Now the relation between the functional and bra-ket representations is given by

$$\begin{aligned}&\langle z|\Phi(x,\bar{c}_0;\alpha)\rangle = \Phi[Z]\ ,\\ &\langle\Phi(x,\bar{c}_0;\alpha)|z\rangle = \Phi^\dagger[Z]\ .\end{aligned} \tag{2.12}$$

With these relations understood, it is easy to rewrite the relations in one representation into those in another.

The field Φ and its Hermitian conjugate $\Phi^\dagger$ are not independent. We impose the following Hermiticity requirement:

$$\Phi^\dagger[X_\mu(\sigma),c(\sigma),\bar{c}(\sigma);\alpha] = (\Omega\Phi)[X_\mu(\sigma),c(\sigma),\bar{c}(\sigma);-\alpha] = \Phi[X_\mu(\pi-\sigma),-c(\pi-\sigma),\bar{c}(\pi-\sigma);-\alpha]\ , \tag{2.13}$$

where Ω is the twist operator:[37]

$$\begin{aligned}&(\Omega\Phi)[X_\mu(\sigma),c(\sigma),\bar{c}(\sigma);\alpha] = \Phi[X_\mu(\pi-\sigma),-c(\pi-\sigma),\bar{c}(\pi-\sigma);\alpha]\ ,\\ &\Omega(\alpha_n,c_n,\bar{c}_n)\Omega^{-1} = (-)^n(\alpha_n,c_n,\bar{c}_n)\quad (n=0,\pm1,\pm2,\ldots),\quad \Omega=\Omega^\dagger=\Omega^{-1},\quad \Omega|0\rangle=|0\rangle\ .\end{aligned} \tag{2.14}$$

In terms of the bra-ket representation, the Hermiticity condition (2.13) is expressed as

$${}_2\langle\Phi(x_2,\bar{c}_0^{(2)};\alpha_2)| = \int dx_1 d\bar{c}_0^{(1)}\frac{d\alpha_1}{2\pi}\langle R(1,2)|\Phi(x_1,\bar{c}_0^{(1)};\alpha_1)\rangle_1\Omega^{(2)}\ , \tag{2.15}$$

where $\langle R(1,2)|$ is given by

$$\langle R(1,2)| = \delta(x_1-x_2)\delta(\bar{c}_0^{(1)}-\bar{c}_0^{(2)})2\pi\delta(\alpha_1+\alpha_2)\,{}_1\langle 0|\,{}_2\langle 0|\exp\left[-\sum_{n\geq 1}\left[\frac{1}{n}\alpha_n^{(1)}\cdot\alpha_n^{(2)} - \bar{c}_n^{(1)}c_n^{(2)} - \bar{c}_n^{(2)}c_n^{(1)}\right]\right] \tag{2.16}$$

and the subscript r in ${}_r\langle 0|$ means that ${}_r\langle 0|$ is a bra vector of the rth oscillators $(\alpha_n^{(r)},c_n^{(r)},\bar{c}_n^{(r)})$. Note that $\langle R(1,2)|$ satisfies the properties

$$\begin{aligned}&\langle R(1,2)|[X^{(1)}(\sigma)-X^{(2)}(\sigma)]=0\ ,\\ &\langle R(1,2)|[c^{(1)}(\sigma)-c^{(2)}(\sigma)]=0\ ,\\ &\langle R(1,2)|[\bar{c}^{(1)}(\sigma)-\bar{c}^{(2)}(\sigma)]=0\ ,\end{aligned} \tag{2.17}$$

where $X^{(r)}$, $c^{(r)}$, and $\bar{c}^{(r)}$ are given by (2.3) with $(\alpha_n,c_n,\bar{c}_n)$ replaced by $(\alpha_n^{(r)},c_n^{(r)},\bar{c}_n^{(r)})$.

In the open-string theory we can incorporate the Yang-Mills group quantum numbers by letting Φ be matrix valued.[38] From the requirement to be explained in Sec. III, there are three possibilities for such groups: namely, U(N), O(N), and USp($2N$) (Ref. 39). In the U(N) case (orientable string) Φ is an $N\times N$ matrix in the fundamental representation. In the case of O(N) or USp($2N$) (nonorientable string), we have to impose the nonorientability constraint on Φ:

$$\Phi^T=\Omega\Phi\quad (|\Phi\rangle^T=\Omega|\Phi\rangle)\ , \tag{2.18}$$

where T denotes the transpose with respect to the matrix index of Φ. When Φ is expanded in terms of component fields of definite mass, Eq. (2.18) implies that the even-mass-level fields ($m^2/2=0,2,4,\ldots$) belong to the antisymmetric representation while the odd-mass-level fields ($m^2/2=-1,1,3,\ldots$) belong to the symmetric representation. When Φ is matrix valued, $\Phi^\dagger$ implies the Hermitian conjugation also with respect to the matrix index; in particular, the bra vector $\langle\Phi|$ carries a Hermitian-conjugated matrix index relative to $|\Phi\rangle$.

The BRS operator Q_B in the first-quantized theory[7,8] plays an essential role also in the second quantization:[6]

$$Q_B=\sqrt{\pi}\int_0^\pi d\sigma\{i\pi_{\bar{c}}[-\tfrac{1}{2}(P^2+X'^2)+i(c'\bar{c}-\pi_{\bar{c}}'\pi_c)] -c(P\cdot X'+c'\pi_c+\pi_{\bar{c}}'\bar{c})\}$$
$$=\frac{\sqrt{\pi}}{4}\sum_{\pm}\int_0^\pi d\sigma\, C_\pm\left[-A_\pm{}^2\mp 2i\frac{dC_\pm}{d\sigma}\bar{C}_\pm\right]. \quad (2.19)$$

In terms of the oscillators, Q_B is written as

$$Q_B=\sum_{n=-\infty}^{\infty}:c_{-n}\left[-\tfrac{1}{2}\sum_{m=-\infty}^{\infty}[\alpha_{n-m}\cdot\alpha_m+(n+m)\bar{c}_{n-m}c_m] +\alpha(0)\delta_{n,0}\right]:, \quad (2.20)$$

where : : denotes the normal ordering and $\alpha(0)$ (intercept parameter) is a constant. Kato and Ogawa[7] found that the nilpotency of Q_B,

$$Q_B{}^2=0, \quad (2.21)$$

is satisfied only when $d=26$ and $\alpha(0)=1$. It is convenient to make the $\bar{c}_0$ and c_0 ($=\partial/\partial\bar{c}_0$) dependence of Q_B explicit and to rewrite it as

$$Q_B=c_0L+\bar{c}_0M+\tilde{Q}_B. \quad (2.22)$$

Here, $\tilde{Q}_B$ stands for the part of Q_B which contains neither c_0 nor $\bar{c}_0$, and

$$L=-\tfrac{1}{2}p^2-\sum_{n\ge 1}[\alpha_{-n}\cdot\alpha_n+n(c_{-n}\bar{c}_n+\bar{c}_{-n}c_n)]+1,$$
$$M=2\sum_{n\ge 1}nc_{-n}c_n. \quad (2.23)$$

Free string action in the second-quantized theory was given by Siegel:[6]

$$S^0=-\delta_B^0\int d1\,\mathrm{tr}\left\langle\Phi(1)\left|\bar{c}_0\frac{\partial}{\partial\bar{c}_0}\right|\Phi(1)\right\rangle$$
$$=-\int d1\,\mathrm{tr}\left\langle\Phi(1)\left|\left[\bar{c}_0\frac{\partial}{\partial\bar{c}_0},Q_B\right]\right|\Phi(1)\right\rangle, \quad (2.24)$$

where 1 and $d1$ denote the "zero-mode variables" $(x,\bar{c}_0,\alpha)$ and their integration $dx\,d\bar{c}_0(d\alpha/2\pi)$, respectively, and tr represents the trace of the matrix index of Φ. Here, δ_B^0 is the BRS transformation at the free level defined by

$$\delta_B^0|\Phi\rangle=Q_B|\Phi\rangle, \quad (2.25a)$$

$$\delta_B^0\langle\Phi|=\langle\Phi|Q_B=(\delta_B^0|\Phi\rangle)^\dagger. \quad (2.25b)$$

Because of the property $Q_B^2=0$, δ_B^0 is nilpotent [when $d=26$ and $\alpha(0)=1$]:

$$(\delta_B^0)^2=0. \quad (2.26)$$

The BRS transformation (2.25b) for the bra vector follows from (2.15) by making use of the formulas

$$\langle R(1,2)|(Q_B^{(1)}+Q_B^{(2)})=0 \quad (2.27)$$

and

$$[\Omega,Q_B]=0, \quad (2.28)$$

and by noting the fact that the δ_B^0 operation anticommutes with every Grassmann-odd quantity, in particular, with $d\bar{c}_0$. For a nonorientable string the BRS transformation (2.25) preserves the constraint (2.18) owing to the property (2.28).

By carrying out the $\bar{c}_0$ integration in (2.24), we get

$$S^0=-\delta_B^0\int d1\,\mathrm{tr}\langle\psi(1)|\phi(1)\rangle$$
$$=\int d1\,\mathrm{tr}[\langle\phi(1)|L|\phi(1)\rangle+\langle\psi(1)|M|\psi(1)\rangle], \quad (2.29)$$

where $|\phi\rangle$ and $|\psi\rangle$ is defined by (2.7), and 1 and $d1$ here denote (x,α) and $dx(d\alpha/2\pi)$, respectively. The BRS transformation (2.25) is expressed in terms of $|\phi\rangle$ and $|\psi\rangle$ as

$$\delta_B^0|\phi\rangle=\tilde{Q}_B|\phi\rangle+M|\psi\rangle,$$
$$\delta_B^0|\psi\rangle=-L|\phi\rangle+\tilde{Q}_B|\psi\rangle. \quad (2.30)$$

III. CONSTRUCTION OF NONLINEAR BRS TRANSFORMATION I; 3-STRING VERTEX

A. Nilpotency requirement of the BRS transformation

The purpose of this section is to extend the homogeneous BRS transformation δ_B^0 (2.24) to a nonlinear one δ_B in such a way that the nilpotency $(\delta_B)^2=0$ is preserved. In the light-cone gauge formulation of the open-string field theory,[20–22] it was necessary to introduce the 3-string and 4-string interaction terms in the action in order to reproduce the correct dual amplitudes at the tree level. Furthermore, if one considers the loop diagrams, one has to mix the closed-string system with the open-string one.[21]

Correspondingly, in our manifestly covariant formulation also, we expect $\delta_B\Phi$ to consist of terms which are quadratic and cubic in Φ and possibly also terms which contain the closed-string field. The reason of this will become clear from the relation between the action and $\delta_B\Phi$ discussed in Sec. VI.

In this paper, we try to construct a nilpotent BRS transformation δ_B restricting to pure open-string system in the following form:

$$\delta_B=\delta_B^0+g\delta_B^1+g^2\delta_B^2, \quad (3.1)$$

where g is the coupling constant and $\delta_B^1\Phi$ ($\delta_B^2\Phi$) is quadratic (cubic) in Φ. It turns out that (3.1) is enough to reproduce correct dual amplitudes at the tree level. It is an open question whether one is actually obliged to include the closed-string terms in δ_B when the loop diagrams are taken into account.

Now, from the requirement of nilpotency, $(\delta_B)^2=0$, the following conditions should be satisfied:

$$(\delta_B^0)^2=0 , \qquad (3.2)$$

$$\{\delta_B^0,\delta_B^1\}=0 , \qquad (3.3)$$

$$(\delta_B^1)^2+\{\delta_B^0,\delta_B^2\}=0 , \qquad (3.4a)$$

$$\{\delta_B^1,\delta_B^2\}=0 , \qquad (3.4b)$$

$$(\delta_B^2)^2=0 . \qquad (3.4c)$$

The first condition (3.2) is satisfied when $d=26$ and $\alpha(0)=1$. In this section we construct (a part of) δ_B^1 from the condition (3.3). In the next section δ_B^2 is constructed so that it almost satisfies $\{\delta_B^0,\delta_B^2\}=0$. In Sec. V we shall fully determine δ_B^1 and δ_B^2 from the conditions (3.4). As we shall see, each of the conditions (3.3) and (3.4) holds only when $d=26$.

B. The form of the 3-string vertex

Now, let us consider δ_B^1. We assume the following form for $\delta_B^1\,|\,\Phi\rangle$:

$$\delta_B^1\,|\,\Phi(3)\rangle=\int d1\,d2\langle\Phi(1)\,|\,\langle\Phi(2)|\,|V(1,2,3)\rangle , \qquad (3.5)$$

where 1, 2, and 3 denote the sets of "zero-mode" variables $(x_r,\bar{c}_0^{(r)},\alpha_r)$ $(r=1,2,3)$ and

$$dr\equiv dx_r d\bar{c}_0^{(r)}(d\alpha_r/2\pi) . \qquad (3.6)$$

When Φ carries the Yang-Mills group quantum number, the right-hand side (RHS) of (3.5) also implies the matrix multiplication of $\langle\Phi(1)\,|$ and $\langle\Phi(2)\,|$. [We assume that $|\,V(1,2,3)\rangle$ has no matrix indices.] Here, we cannot take an arbitrary group and representations. First of all, the Hermiticity condition (2.13) or (2.15) requires that the Lie-algebra elements be Hermitian or anti-Hermitian. Second, because of the form of δ_B^1 (3.5) the Lie algebra should close under a simple multiplication (not a commutator). From these conditions we are led to the three allowed groups U(N), O(N), and USp($2N$), and representations[39] explained in Sec. II.

Now, from (2.25) and (3.5), the requirement (3.3) leads to the following condition on the vertex $V(1,2,3)$:

$$\left[\sum_{r=1}^{3} Q_B^{(r)}\right]|\,V(1,2,3)\rangle=0 . \qquad (3.7)$$

A natural choice for V would be the following one suggested by the light-cone gauge string field theory:[20–22]

$$V(1,2,3)\propto\delta\left[\sum_{r=1}^{3}\alpha_r\right]\prod_{0\le\sigma\le\pi|\alpha_3|}\delta[\Theta_1X^{(1)}(\sigma_1)+\Theta_2X^{(2)}(\sigma_2)-X^{(3)}(\sigma_3)]\delta[\Theta_1\alpha_1c^{(1)}(\sigma_1)+\Theta_2\alpha_2c^{(2)}(\sigma_2)-\alpha_3c^{(3)}(\sigma_3)]$$
$$\times\delta[\Theta_1\alpha_1{}^{-2}\bar{c}^{(1)}(\sigma_1)+\Theta_2\alpha_2{}^{-2}\bar{c}^{(2)}(\sigma_2)-\alpha_3{}^{-2}\bar{c}^{(3)}(\sigma_3)] ,$$
$$\Theta_1(\sigma)\equiv\theta(\pi\alpha_1-\sigma),\quad \Theta_2(\sigma)\equiv\theta(\sigma-\pi\alpha_1) ,$$
$$\sigma_1(\sigma)\equiv\frac{\sigma}{\alpha_1},\quad \sigma_2(\sigma)\equiv\frac{\sigma-\pi\alpha_1}{\alpha_2},\quad \sigma_3(\sigma)\equiv\frac{\pi|\alpha_3|-\sigma}{|\alpha_3|} . \qquad (3.8)$$

Here, we are considering the case $\alpha_1,\alpha_2>0$ and $\alpha_3<0$. The RHS of (3.8) expresses the splitting of the string 3 into the strings 1 and 2 at the point $\sigma_3=\pi\alpha_2/|\alpha_3|$ $(\alpha_1+\alpha_2=|\alpha_3|)$ [see Fig. 1(A)]. The extention of V to all the regions of α_r is given in Fig. 1. The factors of α_r in front of $c^{(r)}(\sigma)$ and $\bar{c}^{(r)}(\sigma)$ in the δ functionals of (3.8) are chosen so that every term in the BRS charge $Q_B^{(3)}$ operated on V can be properly transformed into those in $Q_B^{(1)}$ and $Q_B^{(2)}$. In fact, the δ functionals in (3.8) also imply the following connection conditions for the conjugate variables $P_\mu(\sigma)$, $\pi_c(\sigma)$, and $\pi_{\bar{c}}(\sigma)$:

$$\Theta_1O^{(1)}(\sigma_1)+\Theta_2O^{(2)}(\sigma_2)-O^{(3)}(\sigma_3)=0$$

with

$$O^{(r)}(\sigma_r)=\alpha_r{}^{-1}P^{(r)}(\sigma_r),\ \alpha_r{}^{-2}\pi_c^{(r)}(\sigma_r)\ ,\text{ or }\alpha_r\pi_{\bar{c}}^{(r)}(\sigma_r) ,$$

and hence also for

$$O^{(r)}(\sigma_r)=\alpha_r{}^{-1}A_\pm^{(r)}(\sigma_r),\ \alpha_rC_\pm^{(r)}(\sigma_r),\ \alpha_r{}^{-2}\bar{C}_\pm^{(r)}(\sigma_r) .$$

From the expression (2.19) of Q_B and these connection conditions, it is naively expected that $V(1,2,3)$ of (3.8) satisfies (3.7).

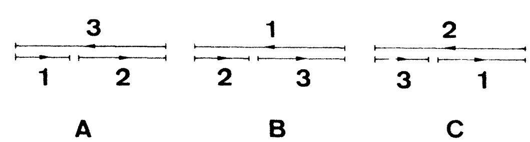


FIG. 1. The structures of overlapping δ functionals in 3-string vertex, for cases (A) $|\alpha_3|=|\alpha_1|+|\alpha_2|$, (B) $|\alpha_1|=|\alpha_2|+|\alpha_3|$, and (C) $|\alpha_2|=|\alpha_3|+|\alpha_1|$, respectively.

C. Oscillator expression of the 3-string δ functional

In order to see if this is really the case, we need a more precise definition of the δ functionals of (3.8); a representation by oscillator modes. The oscillator expression of the bosonic part of the δ functionals of (3.8) has been well known and is given by[20–24]

$$|V_X(1,2,3)\rangle=(2\pi)^d\delta\left[\sum_{r=1}^{3}p_r\right]\exp[E_x(1,2,3)]\,|0\rangle\,, \tag{3.9}$$

$$\begin{aligned}E_X(1,2,3)&\equiv\frac{1}{2}\sum_{\substack{n,m\geq 0\\ r,s}}\bar{N}^{rs}_{nm}\alpha^{(r)}_{-n}\cdot\alpha^{(s)}_{-m}\\&=\frac{1}{2}\sum_{\substack{n,m\geq 1\\ r,s}}\bar{N}^{rs}_{nm}\alpha^{(r)}_{-n}\cdot\alpha^{(s)}_{-m}+\sum_{\substack{n\geq 1\\ r}}\bar{N}^{r}_{n}\alpha^{(r)}_{-n}\cdot\mathbb{P}\\&\quad+\tau_0\sum_r\frac{1}{\alpha_r}\frac{{p_r}^2}{2}\,,\end{aligned} \tag{3.10}$$

where the Neumann functions $\bar{N}^{rs}_{nm}$ are given by

$$\begin{aligned}&\bar{N}^{rs}_{nm}=-\alpha_1\alpha_2\alpha_3\left[\frac{\alpha_r}{n}+\frac{\alpha_s}{m}\right]^{-1}\bar{N}^{r}_{n}\bar{N}^{s}_{m}\quad(n,m\geq 1)\,,\\&\bar{N}^{rs}_{n0}=-c_s\frac{\alpha_1\alpha_2}{\alpha_s}\bar{N}^{r}_{n}\quad(c_1,c_2,c_3)\equiv(1,-1,0)\ (n\geq 1)\,,\\&\bar{N}^{rs}_{00}=\tau_0\left[\frac{\delta_{rs}}{\alpha_r}-\frac{\delta_{r3}}{\alpha_3}-\frac{\delta_{s3}}{\alpha_3}\right],\ \tau_0=\sum_{r=1}^{3}\alpha_r\ln|\alpha_r|\,,\\&\bar{N}^{r}_{n}=\frac{1}{\alpha_r}f_n\left[-\frac{\alpha_{r+1}}{\alpha_r}\right]e^{n\tau_0/\alpha_r}\quad(\alpha_4=\alpha_1,\alpha_0=\alpha_3)\,,\\&f_n(x)=\frac{\Gamma(nx)}{n!\,\Gamma(nx-n+1)}\,,\quad\mathbb{P}=\alpha_rp_{r+1}-\alpha_{r+1}p_r\,.\end{aligned} \tag{3.11}$$

[The properties of $\bar{N}^{rs}_{nm}$ are summarized in Appendix A. The formulas (3.11) correspond to a special choice $Z_1=1$, $Z_2=0$, $Z_3=\infty$ for the parameters Z_r of the Mandelstam mapping[20] (see Appendixes A and D). However, as shown in Sec. 4 of Appendix A, $|V_X\rangle$ of (3.9) and $|V_{FP}\rangle$ of (3.13) are independent of the choice of Z_{1-3}.] In fact, we can show that $|V_X(1,2,3)\rangle$ given by (3.9) satisfies the connection conditions

$$\begin{aligned}&[\Theta_1O^{(1)}(\sigma_1)+\Theta_2O^{(2)}(\sigma_2)-O^{(3)}(\sigma_3)]\,|V_x(1,2,3)\rangle=0\,,\\&O^{(r)}(\sigma_r)=X^{(r)}(\sigma_r),{\alpha_r}^{-1}P^{(r)}(\sigma_r),{\alpha_r}^{-1}A^{(r)}_{\pm}(\sigma_r)\,.\end{aligned} \tag{3.12}$$

The proof is given in Appendix B.

The oscillator expression of the FP ghost coordinate part of the δ functionals in (3.8) is quite similar:

$$|V_{FP}(1,2,3)\rangle=\delta\left[\sum_{r=1}^{3}{\alpha_r}^{-1}\bar{c}^{(r)}_0\right]\times\exp[E_{FP}(1,2,3)]\,|0\rangle\,, \tag{3.13}$$

$$\begin{aligned}E_{FP}(1,2,3)&=i\sum_{\substack{n\geq 1\\ m\geq 0}}\sum_{r,s}\bar{N}^{rs}_{nm}\gamma^{(r)}_{-n}\bar{\gamma}^{(s)}_{-m}\\&=i\sum_{\substack{n,m\geq 1\\ r,s}}\bar{N}^{rs}_{nm}\gamma^{(r)}_{-n}\bar{\gamma}^{(s)}_{-m}+i\sum_{\substack{n\geq 1\\ r}}\bar{N}^{r}_{n}\gamma^{(r)}_{-n}\bar{\Gamma}\,,\end{aligned} \tag{3.14}$$

where the rescaled oscillators $\gamma^{(r)}_n$, $\bar{\gamma}^{(r)}_n$, and $\bar{\Gamma}$ are defined by

$$\begin{aligned}&\gamma^{(r)}_n=in\alpha_rc^{(r)}_n\,,\quad\bar{\gamma}^{(r)}_n={\alpha_r}^{-1}\bar{c}^{(r)}_n\,,\\&\{\gamma^{(r)}_n,\bar{\gamma}^{(s)}_m\}=in\delta_{n+m,0}\delta^{rs}\,,\quad\overset{(-)}{\gamma}{}^{(r)\dagger}_n=\overset{(-)}{\gamma}{}^{(r)}_{-n}\,,\\&\bar{\Gamma}=\alpha_r\bar{\gamma}^{(r+1)}_0-\alpha_{r+1}\bar{\gamma}^{(r)}_0\\&\quad=\frac{\alpha_r}{\alpha_{r+1}}\bar{c}^{(r+1)}_0-\frac{\alpha_{r+1}}{\alpha_r}\bar{c}^{(r)}_0\,.\end{aligned} \tag{3.15}$$

We can also show that $|V_{FP}\rangle$ of (3.13) satisfies the connection condition (see Appendix B)

$$[\Theta_1O^{(1)}(\sigma_1)+\Theta_2O^{(2)}(\sigma_2)-O^{(3)}(\sigma_3)]\,|V_{FP}(1,2,3)\rangle=0\,, \tag{3.16}$$

$$O^{(r)}(\sigma_r)=\alpha_rC^{(r)}_{\pm}(\sigma_r),{\alpha_r}^{-2}\bar{C}^{(r)}_{\pm}(\sigma_r)\,.$$

Despite the above naive expectation, our vertex

$$|V_0(1,2,3)\rangle=2\pi\delta\left[\sum_r\alpha_r\right]|V_X(1,2,3)\rangle\otimes|V_{FP}(1,2,3)\rangle \tag{3.17}$$

turns out not to give the desired vertex $|V(1,2,3)\rangle$ of $O(g)$ BRS transformation (3.5). First of all, $|V_0\rangle$ of (3.17) does not carry the correct FP ghost number: $|V_0\rangle$ has $N_{FP}=-1$, but $|V\rangle$ in (3.5) should carry $N_{FP}=0$ because the field Φ and the measure $d1\,d2\propto d\bar{c}^{(1)}_0d\bar{c}^{(2)}_0$ have $N_{FP}=-1$ and $+2$, respectively, and the BRS transformation ${\delta_B}^1$ should raise N_{FP} by 1. Second, $|V_0\rangle$ does not satisfy the $O(g)$ nilpotency condition (3.7). To see this and to find the correct vertex, let us perform the operation of $\sum_rQ^{(r)}_B$ on $|V_0\rangle$ of (3.17).

D. Calculation of $\sum_rQ^{(r)}_B\,|V_0\rangle$

In calculating $(\sum_rQ^{(r)}_B)\,|V_0\rangle$, we move the annihilation oscillator part in Q_B to the right of $\exp(E_X+E_{FP})$ in $|V_0\rangle$ by making use of the formula

$$Q_Be^E|0\rangle=e^E(Q_B+[Q_B,E]+\tfrac{1}{2}[[Q_B,E],E]+\cdots)\,|0\rangle \tag{3.18}$$

and obtain an expression consisting solely of creation and zero-mode operators. The result of this manipulation is equivalent to making the following replacement, for example, for $(A_\pm)^2$ in Q_B of (2.19):

$$A_\pm A_\pm \to (A_\pm^{(-)}+[A_\pm,E_X])(A_\pm^{(-)}+[A_\pm,E_X]) + [A_\pm,[A_\pm,E_X]] \ , \tag{3.19}$$

where in the RHS $A_\pm^{(-)}$ denotes the creation and zero-mode operator part of $A_\pm$. We now notice an important fact; the quantities in the RHS of (3.19) are singular as they approach the splitting point.[24] For example, $A_\pm^{(1)}(\sigma=\pi-\epsilon/\alpha_1)$ behaves near the splitting point $\sigma=\pi$ as

$$\alpha_1{}^{-1}(A_\pm^{(1)}+[A_\pm^{(1)},E_X])^{(-)}(\sigma)=\frac{1}{\alpha_1}\frac{1}{\sqrt{\pi}}\left[\sum_{n\geq 0}\alpha_{-n}^{(1)}e^{\mp in\sigma}+\sum_{\substack{m\geq 0\\ s}}\left[\sum_{n\geq 1} n\bar{N}_{nm}^{1s}e^{\pm in\sigma}\right]\alpha_{-m}^{(s)}\right]$$

$$\underset{\epsilon\to 0}{\sim} -\frac{1}{\sqrt{\epsilon}}e^{\mp i\pi/4}\frac{1}{(2\pi\,|\,\alpha_1\alpha_2\alpha_3\,|\,)^{1/2}}\left[\mathbb{P}-\alpha_1\alpha_2\alpha_3\sum_{\substack{n\geq 1\\ r}}\frac{n}{\alpha_r}\bar{N}_n^r\alpha_{-n}^{(r)}\right] , \tag{3.20}$$

and we also have

$$\alpha_1{}^{-2}[A_\pm^{(1)},[A_\pm^{(1)},E_X]](\sigma)=\frac{d}{\pi\alpha_1{}^2}\sum_{n,m\geq 1} nm\bar{N}_{nm}^{11}e^{\pm i(n+m)\sigma}\underset{\epsilon\to 0}{\sim}-\frac{d}{16\pi}\frac{1}{\epsilon^2} , \tag{3.21}$$

where use has been made of the formulas

$$\begin{aligned}
&\frac{1}{\alpha_1}\sum_{n\geq 1}e^{in\sigma}n\bar{N}_n^1\underset{\epsilon\to 0}{\sim}-e^{-i\pi/4}\frac{1}{(2\,|\,\alpha_1\alpha_2\alpha_3\,|\,)^{1/2}}\times\frac{1}{\sqrt{\epsilon}} ,\\
&\frac{1}{\alpha_1}\sum_{n\geq 1}e^{in\sigma}n\bar{N}_{nm}^{1s}\underset{\epsilon\to 0}{\sim}-e^{-i\pi/4}\left[\frac{|\,\alpha_1\alpha_2\alpha_3\,|}{2}\right]^{1/2}\frac{m}{\alpha_s}\bar{N}_m^s\times\frac{1}{\sqrt{\epsilon}} .
\end{aligned} \tag{3.22}$$

[These formulas are obtained from (A23) and (A24) in Appendix A.] The quantities $(d/d\sigma)C_\pm$ and $\bar{C}_\pm$ have similar singularity at the splitting point.

In order to treat such singularities more systematically we follow Mandelstam's technique used in his proof of Lorentz invariance in the light-cone gauge string theory.[40] First, note that the BRS charge Q_B of (2.19) remains invariant if we make the substitution

$$\begin{aligned}
&A_\pm(\sigma)\to A_\pm(\sigma\mp i\xi),\quad C_\pm(\sigma)\to C_\pm(\sigma\mp i\xi) ,\\
&\bar{C}_\pm(\sigma)\to\bar{C}_\pm(\sigma\mp i\xi) ,
\end{aligned} \tag{3.23}$$

or equivalently, in oscillator language,

$$(\alpha_n,c_n,\bar{c}_n)\to e^{n\xi}(\alpha_n,c_n,\bar{c}_n) .$$

The replacement (3.23) just corresponds to the time development by an imaginary time $\tau=i\xi$ at the first-quantization level. Then, by attaching the strip of the $\xi<0$ region to each string of Fig. 1(A), we can consider the complex ρ plane of Fig. 2;

$$\begin{aligned}
&\rho=\begin{cases}\alpha_r(\xi_r+i\sigma_r)+i\beta_r, & 0\leq \mathrm{Im}(\rho)\leq\pi\,|\,\alpha_3\,| ,\\ \alpha_r(\xi_r-i\sigma_r)-i\beta_r, & -\pi\,|\,\alpha_3\,|\leq\mathrm{Im}(\rho)\leq 0 ,\end{cases}\\
&(\xi_r\leq 0, 0\leq\sigma_r\leq\pi),\quad (\beta_1,\beta_2,\beta_3)=(0,\pi\alpha_1,\pi\,|\,\alpha_3\,|\,) .
\end{aligned} \tag{3.24}$$

On this plane we define (operator-valued) functions by

$$\begin{aligned}
A(\rho)&=\begin{cases}\dfrac{1}{\alpha_r}A_+^{(r)}(\sigma_r-i\xi_r) ,\\[2mm] \dfrac{1}{\alpha_r}A_-^{(r)}(\sigma_r+i\xi_r) ,\end{cases}\\
C(\rho)&=\begin{cases}\alpha_r C_+^{(r)}(\sigma_r-i\xi_r) ,\\ \alpha_r C_-^{(r)}(\sigma_r+i\xi_r) ,\end{cases}\\
\bar{C}(\rho)&=\begin{cases}\left[\dfrac{1}{\alpha_r}\right]^2\bar{C}_+^{(r)}(\sigma_r-i\xi_r) ,\\[2mm] \left[\dfrac{1}{\alpha_r}\right]^2\bar{C}_-^{(r)}(\sigma_r+i\xi_r) ,\end{cases}
\end{aligned} \tag{3.25}$$

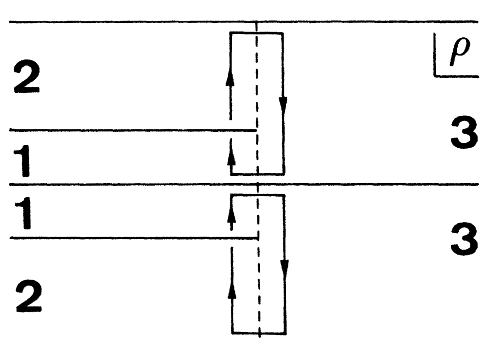


FIG. 2. The original contour C_ρ of integration (3.26) representing $\sum_{r=1}^{3}Q_B^{(r)}$ on the ρ plane (light-cone diagram).

where upper [lower] equalities correspond to the region $0<\mathrm{Im}(\rho)<\pi|\alpha_3|$ $[-\pi|\alpha_3|<\mathrm{Im}(\rho)<0]$. Taking account of the ξ translation invariance of Q_B, the sum of the BRS charges $\sum_{r=1}^{3}Q_B^{(r)}$ can now be rewritten as

$$\sum_{r=1}^{3}Q_B^{(r)}=-i\frac{\sqrt{\pi}}{4}\oint_{C_\rho}d\rho\, C(\rho)\left[-A(\rho)^2+\frac{2dC(\rho)}{d\rho}\bar{C}(\rho)\right], \tag{3.26}$$

where the contour of integration C_ρ is depicted in Fig. 2. (The contribution from the horizontal part of the contour cancels between the two contributions from $\mathrm{Im}\rho \gtrless 0$.) The coordinates $(A(\rho),C(\rho),\bar{C}(\rho))$ should be taken as $(A^{(r)},C^{(r)},\bar{C}^{(r)})$ when ρ is in the strip of the rth string. However, when (3.26) stands in front of $|V_0\rangle$, each of $(A(\rho),C(\rho),\bar{C}(\rho))$ can be regarded as a single (operator-valued) analytic function of ρ with a cut structure of Fig. 2, since it smoothly continues from one strip to another across the boundary.

Since, as we mentioned above, $(A,C,\bar{C})$ is singular at the splitting point and has no singularity elsewhere, the contour C_ρ of Fig. 2 may be deformed to a small circle enclosing the splitting point (Fig. 3). Then taking into account the singularity arising from the first term in the RHS of (3.19) [cf. (3.20)]

$$\{A^{(-)}(\rho)+[A(\rho),E_X]\}^2\sim(\rho-\rho_0)^{-1}, \tag{3.27}$$

where ρ_0 is the splitting point, we find that $\sum_r Q_B^{(r)}|V_0\rangle$ is nonvanishing and contains a piece proportional to $C(\rho_0)|V_0\rangle$. (The lower splitting point at $\rho=\rho_0^*$ contributes $C(\rho_0^*)|V_0\rangle$. However, it is equal to $C(\rho_0)|V_0\rangle$ since the difference $C(\rho_0)-C(\rho_0^*)\propto C_+(\sigma_{\mathrm{int}})-C_-(\sigma_{\mathrm{int}})\propto c(\sigma_{\mathrm{int}})$ vanishes at the splitting point [cf. (2.2)].)

One way to remedy this defect and at the same time raise the FP ghost number of $|V_0\rangle$ by one as is desired is to take

$$|V(1,2,3)\rangle=C(\rho_0)|V_0(1,2,3)\rangle \tag{3.28}$$

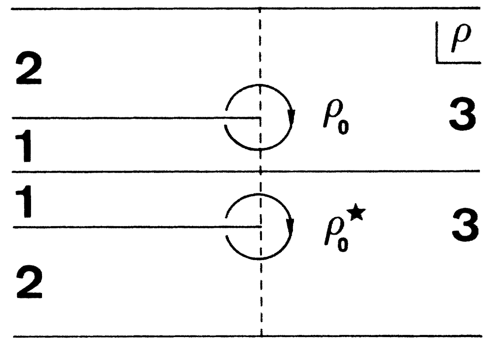


FIG. 3. The contour C_ρ can be deformed to infinitesimal circles enclosing the interaction points ρ_0 and ρ_0^*.

as the vertex of the $O(g)$ BRS transformation (3.5) (Ref. 16). The necessity of a ghost factor in the 3-string vertex was first pointed out by Siegel.[6] [Although $(d/d\rho)C(\rho)|V_0\rangle$ is divergent at $\rho=\rho_0$ like $A(\rho)|V_0\rangle$, $C(\rho_0)|V_0\rangle$ itself has a finite value.] Then the above nonvanishing piece does not contribute to $\sum_r Q_B^{(r)}|V\rangle$ because $[C(\rho_0)]^2=0$.

The singularity arising from the second term on the RHS of (3.19), $[A(\rho),[A(\rho),E_X]]\sim(\rho-\rho_0)^{-2}$, gives another kind of nonvanishing contribution to $\sum_r Q_B^{(r)}|V_0\rangle$. We show below that this is also canceled in $\sum_r Q_B^{(r)}|V\rangle$ when $d=26$.

E. Proof of $\sum_r Q_B^{(r)}|V\rangle=0$

For this purpose it is convenient to make a change of variables from ρ to z connected via a Mandelstam mapping

$$\rho(z)=\sum_{r=1}^{3}\alpha_r\ln(z-Z_r). \tag{3.29}$$

Then we have

$$i\frac{4}{\sqrt{\pi}}\sum_{r=1}^{3}Q_B^{(r)}|V\rangle=\oint_{C_z}dz\left[\frac{d\rho(z)}{dz}\right]^{-1}C(z)\left[-A(z)^2+2\frac{dC(z)}{dz}\bar{C}(z)\right]C(z_0)|V_0\rangle, \tag{3.30}$$

where the contour C_z is depicted in Fig. 4, z_0 is the splitting point $[\rho(z_0)=\rho_0]$, and the new functions $A(z)$, $C(z)$, and $\bar{C}(z)$ of z are defined by

$$A(z)\equiv\left[\frac{d\rho(z)}{dz}\right]A(\rho(z)),\quad C(z)\equiv C(\rho(z)),$$
$$\bar{C}(z)\equiv\left[\frac{d\rho(z)}{dz}\right]\bar{C}(\rho(z)). \tag{3.31}$$

(For notational simplicity, we denote the newly defined functions by the same symbol as the old ones. They should be distinguished by their arguments z or ρ.) By this change of variables the cuts of Figs. 2 and 3 disappear and (3.30) can be evaluated by calculating the residue of the pole at $z=z_0$. Here, it should be noted that a part of the singularity of the integrand (3.30) at $z=z_0$ is contained in $[d\rho(z)/dz]^{-1}$. In fact, since $\rho(z)$ is stationary at the interaction point $z=z_0$,

$$\left.\frac{d\rho(z)}{dz}\right|_{z=z_0}=\sum_r\frac{\alpha_r}{z_0-Z_r}=0, \tag{3.32}$$

we have an expansion

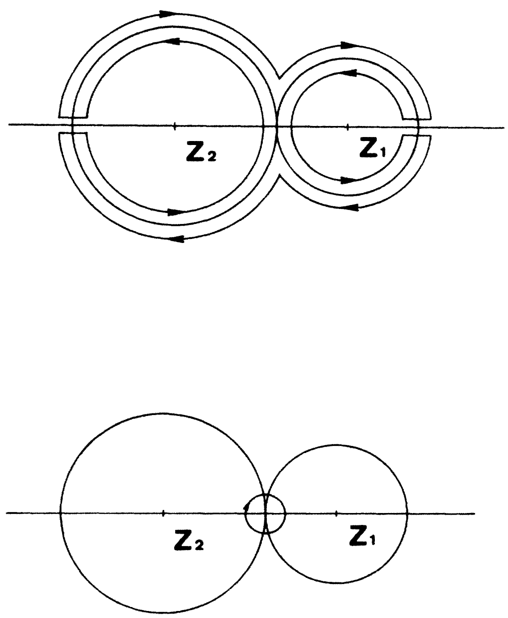


FIG. 4. The contours C_z of integration (3.30) on the z plane: The first one corresponds to C_ρ of Fig. 2 and the second to the reduced one of Fig. 3.

$$\rho_0-\rho(z)=a(z-z_0)^2+b(z-z_0)^3+c(z-z_0)^4+\cdots\ , \tag{3.33}$$

where

$$\rho_0=\rho(z_0),\quad a=\frac{1}{2}\sum_r\frac{\alpha_r}{(z_0-Z_r)^2}\ ,$$
$$b=-\frac{1}{3}\sum_r\frac{\alpha_r}{(z_0-Z_r)^3},\quad c=\frac{1}{4}\sum_r\frac{\alpha_r}{(z_0-Z_r)^4}\ . \tag{3.34}$$

Hence, $[d\rho(z)/dz]^{-1}$ is Laurent expanded as

$$\left[\frac{d\rho(z)}{dz}\right]^{-1}=-\frac{1}{2a}\frac{1}{z-z_0}+\frac{3b}{4a^2}+\left[\frac{c}{a^2}-\frac{9b^2}{8a^3}\right](z-z_0)+\cdots\ . \tag{3.35}$$

In expression (3.30) we make a manipulation corresponding to (3.18) and (3.19). This is equivalent to taking contractions of pairs of factors in

$$C(z)\left[-A(z)^2+2\frac{dC(z)}{dz}\overline{C}(z)\right]C(z_0) \tag{3.36}$$

in all possible ways by making use of the formula

$$\overline{A_\mu(z_1)A_\nu(z_2)}=\frac{1}{\pi}\frac{1}{(z_1-z_2)^2}\eta_{\mu\nu}\ , \tag{3.37a}$$

$$\overline{C(z_1)\overline{C}(z_2)}=-\frac{1}{\pi}\frac{1}{z_1-z_2}\ . \tag{3.37b}$$

The operator O $(=A,C,\overline{C}$, etc.) surviving after the contraction stands for the quantity

$$O^{(-)}+[O,E_X+E_{\rm FP}]\ , \tag{3.38}$$

and consists solely of creation and zero-mode operators. $A(z)$, $C(z)$, $\overline{C}(z)$ and their derivatives with respect to z are now nonsingular at $z=z_0$ due to the factor $(d\rho/dz)$ in (3.31). The singularity of (3.27) has moved to $(d\rho/dz)^{-1}$ in (3.30). The contractions of (3.37) implies the quantity

$$\begin{aligned}\overline{O_1(\rho)O_2(\tilde\rho)}\equiv{}&\theta(\tilde\xi-\xi)\langle 0\,|\,O_1(\rho)O_2(\tilde\rho)\,|\,0\rangle_c\\&\pm\theta(\xi-\tilde\xi)\langle 0\,|\,O_2(\tilde\rho)O_1(\rho)\,|\,0\rangle_c\\&+[O_1(\rho),[O_2(\tilde\rho),(E_X+E_{\rm FP})]]_\mp\ ,\end{aligned} \tag{3.39}$$

where the suffix c denotes the connected part and a $-$ $(+)$ sign in the RHS should be taken when both O_1 and O_2 are fermionic (otherwise). For example, from (2.3), (3.25), and (3.14) we have

$$\begin{aligned}\overline{C(\rho)\overline{C}(\tilde\rho)}=\frac{1}{\pi\alpha_s}\Bigg\{&\Bigg[\theta(\tilde\xi_s-\xi_r)\Bigg[\sum_{n\geq 1}e^{n(\zeta_r-\tilde\zeta_s)}+1\Bigg]\\&-\theta(\xi_r-\tilde\xi_s)\sum_{n\geq 1}e^{n(\tilde\zeta_s-\zeta_r)}\Bigg]\delta^{rs}\\&+\sum_{n,m\geq 0}m\overline{N}{}^{rs}_{nm}e^{n\zeta_r+m\tilde\zeta_s}\Bigg\}\ ,\end{aligned} \tag{3.40}$$

where ρ and $\tilde\rho$ are assumed to lie in the region of the rth and sth string, respectively, and

$$\zeta\equiv\xi+i\sigma\ . \tag{3.41}$$

The RHS of (3.40) can be read off from Eqs. (A7) and (A10) in Appendix A defining $\overline{N}{}^{rs}_{nm}$ to be equal to

$$\frac{1}{\pi\alpha_s}2\frac{\partial}{\partial\tilde\xi_s}N(\rho,\tilde\rho)\Bigg|_{z\ \rm part}=-\frac{1}{\pi}\frac{d\tilde z}{d\tilde\rho}\frac{1}{z-\tilde z}\ . \tag{3.42}$$

Equation (3.37b) is an immediate consequence of (3.42) and (3.31). Derivation of (3.37a) is quite similar.

Now let us turn to the calculation of (3.30). First, we have a term with no contractions. Taking into account a pole in $(d\rho/dz)^{-1}$ of (3.35), it gives a contribution to (3.30)

$$2\pi i\frac{1}{2a}C(z_0)\left[-A(z_0)^2+2\frac{dC}{dz}(z_0)\overline{C}(z_0)\right]C(z_0)\ , \tag{3.43}$$

which vanishes due to $[C(z_0)]^2=0$ as we mentioned before.

There are three kinds of terms with one contraction:

$$\oint dz\left[\frac{d\rho}{dz}(z)\right]^{-1}C(z)\left[-\overline{A(z)A(z)}+2\overline{\frac{dC}{dz}(z)\overline{C}(z)}\right]C(z_0)\ , \tag{3.44a}$$

$$\oint dz\left[\frac{d\rho}{dz}(z)\right]^{-1}\overline{C(z)2\frac{dC}{dz}(z)\overline{C}(z)C(z_0)}\ , \tag{3.44b}$$

$$\oint dz \left[\frac{d\rho}{dz}(z)\right]^{-1} C(z)2\frac{dC}{dz}(z)\overline{C}(z)C(z_0) \ . \tag{3.44c}$$

Care must be taken in evaluating (3.44a) and (3.44b) since they contain the contraction of operators at the coincident point. In order to separate out the short-distance operator-product singularity, we go back to the expression (3.26) in ρ variable and shift the coordinate ρ of one of the contracted operators to $\rho - a\delta$ (δ=const).[40] Then we have, instead of (3.44a) and (3.44b),

$$\oint dz \left[\frac{d\rho(z')}{dz'}\right]^{-1} C(z) \left[-A(z)A(z') + 2\frac{dC}{dz}(z)\overline{C}(z')\right] C(z_0) \ , \tag{3.45a}$$

$$\oint dz \left[\frac{d\rho(z)}{dz}\right]^{-1} C(z')2\frac{dC}{dz}(z)\overline{C}(z)C(z_0) \ , \tag{3.45b}$$

where z' and z are related by

$$\rho(z') = \rho(z) - a\delta \ . \tag{3.46}$$

Note that the factor $[d\rho(z)/dz]^{-1}$ in (3.44a) has changed to $[d\rho(z')/dz']^{-1}$ in (3.45a). Pole residues of (3.45a), (3.45b), and (3.44c) are calculated by making use of the formula

$$\left[\frac{d\rho(z')}{dz'}\right]^{-1} \frac{1}{(z'-z)^2} = -\frac{1}{8a}\frac{1}{\epsilon^3} + \frac{b}{16a^2}\frac{1}{\epsilon^2} + \left[-\frac{3b^2}{16a^3} + \frac{c}{4a^2}\right]\frac{1}{\epsilon} + O(1) + O\left[\frac{\epsilon}{\delta^2}, \frac{1}{\delta}\right] + O\left[\frac{\delta}{\epsilon}\right] , \tag{3.47a}$$

$$\left[\frac{d\rho(z)}{dz}\right]^{-1} \frac{1}{z'-z} = -\frac{1}{4a}\frac{1}{\epsilon^2} + O\left[\frac{1}{\delta}\right] + O\left[\frac{\delta}{\epsilon}\right] \quad (\epsilon = z - z_0) \tag{3.47b}$$

and (3.35), respectively. [Equations (3.47) are shown in Appendix C.] We evaluate (3.45) by performing the contour integration first with δ kept finite, and hence the $O(\epsilon/\delta^2, 1/\delta)$ term does not contribute. The $O(\delta/\epsilon)$ term vanishes after taking the limit $\delta \to 0$. Thus we have

$$(3.45\text{a}) = -i\frac{d-2}{2}\left[\frac{1}{4a}\frac{d^2C}{dz^2}(z_0) - \frac{b}{4a^2}\frac{dC}{dz}(z_0) + \left[\frac{3b^2}{4a^3} - \frac{c}{a^2}\right]C(z_0)\right]C(z_0) \ , \tag{3.48a}$$

$$(3.45\text{b}) = i\frac{1}{a}\frac{d^2C}{dz^2}(z_0)C(z_0) \ , \tag{3.48b}$$

$$(3.44\text{c}) = i\frac{2}{a}\frac{d^2C}{dz^2}(z_0)C(z_0) - i\frac{3b}{a^2}\frac{dC}{dz}(z_0)C(z_0) \ . \tag{3.48c}$$

Summing up (3.48), the contribution to (3.30) of terms with one contraction is found to be given by

$$(d-26)i\left[-\frac{1}{8a}\frac{d^2C}{dz^2}(z_0)C(z_0) + \frac{b}{8a^2}\frac{dC}{dz}(z_0)C(z_0)\right] . \tag{3.49}$$

This indeed vanishes when $d=26$. We have now completed the proof that the vertex $|V(1,2,3)\rangle$ given by (3.28) actually satisfies the $O(g)$ nilpotency condition (3.7).

In the above proof the normal ordering of the BRS charge Q_B was automatically incorporated by the procedure of performing the contour integration before letting $\delta \to 0$. This procedure also implies $\alpha(0)$ (intercept parameter) $=1$ very implicitly.[40,41] In Appendix E we present yet another proof of (3.7), in which $\sum_r Q_B^{(r)}|V\rangle$ is evaluated directly by using its oscillator expression. In this proof we can freely vary $\alpha(0)$ in Q_B, and we find that both $d=26$ and $\alpha(0)=1$ is necessary (and sufficient) for the vanishing of $\sum_r Q_B^{(r)}|V\rangle$.

F. Final form of 3-string vertex

Now that the vertex $|V(1,2,3)\rangle$ is found, our next task is to rewrite it into a form which is manifestly symmetric under the cyclic permutation of three strings. The prefactor $C(\rho_0)$ in (3.28) may be any of $\alpha_r i\pi_{\bar{c}}^{(r)}(\sigma)$ $(r=1,2,3)$ at the splitting point $\sigma = \sigma_I^{(r)}$ (e.g., $\sigma_I^{(r)} = \pi, 0, \pi\alpha_2/|\alpha_3|$ for $r=1,2,3$, respectively, in the case $\alpha_1, \alpha_2 > 0, \alpha_3 < 0$ of Fig. 2). In front of $|V_{\rm FP}(1,2,3)\rangle$ of (3.13), $i\pi_{\bar{c}}^{(r)}(\sigma_I^{(r)})$ is rewritten as

$$\begin{aligned}\sqrt{\pi}\, i\pi_{\bar{c}}^{(r)}(\sigma_I^{(r)}) &= \frac{\partial}{\partial \bar{c}_0^{(r)}} + \sum_{n\ge 1}(c_n^{(r)} + c_{-n}^{(r)})\cos(n\sigma_I^{(r)}) \\ &\to \frac{\partial}{\partial \bar{c}_0^{(r)}} + \sum_{n\ge 1} c_{-n}^{(r)}\cos(n\sigma_I^{(r)}) + \left[\sum_{n\ge 1} c_n^{(r)}\cos(n\sigma_I^{(r)}), E_{\rm FP}(1,2,3)\right] \\ &= \frac{\partial}{\partial \bar{c}_0^{(r)}} + i\frac{1}{\alpha_r}\sum_{\substack{n\ge 1\\ s}}\left[\frac{1}{n}\delta^{rs}\cos(n\sigma_I^{(r)}) - \sum_{m\ge 1}\overline{N}_{mn}^{rs}\cos(m\sigma_I^{(r)})\right]\gamma_{-n}^{(s)} \\ &\equiv \frac{\partial}{\partial \bar{c}_0^{(r)}} + w^{(r)} \ ,\end{aligned} \tag{3.50}$$

and $w^{(r)}$ is shown in Appendix D to be equal to

$$w^{(r)}=i\sum_{\substack{n\geq 1\\ s}}\left[\chi^{rs}\bar{N}^{s}_{n}+\frac{1}{\alpha_r}\sum_{m=1}^{n-1}\bar{N}^{ss}_{n-m,m}\right]\gamma^{(s)}_{-n}\ ,$$
$$\chi^{rs}=\delta^{rs}(\alpha_{r-1}-\alpha_{r+1})/\alpha_r+\sum_{t=1}^{3}\epsilon^{rst}\ \ (\epsilon^{123}=+1)\ . \tag{3.51}$$

From (3.13) and (3.51), the FP ghost coordinate part of the vertex $|V(1,2,3)\rangle$ becomes

$$\sqrt{\pi}\alpha_r i\pi^{(r)}_{\bar{c}}(\sigma^{(r)}_I)\,|V_{\rm FP}(1,2,3)\rangle=\alpha_r\left[\frac{\partial}{\partial\bar{c}^{(r)}_0}+w^{(r)}\right]\delta\left[\sum_{s=1}^{3}\frac{1}{\alpha_s}\bar{c}^{(s)}_0\right]\exp\left[i\sum_{\substack{n,m\geq 1\\ s,t}}\bar{N}^{st}_{nm}\gamma^{(s)}_{-n}\bar{\gamma}^{(t)}_{-m}+i\sum_{\substack{n\geq 1\\ s,t}}\bar{N}^{st}_{n0}\gamma^{(s)}_{-n}\frac{\bar{c}^{(t)}_0}{\alpha_t}\right]|0\rangle$$
$$=\prod_{s=1}^{3}(1-\bar{c}^{(s)}_0w^{(s)})\exp\left[i\sum_{\substack{n,m\geq 1\\ s,t}}\bar{N}^{st}_{nm}\gamma^{(s)}_{-n}\bar{\gamma}^{(t)}_{-m}\right]|0\rangle \tag{3.52}$$

for any choice of r $(=1,2,3)$. This is due to the following relations for $w^{(r)}$:

$$\alpha_r w^{(r)}-i\sum_{\substack{n\geq 1\\ t}}\bar{N}^{tr}_{n0}\gamma^{(t)}_{-n}=\alpha_s w^{(s)}-i\sum_{\substack{n\geq 1\\ t}}\bar{N}^{ts}_{n0}\gamma^{(t)}_{-n},\quad \sum_{r=1}^{3}\alpha_r{}^2w^{(r)}=0\ . \tag{3.53}$$

Therefore, we reach the final expression for the total vertex $|V(1,2,3)\rangle$ (Ref. 16):

$$|V(1,2,3)\rangle=\mu(\alpha_1,\alpha_2,\alpha_3)\sqrt{\pi}[\alpha_r i\pi^{(r)}_{\bar{c}}(\sigma^{(r)}_I)]\,|V_0(1,2,3)\rangle$$
$$=\mu(\alpha_1,\alpha_2,\alpha_3)\prod_{r=1}^{3}(1-\bar{c}^{(r)}_0w^{(r)})\exp[F(1,2,3)]\,|0\rangle\delta(1,2,3)\ , \tag{3.54}$$

where

$$F(1,2,3)=E_X(1,2,3)+i\sum_{\substack{n,m\geq 1\\ r,s}}\bar{N}^{rs}_{nm}\gamma^{(r)}_{-n}\bar{\gamma}^{(s)}_{-m}\ , \tag{3.55}$$

$$\delta(1,2,3)=(2\pi)^{d+1}\delta\left[\sum_{r=1}^{3}p_r\right]\delta\left[\sum_{r=1}^{3}\alpha_r\right]\ . \tag{3.56}$$

In Eq. (3.54) we have multiplied a function of α_r, $\mu(\alpha_1,\alpha_2,\alpha_3)$, which is not determined from the requirement $\sum_r Q^{(r)}_B\,|V\rangle=0$ alone. In Sec. V, we show that another condition $(\delta^1_B)^2+\{\delta^0_B,\delta^2_B\}=0$ of (3.4a) determines $\mu(\alpha)$ to be given by

$$\mu(\alpha_1,\alpha_2,\alpha_3)=\exp\left[-\tau_0\sum_{r=1}^{3}\frac{1}{\alpha_r}\right] \tag{3.57}$$

with τ_0 defined in (3.11).

The vertex (3.54) clearly satisfies the cyclic symmetry:

$$|V(1,2,3)\rangle=|V(2,3,1)\rangle=|V(3,1,2)\rangle\ , \tag{3.58}$$

which is indeed a very important property in constructing gauge-invariant action[18] and BRS-invariant gauge-fixed action.[16]

In the case of δ^0_B in (2.25) the BRS transformation for the bra vector, $\delta^0_B\langle\Phi|$, obtained from the Hermiticity condition (2.15) coincided with $(\delta^0_B\,|\Phi\rangle)^\dagger$. This must also be the case for the present $O(g)$ BRS transformation δ^1_B; namely, the following relation should hold:

$$\delta^1_B\langle\Phi(3)|=(\delta^1_B\,|\Phi(3)\rangle)^\dagger=\int d2\,d1\langle V(1,2,3)||\Phi(2)\rangle\,|\Phi(1)\rangle\ . \tag{3.59}$$

The LHS of (3.59) is calculated from (2.15):

$$\delta^1_B\langle\Phi(3)|=\int d3'\langle R(3',3)|(\delta^1_B\,|\Phi(3')\rangle)\Omega^{(3)}$$
$$=\int d1\,d2\int d1'd2'd3'\langle R(3',3)|\langle R(2',2)|\langle R(1',1)||V(1,',2',3')\rangle\Omega^{(1)}\Omega^{(2)}\Omega^{(3)}\,|\Phi(1)\rangle\,|\Phi(2)\rangle\ , \tag{3.60}$$

where in the second equality we have expressed $\langle\Phi(r)|$ in $\delta^1_B\,|\Phi(3')\rangle$ in terms of $|\Phi(r)\rangle$ again via the constraint (2.15)

$$\langle\Phi(r')| = \int dr\langle R(r,r')|\Phi(r)\rangle\Omega^{(r')} = -\int dr\langle R(r',r)|\Omega^{(r)}|\Phi(r)\rangle \ . \tag{3.61}$$

Since $\langle R(1,2)|$ of (2.16) enjoys the property

$$\langle R(1,2)|(a_n^{(1)}+a_{-n}^{(2)})=0, \quad a_n=\alpha_n,\gamma_n,\bar{\gamma}_n \ , \tag{3.62}$$

and the coefficient of γ_{-n} in $w^{(r)}$ of (3.51) changes its sign under $\alpha_r\to-\alpha_r$ $(r=1,2,3)$ while $\mu(-\alpha_1,-\alpha_2,-\alpha_3)=\mu(\alpha_1,\alpha_2,\alpha_3)$, the following relation holds:

$$\int d1'd2'd3'\langle R(3',3)|\langle R(2',2)|\langle R(1',1)||V(1',2',3')\rangle = \langle V(1,2,3)| \ . \tag{3.63}$$

From (3.60) and (3.63), Eq. (3.59) reduces to

$$\int d1d2\langle V(1,2,3)|\Omega^{(1)}\Omega^{(2)}\Omega^{(3)}|\Phi(1)\rangle|\Phi(2)\rangle = \int d2d1\langle V(1,2,3)||\Phi(2)\rangle|\Phi(1)\rangle \ . \tag{3.64}$$

If we remember the original meaning of the twist operation (2.14), it is clear that the operation of $\Omega^{(1)}\Omega^{(2)}\Omega^{(3)}$ on $|V(1,2,3)\rangle$ simply implies to reverse the cyclic order of strings 1,2,3; i.e., we have

$$\Omega^{(1)}\Omega^{(2)}\Omega^{(3)}|V(1,2,3)\rangle = |V(3,2,1)\rangle \ . \tag{3.65}$$

Hence Eq. (3.64) actually holds by this relation. For a nonorientable string, our $O(g)$ BRS transformation δ_B^1 preserves the constraint (2.18) again owing to Eq. (3.65).

IV. CONSTRUCTION OF NONLINEAR BRS TRANSFORMATION II; 4-STRING VERTEX

A. Form of the 4-string vertex

Construction of $O(g^2)$ BRS transformation δ_B^2 is quite similar to the previous one δ_B^1 although somewhat more laborious. First, we assume the following form for $\delta_B^2\Phi$:

$$\delta_B^2|\Phi(4)\rangle = -\int d1d2d3\langle\Phi(1)|\langle\Phi(2)|\langle\Phi(3)||V^{(4)}(1,2,3,4)\rangle \ . \tag{4.1}$$

Then, we have

$$\{\delta_B^0,\delta_B^2\}|\Phi(4)\rangle = \int d1d2d3\langle\Phi(1)|\langle\Phi(2)|\langle\Phi(3)|\left[\sum_{r=1}^{4}Q_B^{(r)}\right]|V^{(4)}(1,2,3,4)\rangle \ . \tag{4.2}$$

In this section we first try to construct $|V^{(4)}(1,2,3,4)\rangle$ so that (4.2) vanishes, i.e.,

$$\left[\sum_{r=1}^{4}Q_B^{(r)}\right]|V^{(4)}(1,2,3,4)\rangle = 0 \ . \tag{4.3}$$

However, it turns out that the resulting vertex $|V^{(4)}\rangle$ almost satisfies (4.3) but gives a bit of a nonvanishing contribution to (4.2). In the next section we shall calculate $(\delta_B^1)^2$ and show that it just cancels the nonvanishing piece of (4.2) and hence that the nilpotency condition (3.4a) is satisfied.

In analogy with the 4-string vertex in the light-cone gauge string field theory,[21,22] the vertex $|V^{(4)}\rangle$ is expected to satisfy the following connection condition (Fig. 5):

$$\begin{aligned}
&[O^{(4)}(\sigma_4)-O^{(1)}(\sigma_1)]|V^{(4)}\rangle=0, \quad \sigma_4=\frac{\pi|\alpha_4|-\sigma}{|\alpha_4|}, \quad \sigma_1=\frac{\sigma}{\alpha_1} \quad (0\le\sigma\le\sigma_0) \ ,\\
&[O^{(1)}(\sigma_1)-O^{(2)}(\sigma_2)]|V^{(4)}\rangle=0, \quad \sigma_1=\frac{\sigma}{\alpha_1}, \quad \sigma_2=\frac{\pi\alpha_1-\sigma}{|\alpha_2|} \quad (\sigma_0\le\sigma\le\pi\alpha_1) \ ,\\
&[O^{(2)}(\sigma_2)-O^{(3)}(\sigma_3)]|V^{(4)}\rangle=0, \quad \sigma_2=\frac{\pi|\alpha_2|-\eta}{|\alpha_2|}, \quad \sigma_3=\frac{\eta}{\alpha_3} \quad (0\le\eta\le\eta_0) \ ,\\
&[O^{(3)}(\sigma_3)-O^{(4)}(\sigma_4)]|V^{(4)}\rangle=0, \quad \sigma_3=\frac{\eta}{\alpha_3}, \quad \sigma_4=\frac{\pi\alpha_3-\eta}{|\alpha_4|} \quad (\eta_0\le\eta\le\pi\alpha_3) \ ,\\
&\qquad\qquad \eta_0=\sigma_0-\pi(|\alpha_4|-\alpha_3) \ ,\\
&O^{(r)}=X_\mu^{(r)},\alpha_r{}^{-1}A_\pm^{(r)},\alpha_r C_\pm^{(r)},\alpha_r{}^{-2}\bar{C}_\pm^{(r)} \ .
\end{aligned} \tag{4.4}$$

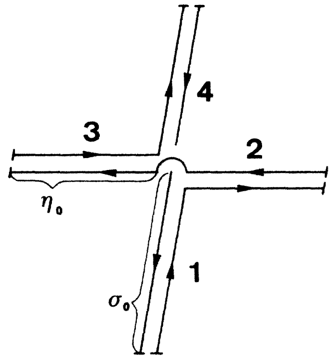


FIG. 5. The structure of overlapping δ functionals in 4-string vertex.

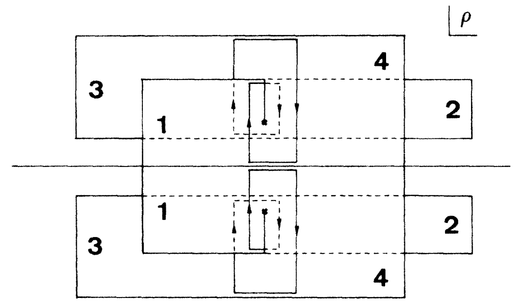


FIG. 6. The light-cone diagram of 4-string vertex and the integration contour C_ρ in (3.26) for the 4-string case.

Here, we are considering the case $\alpha_1,\alpha_3>0$, $\alpha_2,\alpha_4<0$. In the general case, α_r cannot be arbitrary but should satisfy an alternating sign rule,

$$\mathrm{sgn}(\alpha_1,\alpha_2,\alpha_3,\alpha_4)=(+,-,+,-) \quad \text{or}(-,+,-,+)\,, \tag{4.5}$$

in addition to the conservation condition $\sum_r \alpha_r=0$. In (4.4), σ_0 parametrizes the position of the interaction point and varies over

$$\sigma_-\leq\sigma_0\leq\sigma_+\,,$$
$$\sigma_-\equiv\pi\max(0,|\alpha_4|-\alpha_3)\,,$$
$$\sigma_+\equiv\pi\min(\alpha_1,|\alpha_4|)\,.$$

Now, the argument for finding the correct $|V^{(4)}\rangle$ is quite parallel to the one in the previous subsection. First, the simple "δ functional" $|V_0^{(4)}(1,2,3,4)\rangle$ satisfying the connection condition (4.4) is given by

$$|V_0^{(4)}(1,2,3,4;\sigma_0)\rangle=(2\pi)^{d+1}\delta\left[\sum_{r=1}^{4}p_r\right]\delta\left[\sum_{r=1}^{4}\alpha_r\right]\times\delta\left[\sum_{r=1}^{4}\frac{1}{\alpha_r}\bar{c}_0^{(r)}\right]\times\exp(E_X^{(4)}+E_{\mathrm{FP}}^{(4)})\,|0\rangle\rangle\,,$$

$$E_X^{(4)}=\tfrac{1}{2}\sum_{\substack{n,m\geq0\\ r,s}}\bar{N}_{nm}^{(4)rs}\alpha_{-n}^{(r)}\alpha_{-m}^{(s)}\,, \tag{4.6}$$

$$E_{\mathrm{FP}}^{(4)}=i\sum_{\substack{n\geq1\\ m\geq0\\ r,s}}\bar{N}_{nm}^{(4)rs}\gamma_{-n}^{(r)}\bar{\gamma}_{-m}^{(s)}\,,$$

where $\bar{N}_{nm}^{(4)rs}$ is the Neumann function for the 4-string configuration (see Appendix A), and $\gamma_n^{(r)}$ and $\bar{\gamma}_n^{(r)}$ are defined by (3.15). This vertex $|V_0^{(4)}\rangle$ again is not the desired vertex $|V^{(4)}\rangle$ of (4.1) because (i) it lacks the FP ghost number by one, and (ii) it does not satisfy the condition (4.3). The reason why $|V_0^{(4)}\rangle$ in (4.6) fails to satisfy (4.3) is the singularity at the interaction point. Calculations along the lines of the previous section using the expression (3.26) show that $\sum_r Q_B^{(r)}|V_0^{(4)}\rangle$ contains two nonvanishing pieces proportional to $C(\rho_0)$ and $C(\rho_0^*)$, respectively. In the present case the contour C_ρ is given in Fig. 6. Differently from the 3-string case $C(\rho_0)$ and $C(\rho_0^*)$ are not equal in this case because the interaction point is not the end point of some string (except when $\sigma_0=\sigma_-$ or σ_+) and hence the difference $C(\rho_0)-C(\rho_0^*)\propto c(\sigma_{\mathrm{int}})$ does not vanish. Therefore, the above nonvanishing pieces of $\sum_r Q_B^{(r)}|V_0^{(4)}\rangle$ are not totally canceled by multiplying either $C(\rho_0)$ or $C(\rho_0^*)$. However, we show below that the vertex

$$|V^{(4)}(1,2,3,4)\rangle=\frac{\sqrt{\pi}}{2}\int_{\sigma_-}^{\sigma_+}d\sigma_0 f(\sigma_0)[C(\rho_0)+C(\rho_0^*)]\times|V_0^{(4)}(1,2,3,4;\sigma_0)\rangle \tag{4.7}$$

with a suitable measure $f(\sigma_0)$ is annihilated by $\sum_r Q_B^{(r)}$ leaving only nonvanishing contributions from $\sigma_0=\sigma_\pm$. [The factor $\sqrt{\pi}/2$ in (4.7) is for later convenience.]

B. Calculation of $\sum_r Q_B^{(r)}|V^{(4)}\rangle$

The calculation of

$$\left[\sum_{r=1}^{4}Q_B^{(r)}\right][C(\rho_0)+C(\rho_0^*)]\,|V_0^{(4)}(1,2,3,4;\sigma_0)\rangle \tag{4.8}$$

can be done in almost the same manner as for the 3-string vertex in the previous section. Corresponding to (3.30) we have

$$i\frac{4}{\sqrt{\pi}}\sum_{r=1}^{4}Q_B^{(r)}[C(z_0)+C(z_0^*)]\,|V_0^{(4)}\rangle=\left[\oint_{z_0}+\oint_{z_0^*}\right]dz\left[\frac{d\rho(z)}{dz}\right]^{-1}C(z)\left[-A(z)^2+2\frac{dC(z)}{dz}\bar{C}(z)\right][C(z_0)+C(z_0^*)]\,|V_0^{(4)}\rangle\,. \tag{4.9}$$

Note that in the 4-string case the interaction point z_0, which is a solution of (3.32) for

$$\rho(z)=\sum_{r=1}^{4}\alpha_r \ln(z-Z_r) \tag{4.10}$$

is *complex*,[21,22] and the contour of z integration of (4.9) should be performed around z_0 *and* z_0^* (Fig. 7). The pole residues of (4.9) are calculated by taking contraction of all possible pairs of factors in (4.9) by making use of the formulas (3.37). Let us write down the contributions of various contractions to the integral (4.9) around the point z_0. First, the term with no contraction contributes

$$\frac{i\pi}{a}C\left[-A^2+2\frac{dC}{dz}\bar{C}\right](C+C^*)\ , \tag{4.11}$$

where A,C,C^*, etc., stand for $A(z_0)$, $C(z_0)$, and $C(z_0^*)$, respectively. Corresponding to the contractions (3.45a), (3.45b), and (3.44c) we have, in the present case,

$$-i\frac{d-2}{2}\left[\frac{1}{4a}\frac{d^2C}{dz^2}-\frac{b}{4a^2}\frac{dC}{dz}+\left[\frac{3b^2}{4a^3}-\frac{c}{a^2}\right]C\right](C+C^*)\ , \tag{4.12a}$$

$$i\frac{1}{a}\frac{d^2C}{dz^2}(C+C^*)\ , \tag{4.12b}$$

$$i\frac{2}{a}\frac{d^2C}{dz^2}C-i\frac{3b}{a^2}\frac{dC}{dz}C\ , \tag{4.12c}$$

respectively. Besides these three, there is another kind of contraction

$$\oint_{z_0}dz\left[\frac{d\rho(z)}{dz}\right]^{-1}C(z)2\frac{dC}{dz}(z)\overline{\bar{C}(z)C(z_0^*)}\ , \tag{4.13}$$

which gives

$$-\frac{2i}{a}\frac{1}{z_0-z_0^*}C\frac{dC}{dz}\ . \tag{4.14}$$

In the above formulas a, b, and c are defined by (3.34) with r summation from 1 to 4. Integration around z_0^* gives the terms (4.11), (4.12), and (4.14) with z_0 replaced by z_0^* (and hence a, b, and c replaced by their complex conjugates). Summing up all these terms and putting $d=26$ we are left with a nonvanishing result:

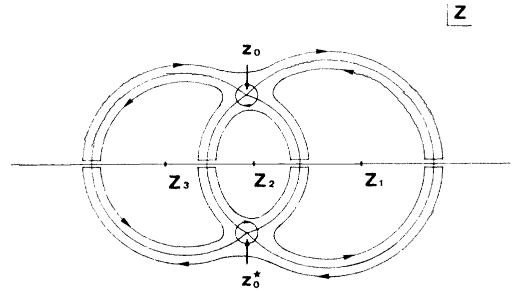


FIG. 7. The contour C_z on the z plane corresponding to C_ρ of Fig. 6. It can be deformed to infinitesimal circles enclosing the interaction points z_0 and z_0^*.

$$\begin{aligned}
&i\frac{4}{\sqrt{\pi}}\sum_{r=1}^{4}Q_B^{(r)}[C(z_0)+C(z_0^*)]\,|V_0^{(4)}\rangle=(\mathrm{I})+(\mathrm{II})+(\mathrm{III})\ ,\\
&(\mathrm{I})=-i\pi CC^*\left[\frac{1}{a}\left[A^2-2\frac{dC}{dz}\bar{C}\right]-\frac{1}{a^*}\left[A^{*2}-2\frac{dC^*}{dz}\bar{C}^*\right]\right]|V_0^{(4)}\rangle\ ,\\
&(\mathrm{II})=\left[-i\left[\frac{2}{a}\frac{d^2C}{dz^2}-\frac{3b}{a^2}\frac{dC}{dz}+\frac{2}{a^*}\frac{1}{z_0-z_0^*}\frac{dC^*}{dz}\right]C^*+iC\left[\frac{2}{a^*}\frac{d^2C^*}{dz^2}-\frac{3b^*}{a^{*2}}\frac{dC^*}{dz}+\frac{2}{a}\frac{1}{z_0^*-z_0}\frac{dC}{dz}\right]\right]|V_0^{(4)}\rangle\ ,\\
&(\mathrm{III})=2\,\mathrm{Im}\left[\frac{9b^2}{a^3}-\frac{12c}{a^2}\right]CC^*\,|V_0^{(4)}\rangle\ .
\end{aligned} \tag{4.15}$$

However, we can prove the following remarkable relations:

$$\frac{1}{a}\left[A^2-2\frac{dC}{dz}\bar{C}\right]-\frac{1}{a^*}\left[A^{*2}-2\frac{dC^*}{dz}\bar{C}^*\right]=\frac{4i}{\pi}\frac{d}{d\sigma_0}(E_X^{(4)}+E_{\mathrm{FP}}^{(4)})\ , \tag{4.16}$$

$$\frac{2}{a}\frac{d^2C}{dz^2}-\frac{3b}{a^2}\frac{dC}{dz}+\frac{2}{a^*}\frac{1}{z_0-z_0^*}\frac{dC^*}{dz}=4i\frac{d}{d\sigma_0}C\ , \tag{4.17}$$

where σ_0 is the position of the interaction point [see (4.4)]. These relations are shown in Appendix F. [The position of the interaction point is determined given a set of values Z_{1-4} of the Mandelstam mapping (4.10). Here, we are taking a special parametrization

$$Z_r=Z_r(\sigma_0) \tag{4.18}$$

by $\sigma_0=\mathrm{Im}\rho(z_0)$ such that the interaction "time" in the ρ plane (Fig. 6), $\tau_0=\mathrm{Re}\rho(z_0)$, remains unchanged when we vary σ_0 in (4.18). The Neumann function $\bar{N}_{nm}^{(4)rs}$, which is determined by (A7), is a function of σ_0 (and α_r).] From Eqs. (4.16) and (4.17), (4.15) is rewritten as

$$i\frac{4}{\sqrt{\pi}}\sum_{r=1}^{4} Q_B^{(r)}[C(z_0)+C(z_0^*)]\,|\,V_0^{(4)}\rangle$$
$$=4\frac{d}{d\sigma_0}[C(z_0)C(z_0^*)\,|\,V_0^{(4)}\rangle]$$
$$+2\,\mathrm{Im}\left[\frac{9b^2}{a^3}-\frac{12c}{a^2}\right]C(z_0)C(z_0^*)\,|\,V_0^{(4)}\rangle\ . \quad (4.19)$$

Therefore, if the σ_0 integration measure $f(\sigma_0)$ in (4.7) satisfies

$$\frac{d}{d\sigma_0}f(\sigma_0)=\mathrm{Im}\left[\frac{9b^2}{2a^3}-\frac{6c}{a^2}\right]f(\sigma_0)\ , \quad (4.20)$$

then $\sum_r Q_B^{(r)}\,|\,V^{(4)}\rangle$ with $|\,V^{(4)}\rangle$ given by (4.7) becomes a "surface integral":

$$\sum_{r=1}^{4} Q_B^{(r)}\,|\,V^{(4)}\rangle=-i\frac{\pi}{2}\int_{\sigma_-}^{\sigma_+}d\sigma_0\frac{d}{d\sigma_0}[f(\sigma_0)C(z_0)C(z_0^*)\,|\,V_0^{(4)}\rangle]=-i\frac{\pi}{2}[f(\sigma_0)C(z_0)C(z_0^*)\,|\,V_0^{(4)}\rangle]_{\sigma_0=\sigma_-}^{\sigma_0=\sigma_+}\ . \quad (4.21)$$

In order to calculate (4.21) we need to know a concrete expression of $f(\sigma_0)$. The measure $f(\sigma_0)$ cannot be freely determined from the condition (4.20) alone. It must be chosen so that the conditions for the nilpotency of the BRS transformation, (3.4), are satisfied. As we shall see in the next section, these conditions are equivalent to the requirement that our theory reproduces the dual amplitudes[37] correctly, and it turns out that we should take the following $f(\sigma_0)$:

$$f(\sigma_0)=J(\sigma_0)\exp\left[-\sum_{r=1}^{4}\overline{N}_{00}^{(4)rr}\right]\ , \quad (4.22)$$

$$J(\sigma_0)=\begin{vmatrix}1 & Z_1 & Z_1{}^2 & Z_1'\\ 1 & Z_2 & Z_2{}^2 & Z_2'\\ 1 & Z_3 & Z_3{}^2 & Z_3'\\ 1 & Z_4 & Z_4{}^2 & Z_4'\end{vmatrix}\quad\left[Z_r'\equiv\frac{d}{d\sigma_0}Z_r(\sigma_0)\right]\ . \quad (4.23)$$

The meaning of the determinant $J(\sigma_0)$ is as follows: Consider the Koba-Nielsen amplitude[42] of the 4-string scattering

$$\int\prod_{r=1}^{4}dZ_rS(Z_r)\ . \quad (4.24)$$

As is well known, we must fix the gauge freedom existing in (4.24) under the projective transformation[37]

$$Z_r\to Z_r^g=\frac{AZ_r+B}{CZ_r+D}\ . \quad (4.25)$$

The infinitesimal form of (4.25) with parameters $\delta\alpha$, $\delta\beta$, and $\delta\gamma$ is given by

$$\delta Z_r=\delta\alpha+\delta\beta Z_r+\delta\gamma Z_r{}^2\ . \quad (4.26)$$

In the region of Z_r which corresponds to Fig. 6, we take some gauge $Z_r=Z_r(\sigma_0)$ parametrized by the position of the interaction point. By following the standard Faddeev-Popov technique[43] of inserting

$$J[Z_r]\int dg\int d\sigma_0\prod_{r=1}^{4}\delta(Z_r^g-Z_r(\sigma_0))=1 \quad (4.27)$$

and factoring out the gauge volume $\int dg$, we are led to the expression of the amplitude

$$\int d\sigma_0 J(\sigma_0)S[Z_r(\sigma_0)]\ , \quad (4.28)$$

with $J(\sigma_0)$ given by (4.23). Hence, $J(\sigma_0)$ is the Faddeev-Popov determinant for the gauge fixing of projective invariance.

The measure $f(\sigma_0)$ in (4.22) is invariant under the projective transformation (4.25) (cf. Sec. 4 of Appendix A). When we take a special gauge which fixes the three of Z_r (Z_a,Z_b,Z_c) to constants, $f(\sigma_0)$ becomes

$$f(\sigma_0)=\left|\frac{\prod_{i=1}^{4}dZ_i}{dV_{abc}d\sigma_0}\right|\exp\left[-\sum_{r=1}^{4}\overline{N}_{00}^{(4)rr}\right]\ , \quad (4.29)$$

$$dV_{abc}=dZ_adZ_bdZ_c/\,|\,Z_a-Z_b\,||Z_b-Z_c||Z_c-Z_a\,|\ . \quad (4.30)$$

We show in Appendix G that $f(\sigma_0)$ given by (4.22) or (4.29) actually satisfies the condition (4.20).

Care must be taken in evaluating (4.21). At first sight one might think that it vanishes because when $\sigma_0=\sigma_\pm$, the interaction point is the end point of some string at which $C(z_0)-C(z_0^*)\propto c(\sigma_{\mathrm{int}})=0$ and hence $C(z_0)C(z_0^*)\,|_{\sigma_0=\sigma_\pm}=0$ (cf. Figs. 12 and 13 in Sec. V). [Actually when $\sigma_0=\sigma_\pm$, we have $z_0=z_0^*$, namely, $d\rho(z)/dz=0$ has a *double* root at $z=z_0$.] However, this argument is incorrect because $f(\sigma_0)$ is in fact *divergent* at $\sigma_0=\sigma_\pm$, and a careful calculation gives a finite nonvanishing result for (4.21). We defer the actual calculation of (4.21) to the next section since we encounter the same situation (i.e., $0\times\infty$) also when we consider $(\delta_B^1)^2$.

It should be commented finally that the consistency of the $O(g^2)$ BRS transformation with the Hermiticity condition, $\delta_B^2\langle\Phi\,|=(\delta_B^2\,|\,\Phi\rangle)^\dagger$, is again guaranteed by the property

$$\Omega^{(1)}\Omega^{(2)}\Omega^{(3)}\Omega^{(4)}\,|\,V^{(4)}(1,2,3,4)\rangle=|\,V^{(4)}(4,3,2,1)\rangle\ , \quad (4.31)$$

which implies that the orientation of string is reversed by the twist. Also important is the cyclic symmetry of $V^{(4)}$:

$$|V^{(4)}(1,2,3,4)\rangle = |V^{(4)}(4,1,2,3)\rangle = |V^{(4)}(3,4,1,2)\rangle = |V^{(4)}(2,3,4,1)\rangle \; . \tag{4.32}$$

[Note that $f(\sigma_0)$ as well as the other parts in $|V^{(4)}\rangle$ in (4.7) has cyclic symmetry as can be seen from (4.29).]

V. COMPLETION OF THE NILPOTENCY PROOF OF BRS TRANSFORMATION

We complete the proof of nilpotency of our BRS transformation in this section. We shall see that the nilpotency is guaranteed in fact by a particular mechanism realizing *duality;* that is, we know from light-cone gauge string field theory that a full dual amplitude is generally realized by a sum of several distinct types of diagrams, each of which contributes to a different part of integration region of Koba-Nielsen variables.[21] This implies, in particular, that the Koba-Nielsen integrand at a boundary of two integration regions is realized commonly by two distinct types of light-cone diagrams in their limits that the interaction "times" of two vertices coincide. It is exactly such diagrams that appear in the BRS transformation taken twice. Therefore cancellations can occur between those pairs of diagrams and the nilpotency of the BRS transformation is satisfied.

A. OSp($d/2$) structure of vertices

Before going to the nilpotency proof, we make here a comment on the OSp($d/2$) symmetry which the exponents of both the 3- and 4-string vertices possess. The 3- and 4-string vertices were given in (3.54) and (4.7) in the previous sections in the forms

$$|V(1,2,3)\rangle = \mu(\alpha_1,\alpha_2,\alpha_3)G(\sigma_I)\,|E(1,2,3)\rangle\bar{\delta}(1,2,3)\;, \tag{5.1a}$$

$$|V^{(4)}(1,2,3,4)\rangle = \int_{\sigma_-}^{\sigma_+} d\sigma_0 f(\sigma_0)G(\sigma_I)\,|E(1,2,3,4)\rangle\bar{\delta}(1,2,3,4)\;, \tag{5.1b}$$

$$\bar{\delta}(1,2,3,(4)) = (2\pi)^{d+1}\delta\left[\sum_r p_r\right]\delta\left[\sum_r \alpha_r\right]\delta\left[\sum_r \frac{1}{\alpha_r}\bar{c}_0^{(r)}\right]\;, \tag{5.2a}$$

$$|E(1,2,3,(4))\rangle = \exp[(E_X+E_{\rm FP})(1,2,3,(4))]\,|0\rangle\;, \tag{5.2b}$$

$$(E_X+E_{\rm FP})(1,2,3,(4)) = \sum_{r,s=1}^{3\text{ or }4}\left[\sum_{n,m=0}^{\infty}\bar{N}^{rs}_{nm}\left(\tfrac{1}{2}\alpha^{(r)}_{-n}\cdot\alpha^{(s)}_{-m} + i\gamma^{(r)}_{-n}\bar{\gamma}^{(s)}_{-m}\right)\right]\;. \tag{5.2c}$$

Here $G(\sigma_I)$ stands for the ghost factor at the interaction point:

$$G(\sigma_I) = i\sqrt{\pi}\alpha_r \pi^{(r)}_{\bar{c}}(\sigma^{(r)}_I) \tag{5.3}$$

with r being any of the strings participating to the vertex.

The exponents (5.2c) actually have OSp($d/2$) symmetry and can be written as

$$E_X+E_{\rm FP} = \sum_{\substack{n,m\ge 0\\ r,s}} \bar{N}^{rs}_{mn}\tfrac{1}{2}a^{M(r)}_{-m}\eta_{MN}a^{N(s)}_{-n} \tag{5.4}$$

in terms of the OSp($d/2$) metric

$$\eta_{MN} = \begin{bmatrix} \eta_{\mu\nu} & & 0 \\ & 0 & i \\ 0 & -i & 0 \end{bmatrix} = \eta^{MN}\;.$$

In this notation the creation and annihilation operators $\alpha^\mu_m, \gamma_m, \bar{\gamma}_m$ are combined into an OSp($d/2$) vector a^M_m

$$a^M_m = (\alpha^\mu_m, \gamma_m, \bar{\gamma}_m)\;,$$

which indeed satisfies OSp($d/2$)-invariant (anti)commutation relations by (2.4) and (3.15):

$$[a^M_m, a^N_n]_{\mp} = m\eta^{MN}\delta_{m+n,0}\;.$$

This OSp($d/2$) symmetry is important below. Since the fermionic degrees of freedom play a role of negative dimensions, as is well known,[44–46] the internal degrees of freedom in this covariant theory effectively reduce to $d-2$ dimensional and, if $d=26$, coincide with the physical dimension 24 in the light-cone gauge string field theory. [It may be necessary to remark that the OSp($d/2$) symmetry for the zero modes is illusory; indeed $a^M_{m=0} = (\alpha^\mu_0 = p^\mu, \gamma_0, \bar{\gamma}_0)$ is not truly a covariant vector since the γ_0 component alone is zero by $\gamma_m = im\alpha c_m$.]

B. $(\delta^1_B)^2$

We now calculate $(\delta^1_B)^2$ and show that it vanishes leaving contributions of particular diagrams which just cancel the nonvanishing surface terms of $\{\delta^0_B, \delta^2_B\}$ in the previous section.

The $O(g)$ BRS transformation δ^1_B take the form, by (3.5) and (5.1a),

$$\delta^1_B\,|\Phi(3)\rangle = \int d1\,d2\langle\Phi(1)|\langle\Phi(2)|G(\sigma_I)|E(1,2,3)\rangle \times(\mu\bar{\delta})(1,2,3)\;, \tag{5.5}$$

with $(\mu\bar{\delta})(1,2,3) \equiv \mu(\alpha_1,\alpha_2,\alpha_3)\bar{\delta}(1,2,3)$. This is equivalently rewritten for the bra state as

$$\delta_B^1\langle\Phi(3)| = \int d2\,d1\langle V(1,2,3)||\Phi(2)\rangle\,|\Phi(1)\rangle$$
$$= \int d2'd1'\langle\Phi(2')|\,\langle\Phi(1')|\int d2\,d1\langle V(2,1,3)|\,\Omega^{(3)}\,|R(1,1')\rangle\,|R(2,2')\rangle$$
$$= \int d1\,d2\langle\Phi(2)|\,\langle\Phi(1)|\,G(\sigma_I)\int d2'd1'\langle E(2',1',3)|\,(\mu\bar{\delta})(2',1',3)\Omega^{(3)}\,|R(1',1)\rangle\,|R(2',2)\rangle\ , \tag{5.6}$$

where use has been made of Eqs. (3.59), (2.15), (3.65), and (5.1a) as well as $G^\dagger=G$ and

$$G(1)\,|R(1,2)\rangle = G(2)\,|R(1,2)\rangle\ . \tag{5.7}$$

The second operator of δ_B^1 on (5.5) yields two terms

$$(\delta_B^1)^2\,|\Phi(4)\rangle = \int d1\,d2[(\delta_B^1\langle\Phi(1)|\,)\langle\Phi(2)| - \langle\Phi(1)|\,(\delta_B^1\langle\Phi(2)|\,)]G(\sigma_I^{124})\,|E(1,2,4)\rangle(\mu\bar{\delta})(1,2,4)\ . \tag{5.8}$$

which have relative minus sign coming from the Grassmann-odd property of $\langle\Phi(1)|$. Let us consider only the first term for a while. It is written by using (5.6) as

$$\int d1\,d2\,d3\langle\Phi(1)|\,\langle\Phi(2)|\,\langle\Phi(3)|\,G(\sigma_I^{125})G(\sigma_I^{534})\,|\Delta(1,2,3,4)\rangle\ , \tag{5.9}$$

with an effective 4-string vertex

$$|\Delta(1,2,3,4)\rangle = \int d2'd1'd5\langle E(1',2',5)|\,(\mu\bar{\delta})(1',2',5)\Omega^{(5)}(\mu\bar{\delta})(5,3,4)\,|E(5,3,4)\rangle\,|R(1',1)\rangle\,|R(2',2)\rangle\ . \tag{5.10}$$

Taking into account the zero-mode dependence in $|R\rangle$'s also, we can easily perform the integration $\int d5$ over zero modes and α of the intermediate string 5, and obtain

$$|\Delta(1,2,3,4)\rangle = \frac{1}{\alpha_5}\bar{\delta}(1,2,3,4)\mu(-\alpha_1,-\alpha_2,\alpha_5)\mu(\alpha_5,\alpha_3,\alpha_4)\int d2'd1'\langle E(1',2',5)|\,\Omega^{(5)}\,|E(5,3,4)\rangle\,|R(1',1)\rangle\,|R(2',2)\rangle\ , \tag{5.11}$$

where the factor $1/\alpha_5$ comes from the $d\bar{c}_0^{(5)}$ integration and p_5, $\bar{c}_0^{(5)}/\alpha_5$, and α_5 are now understood to be p_1+p_2, $\bar{c}_0^{(1)}/\alpha_1+\bar{c}_0^{(2)}/\alpha_2$, and $\alpha_1+\alpha_2$, respectively.

Now we must perform the contraction of oscillator modes of string 5 in $\langle E(1',2',5)|\,\Omega^{(5)}\,|E(5,3,4)\rangle$. This can be done by the help of the following general formula:[21–24]

$$\left\langle 0\left|\exp\left[\tfrac{1}{2}\sum_{m,n\geq 1}\beta_m N_{mn}^{(1)}\beta_n + \sum_{m\geq 1}B_m^{(1)}\beta_m\right]\exp\left[\tfrac{1}{2}\sum_{m,n\geq 1}\beta_{-m}N_{mn}^{(2)}\beta_{-n} + \sum_{m\geq 1}B_m^{(2)}\beta_{-m}\right]\right|0\right\rangle$$
$$= (\det M)^{-1/2}\exp(\tfrac{1}{2}\tilde{B}^{(1)}\tilde{N}^{(2)}M^{-1}\tilde{B}^{(1)} + \tilde{B}^{(2)}M^{-1}\tilde{B}^{(1)} + \tfrac{1}{2}\tilde{B}^{(2)}M^{-1}\tilde{N}^{(1)}\tilde{B}^{(2)})\ , \tag{5.12}$$

$$\tilde{N}_{mn}^{(i)} = \sqrt{m}\,N_{mn}^{(i)}\sqrt{n}\ ,\quad M_{mn} = (1-\tilde{N}^{(1)}\tilde{N}^{(2)})_{mn},\quad \tilde{B}_m^{(i)} = \sqrt{m}\,B_m^{(i)}\ ,$$

which is valid for the bosonic oscillators β_m satisfying

$$[\beta_m,\beta_n] = m\delta_{m+n,0}\ . \tag{5.13}$$

A similar formula holds also for the fermionic oscillators (ghosts in this case) if the determinant factor is replaced by $(\det M)^{+1/2}$. In the present problem, β_m's are oscillators of string 5 and the matrices $N_{mn}^{(i)}$ are the Neumann function $\bar{N}_{nm}^{55}$. $B_m^{(1)}$ and $B_m^{(2)}$ are linear combinations of oscillators and zero modes of strings 1′ and 2′, and 3 and 4, respectively. Thus (5.11) now takes the form

$$|\Delta(1,2,3,4)\rangle = \frac{1}{\alpha_5}[\det(1-\tilde{N}^{66}\tilde{N}^{55})]^{-(d-2)/2}\bar{\delta}(1,2,3,4)\mu(\alpha_1,\alpha_2,-\alpha_5)\mu(\alpha_5,\alpha_3,\alpha_4)$$
$$\times\int d2'd1'\,{}_{1'}\langle 0|\,{}_{2'}\langle 0|\exp[Q(1',2',3,4)]\,|0\rangle_3\,|0\rangle_4\,|R(1',1)\rangle\,|R(2',2)\rangle\ , \tag{5.14}$$

$$Q(1',2',3,4) = \{\text{quadratic form in oscillator modes and zero modes of strings } 1',2',3 \text{ and } 4\}\ ,$$
$$\tilde{N}_{mn}^{55} = \sqrt{m}\,\bar{N}_{mn}^{55}(\alpha_5,\alpha_3,\alpha_4)\sqrt{n}\ ,\quad \tilde{N}_{mn}^{66} = (-)^m\sqrt{m}\,\bar{N}_{mn}^{55}(\alpha_1,\alpha_2,-\alpha_5)\sqrt{n}\,(-)^n\ . \tag{5.15}$$

We must further take the contractions of oscillator modes of strings 1′ and 2′ in the part

$$\int d2'd1'\,{}_{1'}\langle 0|\,{}_{2'}\langle 0|\exp[Q(1',2',3,4)]|0\rangle_3|0\rangle_4 \times |R(1',1)\rangle|R(2',2)\rangle\ . \qquad (5.16)$$

It is, however, not necessary to make a detailed calculation fortunately. Noting the form (2.16) of $|R(1,2)\rangle$, we can again use the formula (5.12) and easily find that (5.16) takes the form

$$\exp[Q'(1,2,3,4)]|0\rangle_1|0\rangle_2|0\rangle_3|0\rangle_4\ , \quad (Q'\text{: another quadratic form}) \qquad (5.17)$$

with coefficient 1. This gives sufficient information since we know that the effective vertex $|\Delta(1,2,3,4)\rangle$ is proportional simply to the 4-string δ functional, as is clear from the δ-functional meaning of the 3-string vertices $\langle V(1',2',5)|$ and $|V(5,3,4)\rangle$.

Since the 3-string vertex $|V(1,2,3)\rangle$ represents three different string configurations according to the relative sign relations of α_{1-3}

$$\text{(A) } |\alpha_3|=|\alpha_1|+|\alpha_2|\,,\quad \text{(B) } |\alpha_1|=|\alpha_2|+|\alpha_3|\,,\quad \text{(C) } |\alpha_2|=|\alpha_3|+|\alpha_1|\ , \qquad (5.18)$$

as shown in Fig. 1, respectively, the structure of a 4-string δ functional implied by $|\Delta(1,2,3,4)\rangle$ is represented by $3\times 3=9$ distinct diagrams as depicted in Fig. 8 according to the signs of α_{1-4}. For instance, the diagram BA-1 in Fig. 8 represents the case in which the 3-string vertices $|V(5,3,4)\rangle$ and $|V(1,2,5)\rangle$ take the A-type and B-type configurations (i.e., $|\alpha_4|=|\alpha_3|+|\alpha_5|$ and $|\alpha_1|=|\alpha_2|+|\alpha_5|$), respectively. We can indeed confirm that $|\Delta(1,2,3,4)\rangle$ satisfies the connection conditions implying the 4-string δ-functional structure by the help of expression (5.10); actually, for the case $\alpha_{1-3}>0$, $\alpha_4<0$, corresponding to the AA-type configuration in diagram A^2-1, for instance, it is easy to show that

$$[\Theta_1 O^{(1)}(\sigma_1)+\Theta_2 O^{(2)}(\sigma_2)+\Theta_3 O^{(3)}(\sigma_3) - O^{(4)}(\sigma_4)]|\Delta(1,2,3,4)\rangle=0\ ,$$

$$O^{(r)}=X^{(r)},\alpha_r{}^{-1}A_{\pm}^{(r)},\alpha_r C_{\pm}^{(r)},\alpha_r{}^{-2}\bar{C}_{\pm}^{(r)}\ ,$$

$$\sigma_1=\frac{\sigma}{\alpha_1},\quad \sigma_2=\frac{\sigma-\pi\alpha_1}{\alpha_2},\quad \sigma_3=\frac{\sigma-\pi(\alpha_1+\alpha_2)}{\alpha_3}\ ,$$

$$\sigma_4=\frac{\pi|\alpha_4|-\sigma}{|\alpha_4|},\quad \Theta_1(\sigma)=\theta(\pi\alpha_1-\sigma)\ , \qquad (5.19)$$

$$\Theta_2(\sigma)=\theta(\sigma-\pi\alpha_1)\theta(\pi(\alpha_1+\alpha_2)-\sigma)\ ,$$

$$\Theta_3(\sigma)=\theta(\sigma-\pi(\alpha_1+\alpha_2))\ ,$$

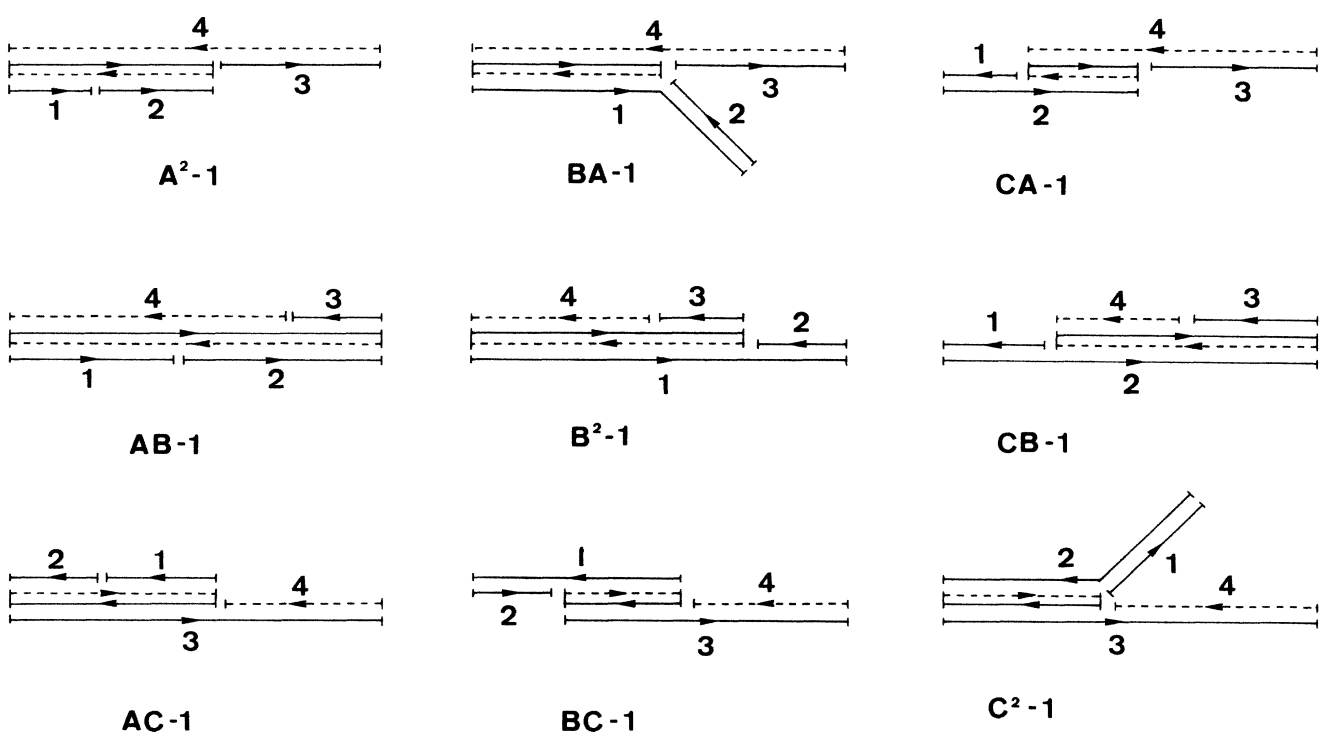


FIG. 8. The 3×3 configurations of 4-string δ functionals appearing in the twice operation of BRS transformation on Φ(4), coming from the first term of (5.8). The solid-dotted double line represents the intermediate string 5 whose coordinates are integrated out.

for expression (5.10) by using the connection conditions (3.12) and (3.16) of a 3-string vertex

$$|V_0(1,2,3)\rangle \propto (\mu\bar{\delta})(1,2,3)\,|E(1,2,3)\rangle$$

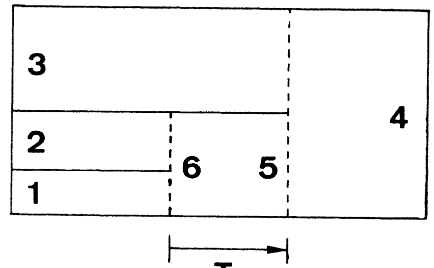
as well as (2.17) of $|R(1,2)\rangle$. Notice in this calculation that the orientation of σ coordinates of strings 1 and 2 was effectively reversed by the presence of twist operator $\Omega^{(5)}$ in (5.10) and the natural answer (5.19) resulted. This situation is already taken into account in Fig. 8 by reversing the arrows of strings.

The 4-string δ functional is uniquely determined by the connection conditions like (5.19) up to a multiplicative constant factor and is generally given in the form

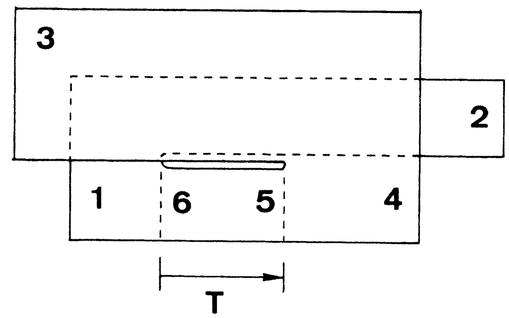
$$|V_0^{(4)}(1,2,3,4)\rangle = e^{E(1,2,3,4)}|0\rangle\bar{\delta}(1,2,3,4)\,, \tag{5.20}$$

$$E(1,2,3,4)=\sum_{r,s=1}^{4}\sum_{n,m\geq 0}^{\infty}\bar{N}_{nm}^{(4)rs}\left(\tfrac{1}{2}\alpha_{-n}^{(r)}\cdot\alpha_{-m}^{(s)}+i\gamma_{-n}^{(r)}\bar{\gamma}_{-m}^{(s)}\right),$$

by using again the 4-string's Neumann function $\bar{N}_{nm}^{(4)rs}$ which is defined generally for N-string diagrams on a ρ plane in Appendix A. Indeed the δ functionals depicted in Fig. 8 correspond one to one to the 4-string diagrams in the light-cone gauge string field theory with the time interval T of two interaction times shrunk to zero. As examples we give such diagrams in Fig. 9 corresponding to diagrams A^2-1 and BA-1 of Fig. 8. So we can obtain $\bar{N}_{nm}^{(4)rs}$ in (5.20) directly from the formula (A7) referring to such diagrams. Now noticing that the factor (5.16) [=(5.17)] contains the vacuum term $|0\rangle=|0\rangle_1|0\rangle_2|0\rangle_3|0\rangle_4$ (i.e., the term independent of the oscillator and zero mode) with weight 1 in coincidence with $e^{E(1,2,3,4)}|0\rangle$ in (5.20), we can determine from (5.14) the proportionality factor of $|\Delta(1,2,3,4)\rangle$ to the δ functional (5.20) and find



A^2-1



BA-1

FIG. 9. The light-cone diagrams which reduce to the 4-string configurations A^2-1 and BA-1 of Fig. 8 in the $T\to 0$ limit.

$$|\Delta(1,2,3,4)\rangle=\frac{1}{\alpha_5}[\det(1-\tilde{N}^{66}\tilde{N}^{55})]^{-(d-2)/2}\times|V_0^{(4)}(1,2,3,4)\rangle\mu(\alpha_1,\alpha_2,-\alpha_5)\times\mu(\alpha_5,\alpha_3,\alpha_4)\,. \tag{5.21}$$

Performing similar calculations also to the second term of (5.8), we finally obtain, from (5.9) and (5.21),

$$(\delta_B^1)^2|\Phi(4)\rangle=\int d1\,d2\,d3\langle\Phi(1)|\langle\Phi(2)|\langle\Phi(3)|\times\Bigg[G(\sigma_I^{125})G(\sigma_I^{534})\frac{1}{\alpha_5}[\det(1-\tilde{N}^{66}\tilde{N}^{55})]^{-(d-2)/2}\mu(\alpha_1,\alpha_2,-\alpha_5)\mu(\alpha_5,\alpha_3,\alpha_4)|V_0^{(4)}(1,2,3,4)\rangle$$
$$+G(\sigma_I^{237})G(\sigma_I^{174})\frac{1}{\alpha_7}[\det(1-\tilde{N}^{88}\tilde{N}^{77})]^{-(d-2)/2}\mu(\alpha_2,\alpha_3,-\alpha_7)\mu(\alpha_1,\alpha_7,\alpha_4)|V_0^{(4)}(1,2,3,4)\rangle\Bigg]\,. \tag{5.22}$$

The effective vertex $|\Delta(1,2,3,4)\rangle$, or the 4-string δ functional $|V_0^{(4)}(1,2,3,4)\rangle$, for the case of second term of (5.8) also has the structure represented by nine distinct diagrams as shown in Fig. 10 similar to Fig. 8 for the first term, and the light-cone diagrams corresponding to diagrams A^2-2 and CA-2 are drawn for illustration in Fig. 11. The number 7 in the second term of (5.22) is the name of the intermediate string with "length" $\alpha_7=\alpha_2+\alpha_3$ and $\tilde{N}^{77}$ and $\tilde{N}^{88}$ are given similarly to (5.15) by

$$\tilde{N}_{mn}^{77}=\sqrt{m}\,\bar{N}_{mn}^{77}(\alpha_1,\alpha_7,\alpha_4)\sqrt{n}\,,\quad \tilde{N}_{mn}^{88}=(-)^m\sqrt{m}\,\bar{N}_{mn}^{77}(\alpha_2,\alpha_3,-\alpha_7)\sqrt{n}\,(-)^n\,. \tag{5.23}$$

Although we have used a common symbol $|V_0^{(4)}(1,2,3,4)\rangle$ both for the first and second terms in (5.22) to denote the

4-string δ functionals, they of course depend on the overlapping structure of strings 1—4 as shown in Figs. 8 and 10. We immediately notice, for instance, the same overlapping structure between diagrams A^2-1 and A^2-2 and hence the corresponding 4-string δ functionals $|V_0^{(4)}(1,2,3,4)\rangle_{A^2\text{-}1}$ and $|V_0^{(4)}(1,2,3,4)\rangle_{A^2\text{-}2}$ must coincide with each other for a common set of values $(\alpha_1,\alpha_2,\alpha_3)$: Comparing the diagrams in Figs. 8 and 10, we thus find

$$\begin{aligned}
&|V_0^{(4)}(1,2,3,4)\rangle_{A^2\text{-}1}=|V_0^{(4)}(1,2,3,4)\rangle_{A^2\text{-}2}\leftrightarrow \alpha_1,\alpha_2,\alpha_3>0\ ,\\
&\left.\begin{array}{l}|V_0^{(4)}(1,2,3,4)\rangle_{\text{CA-1}}\ \text{when}\ |\alpha_1|<|\alpha_2|\\ |V_0^{(4)}(1,2,3,4)\rangle_{\text{BC-1}}\ \text{when}\ |\alpha_1|>|\alpha_2|\end{array}\right\}=|V_0^{(4)}(1,2,3,4)\rangle_{\text{AC-2}}\leftrightarrow \alpha_1<0,\ \alpha_2,\alpha_3>0\ ,\\
&|V_0^{(4)}(1,2,3,4)\rangle_{\text{AB-1}}=\left\{\begin{array}{l}|V_0^{(4)}(1,2,3,4)\rangle_{\text{BA-2}}\ \text{when}\ |\alpha_3|<|\alpha_2|\\ |V_0^{(4)}(1,2,3,4)\rangle_{\text{CB-2}}\ \text{when}\ |\alpha_3|>|\alpha_2|\end{array}\right.\leftrightarrow \alpha_3<0,\ \alpha_1,\alpha_2>0\ ,\\
&|V_0^{(4)}(1,2,3,4)\rangle_{B^2\text{-}1}=|V_0^{(4)}(1,2,3,4)\rangle_{\text{AB-2}}\leftrightarrow \alpha_1>0,\ \ \alpha_2,\alpha_3<0\ ,\\
&|V_0^{(4)}(1,2,3,4)\rangle_{\text{CB-1}}=|V_0^{(4)}(1,2,3,4)\rangle_{\text{BC-2}}\leftrightarrow \alpha_2>0,\ \alpha_1,\alpha_3<0\ ,\\
&|V_0^{(4)}(1,2,3,4)\rangle_{\text{AC-1}}=|V_0^{(4)}(1,2,3,4)\rangle_{C^2\text{-}2}\leftrightarrow \alpha_3>0,\ \alpha_1,\alpha_2<0\ .
\end{aligned}\tag{5.24}$$

Here we have indicated the regions of α values to which these 4-string δ functionals correspond, assuming $\alpha_4<0$. (If $\alpha_4>0$, all the signs of α_{1-3} should be reversed. We assume $\alpha_4<0$ for definiteness, hereafter.) The regions of α_{1-3} in (5.24) exhaust all the possibilities except for the cases of $\alpha_2<0$, $\alpha_1,\alpha_3>0$. These exceptional cases correspond to particular shape of diagrams BA-1, C^2-1, CA-2, B^2-2, in Figs. 8 and 10, each of which possesses a branch. We call such diagrams "horn diagrams." It is exactly these types of configurations that were left nonvanishing as surface terms in the previous calculation of $\sum_r Q_B^{(r)}|V^{(4)}\rangle$ in Sec. IV. We will see below that they actually cancel each other.

Before that, we first show that the other ordinary diagrams all cancel between the first and second terms of (5.22) (Ref. 16). For such ordinary regions of α_{1-3}, the 4-string δ functionals $|V_0^{(4)}(1,2,3,4)\rangle$ of the first and second terms equal each other as noted in (5.24), and further the two ghost factors also coincide up to sign:

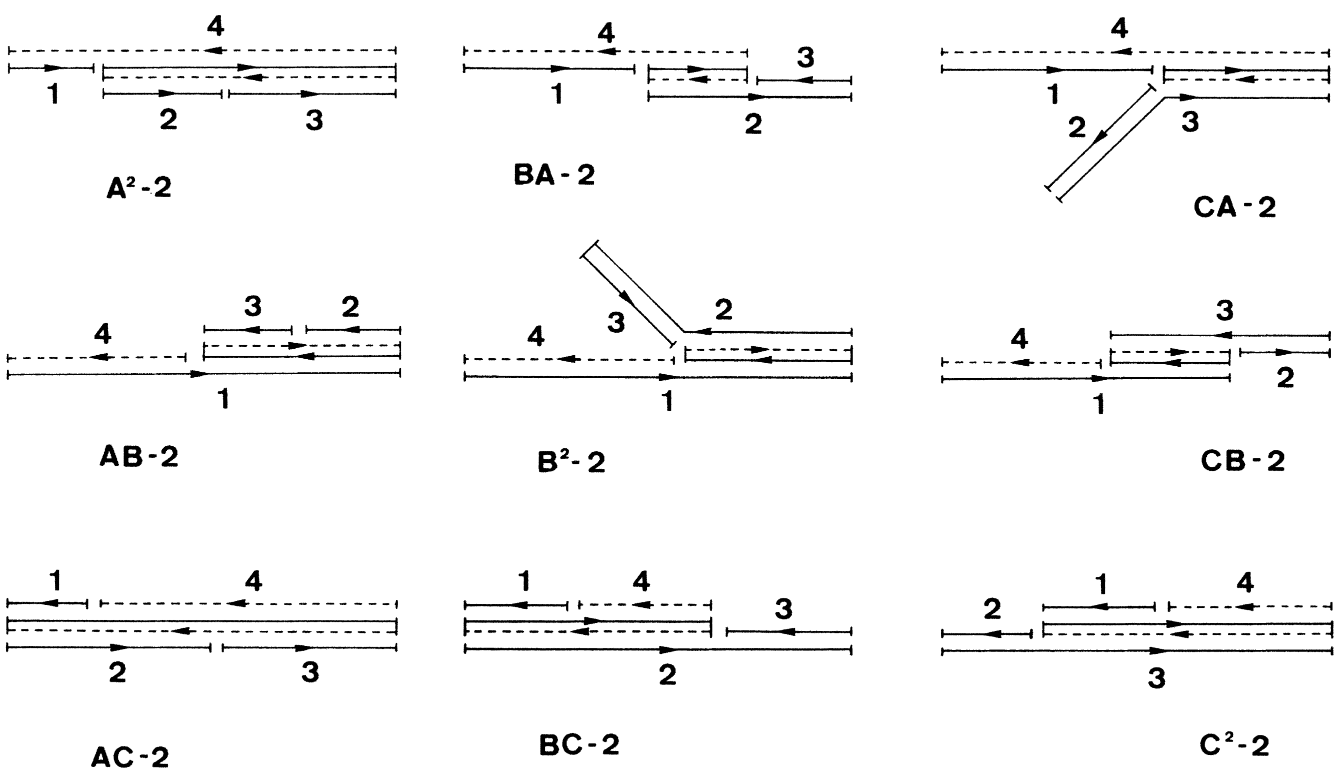


FIG. 10. The other 3×3 configurations of 4-string δ functionals appearing in $(\delta_B^1)^2\Phi(4)$, coming from the second term of (5.8).

$$\mathrm{sgn}(\alpha_5)G(\sigma_I^{125})G(\sigma_I^{534}) = -\mathrm{sgn}(\alpha_7)G(\sigma_I^{237})G(\sigma_I^{174})\ , \tag{5.25}$$

as is easily verified by examining the interaction points of the diagrams in Figs. 8 and 10. Therefore we need to prove the equality

$$\frac{1}{|\alpha_5|}[\det(1-\tilde{N}^{66}\tilde{N}^{55})]^{-(d-2)/2}\mu(\alpha_1,\alpha_2,-\alpha_5)\mu(\alpha_5,\alpha_3,\alpha_4) = \frac{1}{|\alpha_7|}[\det(1-\tilde{N}^{88}\tilde{N}^{77})]^{-(d-2)/2}\mu(\alpha_2,\alpha_3,-\alpha_7)\mu(\alpha_1,\alpha_7,\alpha_4)\ . \tag{5.26}$$

This is indeed a nontrivial equality. Nevertheless it does hold at $d=26$.

Such determinant factors have already appeared in the calculations of 4-string amplitudes in the light-cone gauge string field theory. For instance, consider the decay amplitude of string 4 into strings 1—3 (corresponding to the case $\alpha_1,\alpha_2,\alpha_3>0$), to which just the two light-cone diagrams A^2-1 and A^2-2 drawn in Figs. 9 and 11, respectively, contribute. As was shown by Cremmer and Gervais[22] in detail (and actually almost the same formulas appear also in our covariant framework as will be seen in Sec. VII), the contributions of the diagrams A^2-1 of Fig. 9 and A^2-2 of Fig. 11 to the amplitude are given, respectively, by

$$\mathcal{T}_{\mathrm{A}^2\text{-}1} = \int_0^\infty dT \frac{1}{|\alpha_5|}[\det(1-\tilde{N}^{66}\tilde{N}_T^{55})]^{-(d-2)/2}e^{T/\alpha_5} \times\exp\left[-\sum_{r=1,2,6}\frac{\tau_0(\alpha_1,\alpha_2,\alpha_6)}{\alpha_r}-\sum_{r=5,3,4}\frac{\tau_0(\alpha_5,\alpha_3,\alpha_4)}{\alpha_r}\right]\langle \mathrm{ext}(1-4)\,|\,V_T^{(4)}(1,2,3,4)\rangle\ , \tag{5.27}$$

$$\mathcal{T}_{\mathrm{A}^2\text{-}2} = \int_{-\infty}^0 dT \frac{1}{|\alpha_7|}[\det(1-\tilde{N}_T^{88}\tilde{N}^{77})]^{-(d-2)/2}e^{-T/\alpha_7} \times\exp\left[-\sum_{r=2,3,8}\frac{\tau_0(\alpha_2,\alpha_3,\alpha_8)}{\alpha_r}-\sum_{r=1,7,4}\frac{\tau_0(\alpha_1,\alpha_7,\alpha_4)}{\alpha_r}\right]\langle \mathrm{ext}(1-4)\,|\,V_T^{(4)}(1,2,3,4)\rangle\ , \tag{5.28}$$

where

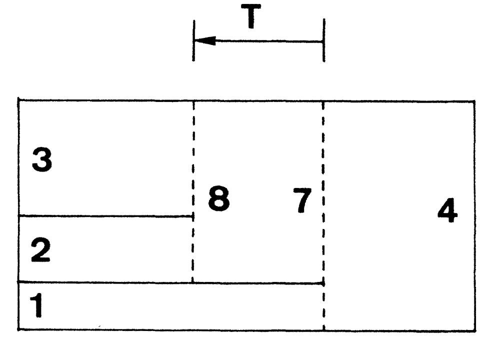
$$(\tilde{N}_T^{rr})_{mn} = \tilde{N}_{mn}^{rr}\exp[-(m+n)T/\alpha_r]\ ,$$
$$\alpha_6=-\alpha_5,\quad \alpha_8=-\alpha_7\ , \tag{5.29}$$
$$\tau_0(\alpha_a,\alpha_b,\alpha_c) = \sum_{r=a,b,c}\alpha_r\ln|\alpha_r|\ .$$

The $\langle \mathrm{ext}(1-4)|$ denotes the external states, and the effective 4-string vertices $|V_T^{(4)}(1,2,3,4)\rangle$ in (5.27) and (5.28) are given by the same equation (5.20) as the previous 4-string δ functional $|V_0^{(4)}(1,2,3,4)\rangle$ if the Neumann functions $\tilde{N}_{nm}^{(4)rs}$ there are replaced by those for the diagrams A^2-1 and A^2-2. [And, of course, the longitudinal and scalar modes $\alpha_m^{(r)\mu=\pm}$ $(m\geq 1)$ as well as the ghosts $\gamma_m^{(r)}$ and $\bar{\gamma}_m^{(r)}$ are set to be zero in the light-cone gauge string field theory. If all the modes are retained, Eqs. (5.27) and (5.28) give the amplitude in our covariant theory (see Sec. VII).]

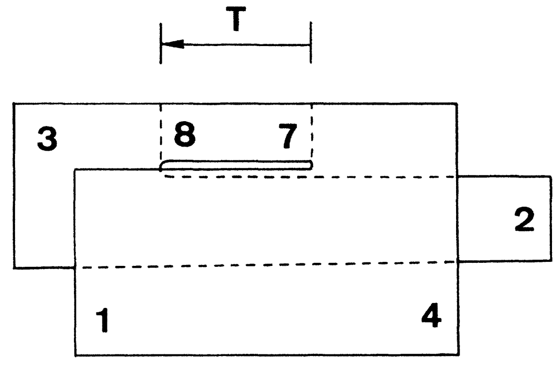
The particular property of the string model is that it reproduces the dual amplitudes.[47,37] Through the Mandelstam mapping

$$\rho(z) = \sum_{i=1}^{N}\alpha_i\ln(z-Z_i)\ , \tag{5.30}$$

each N-string light-cone diagram corresponds to a set of real parameters Z_{1-N} up to the gauge freedom of projective transformations. If we fix this freedom by choosing $Z_2=1$, $Z_3=0$, and $Z_4=\infty$, then the 4-string light-cone diagram is uniquely specified by one parameter $Z_1=x$. By examining the Mandelstam mapping, it is easily seen that the diagrams A^2-1 of Fig. 9 and A^2-2 of Fig. 11, which are also parametrized by T and correspond to the



A²-2



CA-2

FIG. 11. The light-cone diagrams corresponding to the 4-string configurations A^2-2 and CA-2 of Fig. 10.

regions $T>0$ and $T<0$, respectively, correspond to the parameter x in the regions $1\le x\le x_0$ and $x_0\le x<\infty$. Therefore, in order to reproduce the dual amplitude which is given by an integral $\int_1^\infty dx\, f(x)$ of a smooth function $f(x)$ over $1\le x<\infty$, the amplitudes (5.27) and (5.28) should give $\int_1^{x_0} dx\, f(x)$ and $\int_{x_0}^\infty dx\, f(x)$, respectively, with a single function $f(x)$. This is indeed the case at $d=26$. Cremmer and Gervais[22] have actually proven the equalities

$$\left|\frac{\prod_{i=1}^{4} dZ_i}{dV_{abc}dT}\right| \exp\left[-\sum_{j=1}^{4}\bar{N}_{00}^{(4)jj}\right] = \begin{cases} \dfrac{1}{|\alpha_5|}[\det(1-\tilde{N}^{66}\tilde{N}_T^{55})]^{-12}e^{T/\alpha_5}\exp\left[-\sum_{r=1,2,6}\dfrac{\tau_0(\alpha_1,\alpha_2,\alpha_6)}{\alpha_r}-\sum_{r=5,3,4}\dfrac{\tau_0(\alpha_5,\alpha_3,\alpha_4)}{\alpha_r}\right] & (T\ge 0)\,, \\ \dfrac{1}{|\alpha_7|}[\det(1-\tilde{N}_T^{88}\tilde{N}^{77})]^{-12}e^{-T/\alpha_7}\exp\left[-\sum_{r=2,3,8}\dfrac{\tau_0(\alpha_2,\alpha_3,\alpha_8)}{\alpha_r}-\sum_{r=1,7,4}\dfrac{\tau_0(\alpha_1,\alpha_7,\alpha_4)}{\alpha_r}\right] & (T\le 0)\,, \end{cases} \tag{5.31}$$

by a direct calculation. Here again $\bar{N}_{00}^{(4)jj}$ is the Neumann function $\bar{N}_{mn}^{(4)rs}$ (with $r=s=j$, $m=n=0$) for the diagram A^2-1 or A^2-2. This fact in particular implies the equality of the integrands of (5.27) and (5.28) at the boundary $T=0$ ($x=x_0$) at which the diagrams A^2-1 and A^2-2 become the same and the corresponding Neumann functions $\bar{N}_{mn}^{(4)rs}$ coincide. Hence the vertices $|V_T^{(4)}\rangle$ in (5.27) and (5.28) reduce to the same 4-string δ functional $|V_0^{(4)}\rangle$ in (5.20). This equality at $T=0$ just proves the desired Eq. (5.26) in $d=26$ if the α-integration measure is chosen as

$$\mu(\alpha_1,\alpha_2,\alpha_3)=\exp\left[-\sum_{r=1}^{3}\frac{\tau_0(\alpha_1,\alpha_2,\alpha_3)}{\alpha_r}\right] \tag{5.32}$$

as was announced in Sec. III. Although we have discussed explicitly only the case of $\alpha_1,\alpha_2,\alpha_3>0$, clearly Eq. (5.26) is guaranteed by similar equations to (5.31) for any other (nonhorn diagram) cases tabulated in (5.24). Thus we have shown that the duality is the origin of the cancellation of nonhorn diagrams in $(\delta_B^1)^2$. We shall next see that this duality guarantees also the cancellation between the horn-diagram contribution to $(\delta_B^1)^2\Phi$ and the surface term of $\{\delta_B^0,\delta_B^2\}\Phi$.

C. Cancellation between $(\delta_B^1)^2\Phi$ and $\{\delta_B^0,\delta_B^2\}\Phi$

Now let us turn to consider the contributions of horn-diagram configurations to $(\delta_B^1)^2\Phi$ [which correspond to the region $\alpha_2<0$, $\alpha_1,\alpha_3>0$ of integrations $d\alpha_1 d\alpha_2 d\alpha_3$ in (5.22)]. In such a configuration, the positions of two interaction points σ_I^{125} and σ_I^{534} (or, σ_I^{237} and σ_I^{174}) coincide as is clear from the diagrams BA-1 and C^2-1 of Fig. 8 (or CA-2 and B^2-2 of Fig. 10), and hence the ghost factors $G(\sigma_I^{125})G(\sigma_I^{534})$ and $G(\sigma_I^{237})G(\sigma_I^{174})$ vanish. From this fact we claimed in our paper I that the contributions of horn diagrams vanish by themselves and concluded $(\delta_B^1)^2=0$. This is, however, not correct, unfortunately. The fact is much more interesting than was expected. The loophole is that the determinant factors $\det(1-\tilde{N}\tilde{N})$ are in fact divergent for such configurations as will be seen shortly, and (5.22) gives finite result by $0\times\infty$.

In order to obtain a definite answer, we need a regularization. The most natural one is to take

$$G(\sigma_I^{125})G(\sigma_I^{534})\frac{1}{\alpha_5}[\det(1-\tilde{N}^{66}\tilde{N}^{55})]^{-12}|V_0^{(4)}(1,2,3,4)\rangle = \lim_{T\to 0}\pi C(z_0^b)C(z_0^a)\frac{1}{|\alpha_5|}[\det(1-\tilde{N}^{66}\tilde{N}_T^{55})]^{-12}|V_T^{(4)}(1,2,3,4)\rangle\,, \tag{5.33}$$

$$G(\sigma_I^{237})G(\sigma_I^{174})\frac{1}{\alpha_7}[\det(1-\tilde{N}^{88}\tilde{N}^{77})]^{-12}|V_0^{(4)}(1,2,3,4)\rangle = \lim_{T\to 0}\pi C(z_0^b)C(z_0^a)\frac{1}{|\alpha_7|}[\det(1-\tilde{N}_T^{88}\tilde{N}^{77})]^{-12}|V_T^{(4)}(1,2,3,4)\rangle\,, \tag{5.34}$$

by referring to the corresponding light-cone diagrams like diagram BA-1 of Fig. 9 and diagram CA-2 of Fig. 11 with finite time interval T. In (5.33) and (5.34) we have reexpressed the ghost prefactors $G(\sigma_I)$ in terms of $C(z)$ defined by (3.25) as $G(\sigma_I)=\sqrt{\pi}C(z_0)$, and ordered them so that the interaction time of the right factor $C(z_0^a)$ is larger than that of the left factor $C(z_0^b)$, viz.,

$$T_a-T_b=\rho(z_0^a)-\rho(z_0^b)>0\,. \tag{5.35}$$

[We need not take the real part of the RHS since for horn diagrams with finite time interval like diagram BA-1 of Fig. 9 $\rho(z_0^a)$ and $\rho(z_0^b)$ have a common imaginary part.] With this particular ordering of ghost prefactors, we can replace $1/\alpha_5$ and $1/\alpha_7$ on the LHS of (5.33) and (5.34) by their absolute values $1/|\alpha_5|$ and $1/|\alpha_7|$. Note that z_0^a and z_0^b, which are solutions of $d\rho(z)/dz=0$, are *real* and approach a common value as $T\to 0$. [In fact, Eqs. (5.33) and (5.34) hold for ordinary diagrams as well as horn diagrams as is understood by examining Figs. 8 and 10.]

Exactly the same problem of $0\times\infty$ appears in the nonvanishing "surface terms" of $\{\delta_B^0,\delta_B^2\}\,|\Phi(4)\rangle$ in the previous section, which read, by (4.2) and (4.21),

$$\{\delta_B^0,\delta_B^2\}\,|\Phi(4)\rangle=-\int d1\,d2\,d3\langle\Phi(1)|\langle\Phi(2)|\langle\Phi(3)|\left(\frac{i\pi}{2}\right)[f(\sigma_0)C(z_0)C(z_0^*)\,|V_0^{(4)}(\sigma_0)\rangle]_{\sigma_0=\sigma_-}^{\sigma_0=\sigma_+}\,. \tag{5.36}$$

Here, the vertex $|V_0^{(4)}(\sigma_0)\rangle$ is the 4-string δ functional corresponding to the configuration of Fig. 5, and z_0 and z_0^* denote the interaction points at which $\mathrm{Im}\rho(z_0)=\sigma_0$ and $\mathrm{Im}\rho(z_0^*)=-\sigma_0$. At $\sigma_0=\sigma_\pm$, since the two interaction points z_0 and z_0^* coincide, we have $C(z_0)=C(z_0^*)$ and hence the ghost factor $C(z_0)C(z_0^*)$ vanishes. In this case also, however, the measure $f(\sigma_0)$ diverges at $\sigma_0=\sigma_\pm$ (as will be shown shortly). Here again the natural regularization is to use the expression (5.36) itself, i.e., referring to Fig. 6, and to take the limit $\sigma_0\to\sigma_\pm$.

Now we want to show that $(\delta_B^1)^2\,|\Phi(4)\rangle$ of (5.22) (to which now only the horn-diagram configurations contribute) and $\{\delta_B^0,\delta_B^2\}\,|\Phi(4)\rangle$ of (5.36) actually cancel with each other. First we notice that the 4-string vertex $|V_0^{(4)}(\sigma_0)\rangle$ in (5.36) at the points $\sigma_0=\sigma_\pm$ exactly have the same configurations as $|V_0^{(4)}(1,2,3,4)\rangle$ in (5.22). Actually the configuration of $|V_0^{(4)}(\sigma_0)\rangle$ in Fig. 5 reduces to those drawn in Figs. 12 and 13 at $\sigma_0=\sigma_+$ and σ_-, respectively, and we immediately see that the configurations of Figs. 13(a) and 13(b) and Figs. 12(a) and 12(b) just coincide with those of diagrams BA-1 and C^2-1 of Fig. 8 and diagrams CA-2 and B^2-2 of Fig. 10, respectively. Therefore the first term in (5.22) corresponds to the $\sigma_0=\sigma_-$ term of (5.36) and the second term to $\sigma_0=\sigma_+$. Taking account of the regularization (5.33) and (5.34) with (5.35), we now rewrite (5.22) in the form

$$\begin{aligned}(\delta_B^1)^2\,|\Phi(4)\rangle=&\int d1\,d2\,d3\langle\Phi(1)|\langle\Phi(2)|\langle\Phi(3)|\\ &\times\pi\lim_{T\to 0}[g_-(T)C(z_0^b)C(z_0^a)\,|V_0^{(4)}(\sigma_-)\rangle+g_+(T)C(z_0^b)C(z_0^a)\,|V_0^{(4)}(\sigma_+)\rangle]\,,\\ g_-(T)=&\frac{1}{|\alpha_5|}[\det(1-\tilde N^{66}\tilde N^{55}_T)]^{-12}e^{T/\alpha_5}\mu(\alpha_1,\alpha_2,-\alpha_5)\mu(\alpha_5,\alpha_3,\alpha_4)\,,\\ g_+(T)=&\frac{1}{|\alpha_7|}[\det(1-\tilde N^{88}_T\tilde N^{77})]^{-12}e^{-T/\alpha_7}\mu(\alpha_2,\alpha_3,-\alpha_7)\mu(\alpha_1,\alpha_7,\alpha_4)\,,\end{aligned} \tag{5.37}$$

where we have used the fact that we can take the limit $T\to 0$ for the vertex part $|V_T^{(4)}(1,2,3,4)\rangle$ in (5.33) and (5.34) separately. In this form (5.37), the correspondence to (5.36) is clear. Indeed the functions $g_\pm(T)$ in (5.37) are written as

$$g_\pm(T)=\left|\frac{\prod_{i=1}^{4}dZ_i}{dV_{abc}\,dT}\right|\exp\left[-\sum_{j=1}^{4}\bar N_{00}^{(4)jj}\right] \tag{5.38}$$

by the Cremmer-Gervais equality (5.31), while the measure $f(\sigma_0)$ in (5.36) is given by (4.29):

$$f(\sigma_0)=\left|\frac{\prod_{i=1}^{4}dZ_i}{dV_{abc}\,d\sigma_0}\right|\exp\left[-\sum_{j=1}^{4}\bar N_{00}^{(4)jj}\right]. \tag{5.39}$$

This parallelism is by no means accidental. Here again the duality plays a key role. Consider the three diagrams

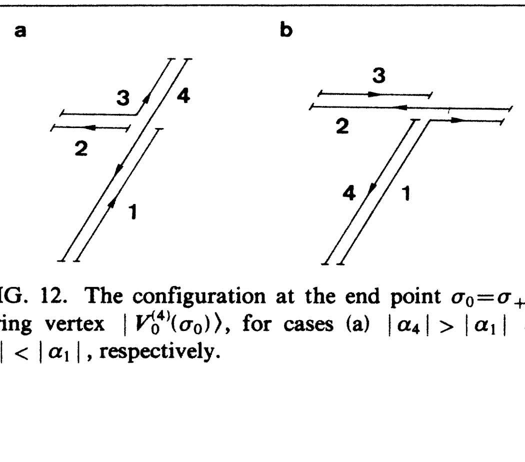


FIG. 12. The configuration at the end point $\sigma_0=\sigma_+$ of the 4-string vertex $|V_0^{(4)}(\sigma_0)\rangle$, for cases (a) $|\alpha_4|>|\alpha_1|$ and (b) $|\alpha_4|<|\alpha_1|$, respectively.

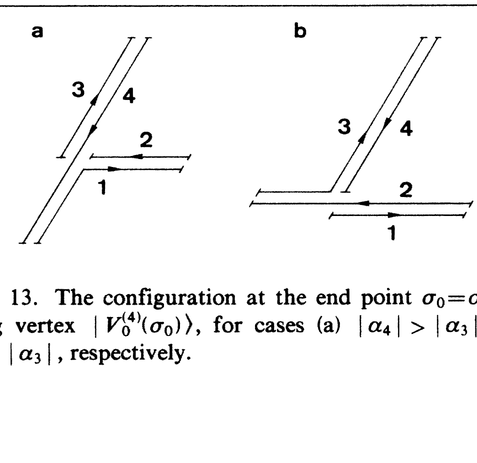


FIG. 13. The configuration at the end point $\sigma_0=\sigma_-$ of the 4-string vertex $|V_0^{(4)}(\sigma_0)\rangle$, for cases (a) $|\alpha_4|>|\alpha_3|$ and (b) $|\alpha_4|<|\alpha_3|$, respectively.

$$\begin{aligned}&\text{(i) BA-1 of Fig. 9 } (0\le T<\infty)\,,\\&\text{(ii) Fig.6 } [\sigma_-=\pi(\alpha_1-|\alpha_2|)\le\sigma\le\sigma_+=\pi\alpha_1]\,,\\&\text{(iii) CA-2 of Fig. 11 } (-\infty<T\le 0)\,.\end{aligned}\tag{5.40}$$

As is well known in the light-cone gauge string field theory,[21,22] these three make up a full dual amplitude for the scattering $1+3\to 2+4$. Indeed if we fix the projective invariance by

$$Z_2=1,\ Z_3=0,\ Z_4=\infty\,,\tag{5.41}$$

the diagrams in (5.40) correspond to the regions of $Z_1=x$, (i) $1\le x\le x_-$, (ii) $x_-\le x\le x_+$, and (iii) $x_+\le x$, respectively. As before, the two apparently different diagrams on both sides of the boundary $x=x_+$ or x_- must give the same integrand on the boundary. It is this duality realizing mechanism again that guarantees the cancellation between (5.36) and (5.37).

Let us see the cancellations between the diagrams at $x=x_+$ and at $x=x_-$:

$$\begin{aligned}&x=x_-:\ \text{BA-1 of Fig. 9 at } T=0 \text{ [1st term of (5.37)]}\leftrightarrow\text{Fig. 6 at } \sigma_0=\sigma_-\,,\\&x=x_+:\ \text{CA-2 of Fig. 11 at } T=0 \text{ [2nd term of (5.37)]}\leftrightarrow\text{Fig. 6 at } \sigma_0=\sigma_+\,.\end{aligned}\tag{5.42}$$

The RHS of Eqs. (5.37) and (5.36) are evaluated as

$$\lim_{T\to 0}\pi g_\pm(T)C(z_0^b)C(z_0^a)\,|\,V_0^{(4)}(\sigma_\pm)\rangle=\lim_{T\to 0}[(z_0^b-z_0^a)g_\pm(T)]\pi C'(z_0^\pm)C(\bar z_0^\pm)\,|\,V_0^{(4)}(\sigma_\pm)\rangle\tag{5.43}$$

and

$$-\lim_{\sigma_0\to\sigma_\pm}\left[\frac{i\pi}{2}\right]f(\sigma_0)C(z_0)C(z_0^*)\,|\,V_0^{(4)}(\sigma_0)\rangle=-\lim_{\sigma_0\to\sigma_\pm}\left[\frac{i}{2}(z_0-z_0^*)f(\sigma_0)\right]\pi C'(z_0^\pm)C(\bar z_0^\pm)\,|\,V_0^{(4)}(\sigma_\pm)\rangle\,,\tag{5.44}$$

respectively, where $C'(z)=dC/dz$ and $\bar z_0^\pm=(z_0^\pm)^*$ are the common interaction points at $x=x_\pm$; $\lim_{T\to 0}z_0^a=\lim_{T\to 0}z_0^b=\lim_{\sigma\to\sigma_\pm}z_0=\lim_{\sigma\to\sigma_\pm}z_0^*=z_0^\pm$. From (5.43), (5.44), and the fact that $g_\pm(T)$ of (5.38) and $f(\sigma_0)$ of (5.39) have a (projective invariant) common factor which takes the form in the "gauge" (5.41)

$$\frac{\prod_{i=1}^4 dZ_i}{dV_{234}}\exp\left[-\sum_{j=1}^4\bar N_{00}^{(4)jj}\right]=dx\,h(x)$$

$$[h(x) \text{ is regular at } x=x_\pm]\tag{5.45}$$

[see (A12) for the expression $\bar N_{00}^{(4)jj}$], we need only to show the equality

$$\lim_{T\to 0}(z_0^b-z_0^a)\left|\frac{dx}{dT}\right|=\pm\frac{i}{2}\lim_{\sigma_0\to\sigma_\pm}(z_-z_0^*)\left|\frac{dx}{d\sigma_0}\right|\,.\tag{5.46}$$

The equality itself can easily be inferred if we recall the relations

$$\begin{aligned}&\rho(z_0^a)-\rho(z_0^b)=T_a-T_b\quad(x\ge x_+,x\le x_-)\,,\\&\rho(z_0)-\rho(z_0^*)=2i\sigma_0\quad(x_-\le x\le x_+)\,,\end{aligned}\tag{5.47}$$

but, in order to show that the limits are actually *finite* and *equal,* we now perform a little calculation.

The interaction points z_0^a and z_0^b or z_0 and z_0^* are the solutions of $d\rho(z)/dz=0$ with

$$\rho(z)=\alpha_1\ln(z-x)+\alpha_2\ln(z-1)+\alpha_3\ln z\,.$$

By differentiating (5.47) with respect to x and using the formula

$$\begin{aligned}d\rho(w)/dx&=[(\partial\rho/\partial z)(dw/dx)+\partial\rho/\partial x]_{z=w}\\&=(\partial\rho/\partial x)_{z=w}\end{aligned}$$

for $w=z_0^{a,b}$ or $z_0^{(*)}$, we find

$$\begin{aligned}&\frac{\alpha_1(z_0^a-z_0^b)}{(z_0^a-x)(z_0^b-x)}=\frac{d}{dx}(T_a-T_b)\quad(x\ge x_+,x\le x_-)\,,\\&\frac{\alpha_1(z_0-z_0^*)}{(z_0-x)(z_0^*-x)}=2i\frac{d\sigma_0}{dx}\quad(x_-\le x\le x_+)\,.\end{aligned}\tag{5.48}$$

From (5.48) and $z_0-z_0^*=i\,|z_0-z_0^*|$, Eq. (5.46) reduces to the following

$$\lim_{T\to 0}\operatorname{sgn}(z_0^b-z_0^a)=\begin{cases}-\ (\sigma_0=\sigma_+)\,,\\+\ (\sigma_0=\sigma_-)\,,\end{cases}$$

$$[T_a-T_b=\rho(z_0^a)-\rho(z_0^b)>0]\tag{5.49}$$

which is indeed valid as can be seen from diagram BA-1 of Fig. 9 and diagram CA-2 of Fig. 11. In addition, Eq. (5.48) says that dT/dx and $d\sigma_0/dx$ are zero at $x=x_\pm$ and hence that the functions $g_\pm(T)\propto|dx/dT|$ and $f(\sigma_0)\propto|dx/d\sigma_0|$ go to infinity at $T=0$ and $\sigma_0=\sigma_\pm$, respectively, as was announced above. The limiting value of (5.46) itself is $\mp|z_0^\pm-x_\pm|^2/|\alpha_1|$ and is indeed finite.

The cancellations for the other pairs of corresponding diagrams are also seen quite similarly. In any case, the cancellation condition reduces to (5.49), which in fact holds for every horn-diagram configuration. Thus we have completed the proof of $O(g^2)$ nilpotency

$$(\delta_B^1)^2+\{\delta_B^0,\delta_B^2\}=0\,,\tag{5.50}$$

under the condition that the measure $f(\sigma_0)$ of a 4-string vertex (4.7) is given by (4.29) [or (4.22), equivalently].

D. A comment on the necessity of string-length parameter

It would be appropriate to add a comment here why such an unphysical parameter α need to be included in the arguments of our string field.[16] The use of the 3-string vertex of the δ-functional form of Fig. 1 does not directly imply the necessity of α. Even without α in the string fields, one can construct the same overlapping δ functional by setting one of α_r's, say α_3, equal to -1 and including an interaction point parameter σ_0 and its measure $\bar{\mu}(\sigma_0)$ *only* in the definition of the vertex

$$|V^{(3)}\rangle=\int d\sigma_0\bar{\mu}(\sigma_0)\,|V^{(3)}(\sigma_0)\rangle$$

(but *not* in the string field). Since our δ functional only depends on the ratios α_1/α_3 and α_2/α_3, one can use the same δ functional by identifying σ_0 with $-\alpha_1/\alpha_3$ (and thus $-\alpha_2/\alpha_3=1-\sigma_0$).

In such a case, however, it becomes impossible even for the non-horn-diagram contributions to $(\delta_B^1)^2\Phi$ to cancel. For instance, diagram A^2-1 of Fig. 8 and diagram A^2-2 of Fig. 10 have the same δ-functional structure and therefore must cancel with each other also in this case. In order to compare these two diagrams with common ratios $\alpha_1{:}\alpha_2{:}\,|\alpha_3|$, we need to make different changes of variables σ_0 and σ_0' of two 3-vertices (see Fig. 14) for the two diagrams A^2-1 and A^2-2. To obtain $\alpha_1{:}\alpha_2{:}\,|\alpha_3|=y{:}(x-y){:}(1-x)$, they are $\sigma_0=y/x$, $\sigma_0'=x$ for the former and $\sigma_0=(x-y)/(1-y)$, $\sigma_0'=y$ for the latter. Taking account of the Jacobian factor for the change of variables $d\sigma_0 d\sigma_0'\to dx\,dy$, we find that the cancellation requires the following equality for the measure $\bar{\mu}(\sigma_0)$:

$$\bar{\mu}\left[\frac{y}{x}\right]\bar{\mu}(x)\frac{1}{x}=\bar{\mu}\left[\frac{x-y}{1-y}\right]\bar{\mu}(y)\frac{1}{1-y}\quad(x\geq y)\,. \tag{5.51}$$

[We here understand that the full measure is this $\bar{\mu}(\sigma_0)$ *times* the conventional measure (5.32).] On the other hand, also in this case, we must include all the three types of δ-functional configurations A, B, and C in Fig. 1, since the cyclic symmetry of the vertex is absolute necessary for constructing gauge-invariant or BRS-invariant (gauge-fixed) actions as will be seen in Sec. VI. So we need also the cancellation between diagram AB-1 of Fig. 8 and diagram BA-2 of Fig. 10, for instance, which are redrawn in Fig. 14. By changing the variables as $\sigma_0=1-x$, $\sigma_0'=y$ for the former and $\sigma_0=y/x$, $\sigma_0'=(1-x)/(1-y)$ for the latter to get common ratios $\alpha_1{:}\alpha_2{:}\,|\alpha_3|=(1-x){:}x{:}y$, we find a requirement

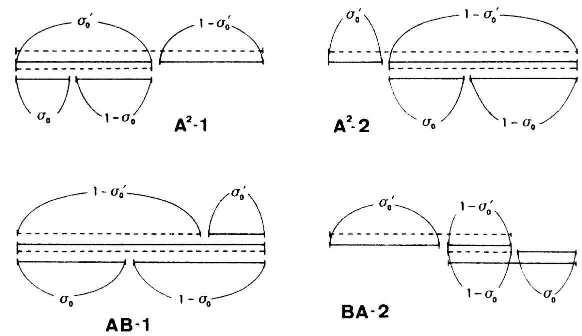


FIG. 14. The diagrams A^2-1 and A^2-2, and AB-1 and BA-2, in $(\delta_B^1)^2\Phi$ for the case when the string field contains no α parameter.

$$\bar{\mu}(1-x)\bar{\mu}(y)=\bar{\mu}\left[\frac{y}{x}\right]\bar{\mu}\left[\frac{1-x}{1-y}\right]\frac{x-y}{x^2(1-y)^2}\quad(x\geq y)\,. \tag{5.52}$$

These two requirements (5.51) and (5.52) already contradict each other except for the trivial solution $\bar{\mu}(\sigma_0)=0$. Indeed multiplying both sides of (5.51) by those of (5.52), we get

$$\bar{\mu}(x)\bar{\mu}(1-x)=\frac{x-y}{x(1-y)^3}\bar{\mu}\left[\frac{x-y}{1-y}\right]\bar{\mu}\left[\frac{1-x}{1-y}\right]$$

as far as $\bar{\mu}\not\equiv 0$. Differentiating this with respect to y and taking the limit $y\to 0$, we obtain an equation for the function $g(x)\equiv\bar{\mu}(x)\bar{\mu}(1-x)$,

$$\left[-\frac{1}{x}+3\right]g(x)+(x-1)g'(x)=0\,,$$

which is solved to give

$$g(x)=\frac{c}{x(1-x)^2}$$

(c is the integration constant). This is, however, asymmetric under $x\leftrightarrow 1-x$ and contradicts the definition of $g(x)$.

E. $O(g^3)$ nilpotency $\{\delta_B^1,\delta_B^2\}=0$

Let us next prove the $O(g^3)$ nilpotency $\{\delta_B^1,\delta_B^2\}=0$. The $O(g^2)$ BRS transformation δ_B^2 is given by (4.1) and (5.1b) as

$$\delta_B^2\,|\Phi(4)\rangle=-\int d1\,d2\,d3\int_{\sigma_-}^{\sigma_+}d\sigma_0 f(\sigma_0)\langle\Phi(1)|\,\langle\Phi(2)|\,\langle\Phi(3)|\,G(\sigma_I)\bar{\delta}(1,2,3,4)\,|E(1,2,3,4)\rangle\,, \tag{5.53}$$

or equivalently for the bra state [by using $\delta_B^2\langle\Phi|=(\delta_B^2|\Phi\rangle)^\dagger$]

$$\begin{aligned}\delta_B^2\langle\Phi(4)|&=\int d1\,d2\,d3\int_{\sigma_-}^{\sigma_+}d\sigma_0 f(\sigma_0)\langle E(1,2,3,4)|\,\bar{\delta}(1,2,3,4)G(\sigma_I)\,|\Phi(3)\rangle\,|\Phi(2)\rangle\,|\Phi(1)\rangle\\&=-\int d1\,d2\,d3\int_{\sigma_-}^{\sigma_+}d\sigma_0 f(\sigma_0)\langle\Phi(1)|\,\langle\Phi(2)|\,\langle\Phi(3)|\,G(\sigma_I)\\&\qquad\times\int d1'd2'd3'\langle E(1',2',3',4)|\,\bar{\delta}(1',2',3',4)\Omega^{(4)}\,|R(3',3)\rangle\,|R(2',2)\rangle\,|R(1'1)\rangle\,,\end{aligned} \tag{5.54}$$

where use has been made of Eqs. (2.15), (4.31), (4.32), and (5.7), and the relations $(d\,1)^\dagger = -d\,1$, $G^\dagger = G$, $\bar{\delta}^\dagger = \bar{\delta}$, $|R(1,2)\rangle = -|R(2,1)\rangle$.

In quite the same way as we obtained (5.22) in Sec. V B we easily reach the following expression by using (5.53) and (5.6):

$$\begin{aligned}
\delta_B^1\delta_B^2|\Phi(5)\rangle = -\int d1\,d2\,d3\,d4\langle\Phi(1)|\langle\Phi(2)|\langle\Phi(3)|\langle\Phi(4)| \\
\times\Bigg[\int d\sigma_0^X f(\sigma_0^X)G(\sigma_I^{X345})G(\sigma_I^{12X})\frac{1}{\alpha_X}[\det(X)]^{-(d-2)/2}\mu(\alpha_1,\alpha_2,-\alpha_X)\,|V_0^{(5)}(1-5)\rangle_{\text{Fig. 15}} \\
+\int d\sigma_0^Y f(\sigma_0^Y)G(\sigma_I^{1Y45})G(\sigma_I^{23Y})\frac{1}{\alpha_Y}[\det(Y)]^{-(d-2)/2}\mu(\alpha_2,\alpha_3,-\alpha_Y)\,|V_0^{(5)}(1-5)\rangle_{\text{Fig. 16}} \\
+\int d\sigma_0^Z f(\sigma_0^Z)G(\sigma_I^{12Z5})G(\sigma_I^{34Z})\frac{1}{\alpha_Z}[\det(Z)]^{-(d-2)/2}\mu(\alpha_3,\alpha_4,-\alpha_Z)\,|V_0^{(5)}(1-5)\rangle_{\text{Fig. 17}}\Bigg], \\
\alpha_X=\alpha_1+\alpha_2,\quad \alpha_Y=\alpha_2+\alpha_3,\quad \alpha_Z=\alpha_3+\alpha_4, \qquad (5.55) \\
\det(x)\equiv\det(1-\tilde{N}^{(3)\bar{x}\bar{x}}\tilde{N}^{(4)xx})\quad \text{for } x=X,Y,Z,\ldots\,.
\end{aligned}$$

Similarly we get by using (5.54) and (5.5)

$$\begin{aligned}
\delta_B^2\delta_B^1|\Phi(5)\rangle = \int d1\,d2\,d3\,d4\langle\Phi(1)|\langle\Phi(2)|\langle\Phi(3)|\langle\Phi(4)| \\
\times\Bigg[\int d\sigma_0^A f(\sigma_0^A)G(\sigma_I^{123A})G(\sigma_I^{A45})\frac{1}{\alpha_A}[\det(A)]^{-(d-2)/2}\mu(\alpha_A,\alpha_4,\alpha_5)\,|V_0^{(5)}(1-5)\rangle_{\text{Fig. 18}} \\
+\int d\sigma_0^Y f(\sigma_0^Y)G(\sigma_I^{1Y45})G(\sigma_I^{23Y})\frac{1}{\alpha_Y}[\det(Y)]^{-(d-2)/2}\mu(\alpha_2,\alpha_3,-\alpha_Y)\,|V_0^{(5)}(1-5)\rangle_{\text{Fig. 16}} \\
\times|V_0^{(5)}(1-5)\rangle_{\text{Fig. 19}}\Bigg],\quad \alpha_A=-(\alpha_4+\alpha_5),\quad \alpha_B=-(\alpha_1+\alpha_5)\,. \qquad (5.56)
\end{aligned}$$

Here in (5.55) and (5.56), $|V_0^{(5)}(1-5)\rangle$ denotes the 5-string δ functional defined by the same form equation as (5.20) with the Neumann function $\bar{N}_{mn}^{(5)rs}$ substituted corresponding to 5-string configurations depicted in Figs. 15–19. The functions $\tilde{N}_{mn}^{(N)\bar{x}\bar{x}}$ and $\tilde{N}_{mn}^{(N)xx}$ (with $N=3,4$) in the determinant factors are given, also similar to the previous one in (5.15), by

$$\begin{aligned}
\tilde{N}_{mn}^{(N)\bar{x}\bar{x}} &= (-)^m\sqrt{m}\,\bar{N}_{mn}^{(N)xx}\sqrt{n}\,(-)^n\,, \\
\tilde{N}_{mn}^{(N)xx} &= \sqrt{m}\,\bar{N}_{mn}^{(N)xx}\sqrt{n}\,,
\end{aligned} \qquad (5.57)$$

in terms of the Neumann functions $\bar{N}_{mn}^{rs}$ of relevant 3- and 4-string vertices.

As before we find the following equations for all the 5-string δ functionals $|V_0^{(5)}\rangle$ corresponding to the configurations shown in Figs. 15–19 by comparing the diagrams:

$$\begin{aligned}
|V_0^{(5)}(\text{A-1})\rangle &= \begin{cases}|V_0^{(5)}(\text{A-5})\rangle\,,\\ |V_0^{(5)}(\text{C-2})\rangle\,,\end{cases} \\
|V_0^{(5)}(\text{A-2})\rangle &= \begin{cases}|V_0^{(5)}(\text{B-1})\rangle\,,\\ |V_0^{(5)}(\text{C-3})\rangle\,,\end{cases} \\
|V_0^{(5)}(\text{A-3})\rangle &= \begin{cases}|V_0^{(5)}(\text{B-2})\rangle\,,\\ |V_0^{(5)}(\text{A-4})\rangle\,,\end{cases} \qquad (5.58) \\
|V_0^{(5)}(\text{B-4})\rangle &= \begin{cases}|V_0^{(5)}(\text{B-3})\rangle\,,\\ |V_0^{(5)}(\text{B-5})\rangle\,,\end{cases} \\
|V_0^{(5)}(\text{C-5})\rangle &= \begin{cases}|V_0^{(5)}(\text{C-4})\rangle\,,\\ |V_0^{(5)}(\text{C-1})\rangle\,,\end{cases}
\end{aligned}$$

where the arguments of $V_0^{(5)}$ indicate the diagrams in

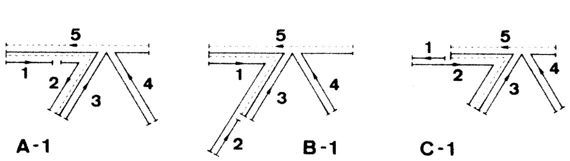


FIG. 15. The 5-string configurations appearing in $\delta_B^1\delta_B^2\Phi(5)$; the first term in (5.55).

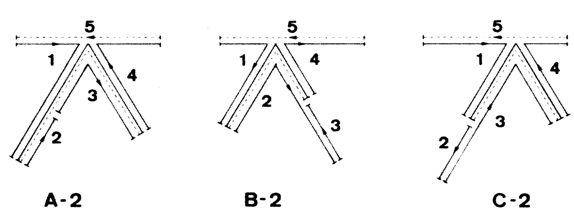


FIG. 16. The 5-string configurations appearing in $\delta_B^1\delta_B^2\Phi(5)$; the second term in (5.55).

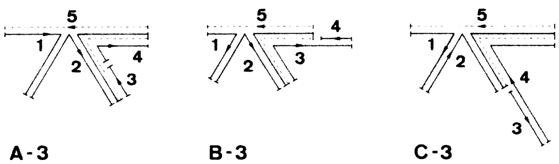


FIG. 17. The 5-string configurations appearing in $\delta_B^1\delta_B^2\Phi(5)$; the third term in (5.55).

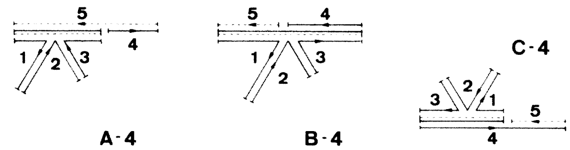


FIG. 18. The 5-string configurations appearing in $\delta_B^2\delta_B^1\Phi(5)$; the first term in (5.56).

Figs. 15—19. Each of these equalities means that the LHS for one region of parameters σ_0 and α's coincide with the upper one of the RHS and the LHS for the other region with the lower one of the RHS. So the cancellations occur between these 2×5 pairs of configurations. As before, by the reason that the calculation of BRS transformations taken twice is almost identical with that of scattering amplitude for particular light-cone diagrams in which the time interval of two interaction points is zero, the factors of determinant and measures in (5.55) and (5.56) are again exactly identical with those in the amplitudes in the light-cone gauge string field theory. More precisely, the following products of factors

$$f(\sigma_0)\frac{1}{|\alpha|}[\det(1-\tilde{N}^{(3)}\tilde{N}^{(4)})]^{-(d-2)/2}\mu(\alpha_i,\alpha_j,\alpha_k) \tag{5.59}$$

for each pair of corresponding configurations are guaranteed to coincide with each other by the duality if $d=26$ and the measures $\mu(\alpha_1,\alpha_2,\alpha_3)$ and $f(\sigma_0)$ are chosen as determined before, since each pair of diagrams correspond to a common boundary point of the Koba-Nielsen variables. This equality of the quantity (5.59) corresponds to Eq. (5.26) of the 4-string case. [As a matter of fact, no literature has appeared which gives a direct estimation of determinant factors appearing in 5- or more string scattering amplitudes, and no direct proof exists for the equalities of the factors (5.59) or similar ones in higher N-string amplitudes. However, Mandelstam[20,48] has proved in another way that the light-cone gauge string field theory actually reproduces the Koba-Nielsen amplitudes for the general N-string case. His proof, therefore, turns out to give an indirect proof of such equalities for the general N-string case. (See Sec. VII.)]

Therefore we need to take care of only the sign of $1/\alpha_r$, i.e., $\mathrm{sgn}(\alpha_r)$ $(r=X,Y,Z,A,B)$ and the order of two ghost factors in (5.55) and (5.56). Since the ghost factors are ordered as $G(\sigma_I^{\text{4-vertex}})G(\sigma_I^{\text{3-vertex}})$ commonly for all the terms in (5.55) and (5.56), we have only to examine $\mathrm{sgn}(\alpha_r)$. As an example, consider the first pair of diagrams of (5.58), A-1 of Fig. 15 and A-5 of Fig. 19. From (5.55) and (5.56), the former is proportional to $\mathrm{sgn}(\alpha_X)=\mathrm{sgn}(\alpha_1+\alpha_2)$ and the latter to $-\mathrm{sgn}(\alpha_B)=\mathrm{sgn}(\alpha_1+\alpha_5)$. We recall that our 4-string vertex $|V^{(4)}(1,2,3,4)\rangle$ constructed in Sec. IV is nonvanishing only for the configurations with alternating signs; i.e., $\mathrm{sgn}(\alpha_1,\alpha_2,\alpha_3,\alpha_4)=(+,-,+,-)$ or $(-,+,-,+)$ [Eq. (4.5)]. So, from diagram A-1 of Fig. 15, we see that $\mathrm{sgn}(\alpha_1+\alpha_2)=-\mathrm{sgn}(\alpha_5)$. On the other hand, in this configuration we have $|\alpha_1|<|\alpha_5|$ as is seen from diagram A-5 of Fig. 19 and hence $\mathrm{sgn}(\alpha_1+\alpha_5)=\mathrm{sgn}(\alpha_5)$. Thus $\mathrm{sgn}(\alpha_X)$ and $-\mathrm{sgn}(\alpha_B)$ are opposite and the contributions of diagrams A-1 and A-5 actually cancel. As another example, consider the other region of σ_0 and α's in which the A-1 configuration becomes identical with C-2. In this case we have $|\alpha_2|<|\alpha_3|$ as is clear in diagram C-2 of Fig. 16. So, by (5.55), the diagram C-2 contributes with sign $\mathrm{sgn}(\alpha_Y)=\mathrm{sgn}(\alpha_2+\alpha_3)=\mathrm{sgn}(\alpha_3)$. The sign $\mathrm{sgn}(\alpha_X)$ of the A-2 contribution is indeed opposite; $\mathrm{sgn}(\alpha_X)=\mathrm{sgn}(\alpha_1+\alpha_2)=-\mathrm{sgn}(\alpha_3)$ again by the alternating sign rule as is seen from diagram A-1 of Fig. 15.

Similarly, it is easy to see the cancellations for all the other cases in (5.58), and the sum of (5.55) and (5.56) vanishes. We thus have finished the proof of $O(g^3)$ nilpotency $\{\delta_B^1,\delta_B^2\}=0$.

F. $O(g^4)$ nilpotency $(\delta_B^2)^2=0$

Now let us go to the final part of the nilpotency proof of our BRS transformation. With quite the same procedure as in the previous cases, it is easy to obtain, from (5.53) and (5.54),

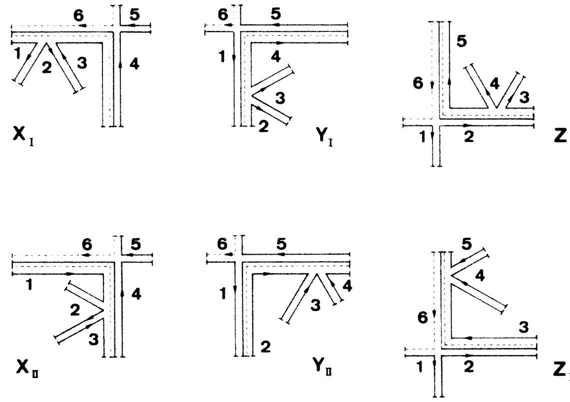


FIG. 20. The 6-string configurations appearing in $(\delta_B^2)^2\Phi(6)$ of (5.60).

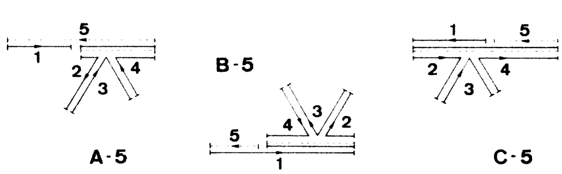


FIG. 19. The 5-string configurations appearing in $\delta_B^2\delta_B^1\Phi(5)$; the second term in (5.56).

$$(\delta_B^2)^2|\Phi(6)\rangle = -\int d1d2d3d4d5\langle\Phi(1)|\langle\Phi(2)|\langle\Phi(3)|\langle\Phi(4)|\langle\Phi(5)|$$

$$\times\int d^2\sigma_0\left[f(\sigma_0^{123X})f(\sigma_0^{X456})G(\sigma_I^{123X})G(\sigma_I^{X456})\frac{1}{\alpha_X}[\det(X)]^{-(d-2)/2}|V_0^{(6)}(1-6)\rangle_X\right.$$

$$+f(\sigma_0^{234Y})f(\sigma_0^{1Y56})G(\sigma_I^{234Y})G(\sigma_I^{1Y56})\frac{1}{\alpha_Y}[\det(Y)]^{-(d-2)/2}|V_0^{(6)}(1-6)\rangle_Y$$

$$\left.+f(\sigma_0^{345Z})f(\sigma_0^{12Z6})G(\sigma_I^{345Z})G(\sigma_I^{12Z6})\frac{1}{\alpha_Z}[\det(Z)]^{-(d-2)/2}|V_0^{6}(1-6)\rangle_Z\right],\tag{5.60}$$

$$\alpha_X=\alpha_1+\alpha_2+\alpha_3,\quad \alpha_Y=\alpha_2+\alpha_3+\alpha_4,\quad \alpha_Z=\alpha_3+\alpha_4+\alpha_5,$$

$$\det(x)\equiv\det(1-\tilde N^{(4)\bar x\bar x}\tilde N^{(4)xx})\quad\text{for } x=X,Y,Z.$$

The 6-string δ functionals $|V_0^{(6)}(1-6)\rangle_{X,Y,Z}$ are defined by Eq. (5.20) with the Neumann functions $\bar N_{mn}^{(6)rs}$ substituted which correspond to the configurations depicted as $X-Z$ in Fig. 20, respectively. Each of the configurations $X-Z$ corresponds to either one of the two diagrams distinguished by indices I and II depending on the values of two σ_0.

Now we can understand the following equalities for the 6-string δ functionals in (5.60) from Fig. 20:

$$|V_0^{(6)}\rangle_{X_{II}}=|V_0^{(6)}\rangle_{Y_I},\quad |V_0^{(6)}\rangle_{Y_{II}}=|V_0^{(6)}\rangle_{Z_I},\quad |V_0^{(6)}\rangle_{Z_{II}}=|V_0^{(6)}\rangle_{X_I}.\tag{5.61}$$

As before duality guarantees that the following products of factors

$$f(\sigma_0)f(\sigma_0')\frac{1}{|\alpha|}[\det(1-\tilde N^{(4)}\tilde N^{(4)})]^{-(d-2)/2}\tag{5.62}$$

in the coefficients of (5.60) are equal for each pair of corresponding configurations given in (5.61), if $d=26$ and $f(\sigma_0)$ is given by (4.22). So we examine only the signs of $\alpha_{X,Y,Z}$ and the order of two ghost factors. For the former we have a simple relation in this case

$$\mathrm{sgn}(\alpha_X)=-\mathrm{sgn}(\alpha_Y)=\mathrm{sgn}(\alpha_Z)=-\mathrm{sgn}(\alpha_6),\tag{5.63}$$

owing to the alternating sign rule (4.5) of our 4-string vertex. From this, for the cancellation between X and Y or Y and Z configurations to occur, the two ghost factors must be placed in the same order, while in the opposite order between Z and X configurations. This is just what happens. For the configurations X_{II} and Y_I, the second 4-vertex (i.e., the vertex of the second BRS transformation) of X_{II} corresponds to the second one of Y_I and the first of X_{II} to the first of Y_I, as is seen from Fig. 20. Therefore the two ghost factors are in the same order and coincide: $G(\sigma_I^{123X})G(\sigma_I^{X456})=G(\sigma_I^{234Y})G(\sigma_I^{1Y56})$. The same is true for Y_{II} and Z_I configurations. For Z_{II} and X_I configurations, however, Fig. 20 tells us that the correspondence of the first and second vertices is opposite and so is the order of two ghost factors.

G. Properties of the 3-string and 4-string vertices

Before closing this section, we summarize now the properties of our vertices which we have proved up to here.

It should be noted that we have actually proved more than the nilpotency $(\delta_B)^2=0$. For instance, in $O(g)$, we proved not merely the nilpotency

$$\{\delta_B^0,\delta_B^1\}|\Phi(3)\rangle=\int d1d2\{[(\langle\Phi(1)|Q_B^{(1)})\langle\Phi(2)|-\langle\Phi(1)|(\langle\Phi(2)|Q_B^{(2)})]|V(1,2,3)\rangle$$
$$-\langle\Phi(1)|\langle\Phi(2)|Q_B^{(3)}|V(1,2,3)\rangle\}=0,\tag{5.64}$$

but the equation

$$\sum_{r=1}^{3}Q_B^{(r)}|V(1,2,3)\rangle=0.\tag{5.65}$$

The latter is clearly stronger since the former claims the identity only when all the three string fields $\langle\Phi(1)|$, $\langle\Phi(2)|$, and $\langle\Phi(3)|$ in (5.64) are the same string but the latter implies that (5.64) holds even when they are replaced by three different string fields.

The same remark applies also for all the previous $O(g^N)$ nilpotency proofs. Actually recall that the $O(g^{N-1})$ term of $(\delta_B)^2|\Phi\rangle$ generally took the form

$$\int d1d2\cdots dN\langle\Phi(1)|\langle\Phi(2)|\cdots\langle\Phi(N)|\times\left[\sum(\text{ghost factors})|V_0^{(N+1)}\rangle\right].\tag{5.66}$$

We have seen there that the cancellations occur between pairs of terms which possess the same δ-functional configurations, hence in particular the same sets of values of α_r $(r=1,2,\ldots,N)$. That is, (5.66) vanishes for each set of values of α_r, i.e., before the $d\alpha_1d\alpha_2\cdots d\alpha_N$ integration.

This implies that (5.66) holds in fact even when $\langle\Phi(1)|-\langle\Phi(N)|$ are replaced by N completely different string fields since $\langle\Phi(r)|$ at two different values of α_r can be completely independent even for a single Φ.

In order to express these stronger identities concisely, it is useful to define 2-string and 3-string products for arbitrary string fields Φ, Ψ, and Λ:

$$|(\Phi*\Psi)(3)\rangle \equiv \int \langle\Phi(1)|\langle\Psi(2)||V(1,2,3)\rangle d1\,d2\,, \tag{5.67a}$$

$$|(\Phi\circ\Psi\circ\Lambda)(4)\rangle \equiv \int \langle\Phi(1)|\langle\Psi(2)|\langle\Lambda(3)| \times |V^{(4)}(1,2,3,4)\rangle d1\,d2\,d3\,. \tag{5.67b}$$

[Notice that the measures $d1\,d2(d3)$ are placed on the end for the convenience to operate δ_B from the left naturally.] Since it is more convenient, in this context, to use the functional representation instead of bra-ket notation, we rewrite these into the functional forms with the help of (2.11) and (2.12):

$$(\Phi*\Psi)[Z_3]=\int \Phi[Z_1]\Psi[Z_2]V[Z_2,Z_1,\tilde{Z}_3][dZ_1dZ_2]\,, \tag{5.68a}$$

$$(\Phi\circ\Psi\circ\Lambda)[Z_4]=\int \Phi[Z_1]\Psi[Z_2]\Lambda[Z_3] \times V^{(4)}[Z_3,Z_2,Z_1,\tilde{Z}_4] \times[dZ_1dZ_2dZ_3]\,. \tag{5.68b}$$

Here we have used the Hermiticity condition (2.13)

$$\Phi^\dagger[Z]=\Phi[\tilde{Z}]\,,\quad \tilde{Z}=(X^\mu(\pi-\sigma),-c(\pi-\sigma),\bar{c}(\pi-\sigma);-\alpha)\,, \tag{5.69}$$

as well as the property (3.65) of a 3-string vertex under the twist operation which reads

$$V[\tilde{Z}_1,\tilde{Z}_2,Z_3]=V[Z_2,Z_1,\tilde{Z}_3]\,,\quad V[Z_1,Z_2,Z_3]\equiv\langle z_1|\langle z_2|\langle z_2||V(1,2,3)\rangle \tag{5.70}$$

in the present functional representation and a similar one (4.31) for the 4-string vertex $V^{(4)}$. Henceforth we adopt (5.68) as a basic definition of string products for general (not necessarily Hermitian) string fields.

Now with these notations (5.68) our full BRS transformation δ_B takes the form

$$\delta_B\Phi=Q_B\Phi+g\Phi*\Phi+g^2\Phi\circ\Phi\circ\Phi\,, \tag{5.71}$$

and it is straightforward to write down several identities implied by its nilpotency

$$\begin{aligned}0&=(\delta_B)^2\Phi\\ &=-Q_B(\delta_B\Phi)+g(\delta_B\Phi*\Phi-\Phi*\delta_B\Phi)\\ &\quad+g^2(\delta_B\Phi\circ\Phi\circ\Phi-\Phi\circ\delta_B\Phi\circ\Phi+\Phi\circ\Phi\circ\delta_B\Phi)\end{aligned} \tag{5.72}$$

at each $O(g^N)$. They, however, hold in stronger forms as explained above. In view of the forms of weaker identities from (5.72), it is easy to see that those stronger identities, which we have actually proved are written in the following form:[18]

$$O(g):\ Q_B(\Phi*\Psi)=Q_B\Phi*\Psi+(-)^{|\Phi|}\Phi*Q_B\Psi\,, \tag{5.73a}$$

$$O(g^2):\ -Q_B(\Phi\circ\Psi\circ\Lambda)+(Q_B\Phi\circ\Psi\circ\Lambda+(-)^{|\Phi|}\Phi\circ Q_B\Psi\circ\Lambda+(-)^{|\Phi|+|\Psi|}\Phi\circ\Psi\circ Q_B\Lambda) =(-)^{|\Phi|+|\Psi|+|\Lambda|}[(\Phi*\Psi)*\Lambda-\Phi*(\Psi*\Lambda)]\,, \tag{5.73b}$$

$$O(g^3):\ (-)^{|\Sigma|}(\Phi\circ\Psi\circ\Lambda)*\Sigma+\Phi*(\Psi\circ\Lambda\circ\Sigma)=(\Phi*\Psi)\circ\Lambda\circ\Sigma-\Phi\circ(\Psi*\Lambda)\circ\Sigma+\Phi\circ\Psi\circ(\Lambda*\Sigma)\,, \tag{5.73c}$$

$$O(g^4):\ (-)^{|\Sigma|+|\Xi|}(\Phi\circ\Psi\circ\Lambda)\circ\Sigma\circ\Xi+(-)^{|\Xi|}\Phi\circ(\Psi\circ\Lambda\circ\Sigma)\circ\Xi+\Phi\circ\Psi\circ(\Lambda\circ\Sigma\circ\Xi)=0\,. \tag{5.73d}$$

These identities hold for arbitrary (matrix-valued) string fields $\Phi,\Psi,\Lambda,\dots$, where $|\Phi|$ is 0 if Φ is Grassmann-even and 1 if Grassmann-odd.

The identities (5.73) represent actually the properties of the 3-string and 4-string vertex functionals which we have constructed in Secs. III and IV. Equation (5.73a) is a distribution law of Q_B operation on the $*$ product, Eq. (5.73d) is an associativity law for the (∘ ∘) product. Equation (5.73b) shows that the distribution law of Q_B operation is violated on the (∘ ∘) product but is compensated by also a breaking of associativity for the $*$ product. [It is interesting to note that Eq. (5.73b) is replaced in the closed-string case[17,18] by a Jacobi-type identity $\Phi*(\Psi*\Lambda)+(-)^{|\Phi|(|\Psi|+|\Lambda|)}\Psi*(\Lambda*\Phi)+(-)^{|\Lambda|(|\Phi|+|\Psi|)}\Lambda*(\Phi*\Psi)=0$.] It should be emphasized again that all those identities hold only if $d=26$, as we have seen in this section.

For later convenience we define bilinear, trilinear, and quadralinear forms of string fields by

$$\Phi\cdot\Psi\equiv\int[dZ]\mathrm{tr}(\Phi[\tilde{Z}]\Psi[Z])=\int[dZ]\mathrm{tr}(\Phi[Z]\Psi[\tilde{Z}]) =\int d1\,d2\,\mathrm{tr}\langle R(2,1)|\Omega^{(2)}|\Phi(1)\rangle|\Psi(2)\rangle\,, \tag{5.74a}$$

$$[\Phi\Psi\Lambda]_3\equiv\Phi\cdot(\Psi*\Lambda)=\int[dZ_1dZ_2dZ_3]\mathrm{tr}(\Phi[Z_1]\Psi[Z_2]\Lambda[Z_3])V[Z_3,Z_2,Z_1]\,, \tag{5.74b}$$

$$[\Phi\Psi\Lambda\Sigma]_4\equiv\Phi\cdot(\Psi\circ\Lambda\circ\Sigma) =(-)^{|\Phi|+|\Psi|+|\Lambda|+|\Sigma|}\int[dZ_1dZ_2dZ_3dZ_4]\mathrm{tr}(\Phi[Z_1]\Psi[Z_2]\Lambda[Z_3]\Sigma[Z_4])V^{(4)}[Z_4,Z_3,Z_2,Z_1]\,. \tag{5.74c}$$

If the string fields are all Hermitian, these are simply rewritten also in the bra-ket notations: The first one is connected to a more familiar inner product,

$$\Phi\cdot\Psi=\int d1\,\mathrm{tr}\langle\Phi(1)\,|\,\Psi(1)\rangle=\int dZ\,\mathrm{tr}(\Phi^{\dagger}[Z]\Psi[Z]) \tag{5.75a}$$

and the latter two take the form

$$[\Phi\Psi\Lambda]_3=\int d1\,d2\,d3\,\mathrm{tr}\langle\Phi(1)\,|\,\langle\Psi(2)\,|\,\langle\Lambda(3)\,|\,|\,V(1,2,3)\rangle\ , \tag{5.75b}$$

$$[\Phi\Psi\Lambda\Sigma]_4=(-)^{|\Phi|+|\Psi|+|\Lambda|+|\Sigma|}\int d1\,d2\,d3\,d4\,\mathrm{tr}\langle\Phi(1)\,|\,\langle\Psi(2)\,|\,\langle\Lambda(3)\,|\,\langle\Sigma(4)||V^{(4)}(1,2,3,4)\rangle\ . \tag{5.75c}$$

Finally, we note that the important cyclic symmetry properties of our 3-string and 4-string vertices, (3.58) and (4.32), are expressed in these notations as[18]

$$\begin{aligned}[]
[\Phi\Psi\Lambda]_3&=(-)^{|\Phi|(|\Psi|+|\Lambda|)}[\Psi\Lambda\Phi]_3\\
&=(-)^{|\Lambda|(|\Phi|+|\Psi|)}[\Lambda\Phi\Psi]_3\ ,\\
[\Phi\Psi\Lambda\Sigma]_4&=(-)^{|\Phi|(|\Psi|+|\Lambda|+|\Sigma|)+1}[\Psi\Lambda\Sigma\Phi]_4\\
&=(-)^{|\Sigma|(|\Phi|+|\Psi|+|\Lambda|)+1}[\Sigma\Phi\Psi\Lambda]_4\ ,
\end{aligned} \tag{5.76}$$

etc.

VI. ACTIONS

Now that we have constructed the full BRS transformation satisfying nilpotency and clarified all the necessary properties of the 3-string and 4-string vertices, it is quite an easy matter to write down a gauge-invariant action as well as a BRS-invariant gauge-fixed action. Although we found originally the latter first[16,17] and next the former,[18] we present them here in the opposite order since we think it more logical and transparent.

A. Gauge-invariant action

In analogy with the usual gauge transformation of Yang-Mills theory,

$$\begin{aligned}
\delta A&=d\epsilon+ig[A,\epsilon]\ ,\\
A&\equiv\sum_a T^a A^a_\mu dx^\mu,\quad \epsilon\equiv\sum_a T^a\epsilon^a\ ,
\end{aligned} \tag{6.1}$$

we are led to consider the following gauge transformation in string field theory:

$$\begin{aligned}
\delta\Phi=&\ Q_B\Lambda+g(\Phi*\Lambda-\Lambda*\Phi)\\
&-g^2(\Phi\circ\Phi\circ\Lambda-\Phi\circ\Lambda\circ\Phi+\Lambda\circ\Phi\circ\Phi)
\end{aligned} \tag{6.2}$$

with Λ being a (string functional) matrix-valued transformation parameter. Since our string field Φ is Grassmann-odd, Hermitian, and carries FP ghost number $N_{\mathrm{FP}}=-1$, the parameter Λ must be Grassmann-even, anti-Hermitian (i.e., satisfies $\Lambda^{\dagger}[Z]=-\Lambda[\tilde{Z}]$), and carry $N_{\mathrm{FP}}=-2$. The last term in (6.2) may look strange in comparison with (6.1) but is found necessary as we will see shortly. [The string gauge transformation (6.2) indeed reproduces in the zero-slope limit the Yang-Mills gauge transformation (6.1) as we will show explicitly in Sec. VIII.]

The gauge-invariant action is now easily found as[18]

$$S=\Phi\cdot Q_B\Phi+\tfrac{2}{3}g\Phi^3+\tfrac{2}{4}g^2\Phi^4\ , \tag{6.3}$$

with notations introduced in the previous section:

$$\Phi\cdot Q_B\Phi=\int d1\,\mathrm{tr}\langle\Phi(1)\,|\,Q_B\,|\,\Phi(1)\rangle\ , \tag{6.4a}$$

$$\begin{aligned}
\Phi^3&\equiv[\Phi\Phi\Phi]_3\\
&=\int d1\,d2\,d3\,\mathrm{tr}\langle\Phi(1)\,|\,\langle\Phi(2)\,|\,\langle\Phi(3)||V(1,2,3)\rangle\ ,
\end{aligned} \tag{6.4b}$$

$$\begin{aligned}
\Phi^4&\equiv[\Phi\Phi\Phi\Phi]_4\\
&=\int d1\,d2\,d3\,d4\,\mathrm{tr}\langle\Phi(1)\,|\,\langle\Phi(2)\,|\,\langle\Phi(3)\,|\,\langle\Phi(4)\,|\\
&\qquad\times|\,V^{(4)}(1,2,3,4)\rangle\ .
\end{aligned} \tag{6.4c}$$

Indeed the invariance of the action (6.3) under the transformation (6.2) is seen as follows: By using the definitions (5.74), the cyclic symmetry (5.76), and also the following properties of the inner product (5.74a) or (5.75a),

$$\begin{aligned}
\Phi\cdot\Psi&=(-)^{|\Phi||\Psi|}\Psi\cdot\Phi\ ,\\
(Q_B\Phi)\cdot\Psi&=(-)^{1+|\Phi|}\Phi\cdot Q_B\Psi\ ,
\end{aligned} \tag{6.5}$$

we first notice that δS can be written in the form

$$\begin{aligned}
\delta S&=2\delta\Phi\cdot(Q_B\Phi+g\Phi*\Phi+g^2\Phi\circ\Phi\circ\Phi)\\
&=2\delta\Phi\cdot\delta_B\Phi\\
&=2\delta_B\Phi\cdot[\,Q_B\Lambda+g(\Phi*\Lambda-\Lambda*\Phi)\\
&\qquad-g^2(\Phi\circ\Phi\circ\Lambda-\Phi\circ\Lambda\circ\Phi+\Lambda\circ\Phi\circ\Phi)]\ ,
\end{aligned} \tag{6.6}$$

with $\delta_B\Phi$ denoting the BRS transform (5.71) of Φ. This is further rewritten with the help of cyclic symmetry (5.76), in particular, as

$$\begin{aligned}
\delta S&=2((Q_B\Lambda)\cdot\delta_B\Phi+g\{[\Lambda(\delta_B\Phi)\Phi]_3-[\Lambda\Phi(\delta_B\Phi)]_3\}+g^2\{[\Lambda(\delta_B\Phi)\Phi\Phi]_4-[\Lambda\Phi(\delta_B\Phi)\Phi]_4+[\Lambda\Phi\Phi(\delta_B\Phi)]_4\})\\
&=2\Lambda\cdot[-Q_B(\delta_B\Phi)+g(\delta_B\Phi*\Phi-\Phi*\delta_B\Phi)+g^2(\delta_B\Phi\circ\Phi\circ\Phi-\Phi\circ\delta_B\Phi\circ\Phi+\Phi\circ\Phi\circ\delta_B\Phi)]\\
&=2\Lambda\cdot\delta_B(\delta_B\Phi)\ .
\end{aligned} \tag{6.7}$$

This vanishes by the nilpotency of our BRS transformation $(\delta_B)^2=0$, i.e., (5.72). It should be noted that the cyclic symmetry property of our vertices was crucially important here.

Next we clarify the group structure of gauge transformations (6.2). Although only the weaker identity $(\delta_B)^2\Phi=0$ was necessary for proving the gauge invariance in the above, we now need all of the stronger identities (5.73). The calculation of the commutator $[\delta(\Lambda_1),\delta(\Lambda_2)]$ of two gauge transformations with parameters Λ_1 and Λ_2 is, of course, straightforward but somewhat cumbersome. So it is better to proceed as follows: We first calculate only the terms containing Q_B in the commutator to guess the final result, and then verify the guess. Now from (6.2) we have

$$[\delta(\Lambda_1),\delta(\Lambda_2)]\Phi=\delta(\Lambda_1)[g(\Phi*\Lambda_2-\Lambda_2*\Phi)-g^2(\Phi\circ\Phi\circ\Lambda_2-\Phi\circ\Lambda_2\circ\Phi+\Lambda_2\circ\Phi\circ\Phi)]-(1\leftrightarrow 2)$$
$$=[g(Q_B\Lambda_1*\Lambda_2-\Lambda_2*Q_B\Lambda_1)-g^2(Q_B\Lambda_1\circ\Phi\circ\Lambda_2+\Phi\circ Q_B\Lambda_1\circ\Lambda_2-Q_B\Lambda_1\circ\Lambda_2\circ\Phi-\Phi\circ\Lambda_2\circ Q_B\Lambda_1$$
$$+\Lambda_2\circ Q_B\Lambda_1\circ\Phi+\Lambda_2\circ\Phi\circ Q_B\Lambda_1)+\cdots]-(1\leftrightarrow 2)\,, \tag{6.8}$$

where the ellipsis represents the other terms containing no Q_B. By the distribution law of Q_B on the $*$ and ($\circ$ $\circ$) products [(5.73a) and (5.37b)] this is rewritten as

$$[\delta(\Lambda_1),\delta(\Lambda_2)]\Phi=Q_B[g(\Lambda_1*\Lambda_2)+g^2(\Phi\circ\Lambda_1\circ\Lambda_2-\Lambda_1\circ\Phi\circ\Lambda_2+\Lambda_1\circ\Lambda_2\circ\Phi)-(1\leftrightarrow 2)]$$
$$-g^2[(Q_B\Phi\circ\Lambda_1\circ\Lambda_2-\Lambda_1\circ Q_B\Phi\circ\Lambda_2+\Lambda_1\circ\Lambda_2\circ Q_B\Phi)-(1\leftrightarrow 2)]+\cdots\,. \tag{6.9}$$

The first term on the RHS takes the form of the $O(g^0)$ gauge transformation $Q_B\Lambda$ and the second is a term proportional to the $O(g^0)$ equation of motion $Q_B\Phi=0$. This suggests that the final answer takes the form

$$[\delta(\Lambda_1),\delta(\Lambda_2)]\Phi=\delta(\Lambda_3)\Phi-g^2[(S_{,\Phi}\circ\Lambda_1\Lambda_2-\Lambda_1\circ S_{,\Phi}\circ\Lambda_2+\Lambda_1\circ\Lambda_2\circ S_{,\Phi})-(1\leftrightarrow 2)]\,, \tag{6.10a}$$

$$\Lambda_3=[g(\Lambda_1*\Lambda_2)+g^2(\Phi\circ\Lambda_1\circ\Lambda_2-\Lambda_1\circ\Phi\circ\Lambda_2+\Lambda_1\circ\Lambda_2\circ\Phi)]-(1\leftrightarrow 2)\,, \tag{6.10b}$$

$$S_{,\Phi}\equiv\tfrac{1}{2}(\partial/\partial\Phi)S=(Q_B\Phi+g\Phi*\Phi+g^2\Phi\circ\Phi\circ\Phi)=\delta_B\Phi\,, \tag{6.10c}$$

where $S_{,\Phi}=0$ is the equation of motion. Equation (6.10) is actually confirmed by calculating all the other terms by the full use of the stronger identities (5.73).

The result (6.10), which was already reported in III, is very interesting: First, it shows that the algebra of gauge transformation closes only on-shell, $S_{,\Phi}=0$. Further, even on-shell, the third gauge transformation parameter Λ_3 given by (6.10b) depends on the string field Φ explicitly, implying that the "structure constant" of this algebra is field dependent. These properties are reminiscent of the same situation in supergravity[49] and may suggest the existence of some auxiliary string-field variables which make the algebra close off-shell. The off-shell closure is, however, not absolutely necessary.

Finally we should comment on a restriction on Φ and Λ, which should be imposed if the above action (6.3) can be said to be a *gauge-invariant action* in the proper sense.

The string field Φ, or its ket representation $|\Phi\rangle$ more precisely, is generally expanded into states of the form

$$\varphi_{l,\mathbf{m},\overline{\mathbf{m}}}^{\mu_1\cdots\mu_n}(x,\alpha)\left[\prod_{i=1}^{n}\alpha_{l_i\mu_i}^{\dagger}\right]\left[\prod_{j=1}^{p}c_{m_j}^{\dagger}\right]\times\left[\prod_{k=1}^{q}\bar{c}_{\overline{m}_k}^{\dagger}\right]|0\rangle\quad(m_j\ge 1,\ \overline{m}_k\ge 0) \tag{6.11}$$

and we call such components possessing p ghost and q antighost oscillator (or zero) modes (p,q)-form, or simply $(p-q)$-form by dropping the discrimination among $(p-q+r,r)$ forms with arbitrary $r\ge 0$ (Ref. 13). Notice the difference between the number $(p-q)$ and the ghost number $N_{\rm FP}$ which we have used up to now in this paper. The number $(p-q)$ is an "internal ghost number" carried by the ghost variables c and $\bar{c}$ alone, while $N_{\rm FP}$ is the net ghost number which may be carried also by the coefficient fields $\varphi_{l,\mathbf{m},\overline{\mathbf{m}}}^{\mu_1\cdots\mu_n}(x,\alpha)$. Therefore, our assumption that the string field Φ carries $N_{\rm FP}=-1$ does not exclude the components with $p-q\neq -1$ but simply specifies the ghost number assignment to the coefficient fields allowing all the (p,q) forms. So the coefficients $\varphi(x)$ of $(p-q)$ forms carry the ghost number $N_{\rm FP}=-(p-q)-1$. Probably the gauge-invariant action in the proper sense should not contain the component fields $\varphi(x)$ carrying nonzero ghost number $N_{\rm FP}$ since it has to be meaningful also as a classical field theory. If one thinks so, one can restrict the string field Φ to its $(p-q)=-1$ form sector Φ_{-1} in the gauge-invariant action S. Indeed this restriction is consistent with the gauge invariance if one also restricts the gauge transformation parameter Λ to the -2 form sector. The above proofs of the gauge invariance and of the group law (6.10) remain unchanged with these restrictions.

The authors, however, do not know whether this restriction is absolutely necessary or not for the "true" gauge-invariant action. For the requirement of gauge invariance alone, no restriction is necessary as we have seen. Further there is another, equally consistent, restriction to require the Φ=odd form and the Λ=even form, in which case the component fields φ appearing in the action are all Grassmann even (i.e., usual Bose fields). See III for more details. This issue will not finally be settled until the gauge-fixing procedure is made clear. The gauge-fixing procedure becomes quite a nontrivial problem in the presence of interaction as was pointed out in III.

B. BRS-invariant gauge-fixed action

Our next task is to construct a gauge-fixed action which is invariant under a nilpotent BRS transformation. In the free-string case, the usual procedure of gauge fixing works and one can obtain the gauge-fixed action from the gauge-invariant one as was made clear by many authors:[14] Namely, starting from the gauge-invariant action (6.3) with Φ restricted to the -1 form sector (and $g=0$), and choosing the gauge $|\psi\rangle=0$, one actually reaches the gauge-fixed action after adding an infinite sequence of ghost's ghosts. Interestingly, the resultant gauge-fixed action turns out to be Siegel's free action[6] (2.24) or (2.29) though historically Siegel's action was found prior to the gauge-invariant one.

This gauge-fixing procedure does not work in the presence of interaction terms as was pointed out in III. Therefore we do not discuss the gauge-fixing problem any more. Fortunately we know already a BRS-invariant gauge-fixed action. It was found in I prior to the gauge-invariant one, just like in the free case. Further it was shown in III to be equivalent to the gauge-invariant one, at least, at the tree level and also as classical field theories.

Our BRS-invariant gauge-fixed action is given from the gauge invariant one (6.3) simply by setting the ψ component equal to zero:

$$\hat{S}[\phi]=(\Phi\cdot Q_B\Phi+\tfrac{2}{3}g\Phi^3+\tfrac{2}{4}g^2\Phi^4)_{\psi=0}\ . \tag{6.12}$$

Here recall that ψ is the $\bar{c}_0$-independent part in Φ as defined in (2.7):

$$|\Phi(x,\bar{c}_0,\alpha)\rangle=-\bar{c}_0\,|\phi(x,\alpha)\rangle+|\psi(x,\alpha)\rangle\ . \tag{6.13}$$

Therefore the action (6.12) contains only the ϕ-component string field which carries the FP ghost number $N_{\rm FP}=0$ and hence is Grassmann even. [However, remember that ϕ contains any (p,q) forms as was noted before.] By performing the integration over $\bar{c}_0$, the action (6.12) can be written in the form

$$\hat{S}[\phi]=\phi\cdot L\phi+\tfrac{2}{3}g\phi^3+\tfrac{2}{4}g^2\phi^4\ , \tag{6.14}$$

with the notations defined by

$$\phi\cdot L\phi=\int d1\,{\rm tr}\langle\phi(1)|L\,|\phi(1)\rangle\ , \tag{6.15a}$$

$$\phi^3=\int d1\,d2\,d3\,{\rm tr}\langle\phi(1)|\langle\phi(2)|\langle\phi(3)|\times|v(1,2,3)\rangle\ , \tag{6.15b}$$

$$\phi^4=\int d\sigma_0\int d1\,d2\,d3\,d4\,{\rm tr}\langle\phi(1)|\langle\phi(2)|\langle\phi(3)|\times\langle\phi(4)||v_{\sigma_0}^{(4)}(1,2,3,4)\rangle\ . \tag{6.15c}$$

Here, r and dr $(r=1-4)$ are now those for zero-mode variables other than $\bar{c}_0$, denoting (p_r,α_r) and $dp_r d\alpha_r/(2\pi)^{d+1}$, respectively, and the reduced vertices $|v(1,2,3)\rangle$ and $|v_{\sigma_0}^{(4)}(1,2,3,4,)\rangle$ are given, from (5.1), (5.2), and (6.4), by

$$|v(1,2,3)\rangle=\mu(\alpha_1,\alpha_2,\alpha_3)\delta(1,2,3)e^{F(1,2,3)}|0\rangle\ , \tag{6.16a}$$

$$|v_{\sigma_0}^{(4)}(1,2,3,4)\rangle=f(\sigma_0)\delta(1,2,3,4)e^{F_{\sigma_0}^{(4)}(1,2,3,4)}|0\rangle\ , \tag{6.16b}$$

$$\delta(1,2,3,(4))=(2\pi)^{d+1}\delta\left[\sum_r\alpha_r\right]\delta\left[\sum_r p_r\right]\ , \tag{6.17a}$$

$$F(1,2,3)=\sum_{r,s=1}^{3}\left[\frac{1}{2}\sum_{n,m\geq 0}\bar{N}^{rs}_{mn}\alpha^{(r)}_{-m}\alpha^{(s)}_{-n}+i\sum_{n,m\geq 1}\bar{N}^{rs}_{mn}\gamma^{(r)}_{-m}\bar{\gamma}^{(s)}_{-n}\right]\ , \tag{6.17b}$$

$$F_{\sigma_0}^{(4)}(1,2,3,4)=\sum_{r,s=1}^{4}\left[\frac{1}{2}\sum_{n,m\geq 0}\bar{N}^{(4)rs}_{mn}\alpha^{(r)}_{-m}\alpha^{(s)}_{-n}+i\sum_{n,m\geq 1}\bar{N}^{(4)rs}_{mn}\gamma^{(r)}_{-m}\bar{\gamma}^{(s)}_{-n}\right]\ . \tag{6.17c}$$

Notice that the functions F and $F_{\sigma_0}^{(4)}$ are just the previous exponents $E_X+E_{\rm FP}$ and $E_X^{(4)}+E_{\rm FP}^{(4)}$ with the ghost zero-mode parts omitted.

The BRS transformation $\hat{\delta}_B\phi$ in this gauge-fixed system is given again by setting $\psi=0$ in the original BRS transformation (5.71) of the ϕ component:

$$\hat{\delta}_B\phi\equiv\delta_B\phi\,|_{\psi=0}=\int d\bar{c}_0\delta_B\phi\Big|_{\psi=0}\ . \tag{6.18}$$

Before giving the explicit form of this in terms of ϕ, we now prove two things: (i) This new BRS transformation $\hat{\delta}_B$ is nilpotent on the mass shell, i.e., nilpotent when the equation of motion $\delta\hat{S}/\delta\phi=0$ holds; (ii) the gauge-fixed action $\hat{S}$, (6.14), is invariant under this BRS transformation.

To show these, we first recall the particular property which the previous gauge-invariant action S, (6.3), had; that is, the change of S under an arbitrary variation $\delta\Phi$ of Φ took the form

$$\delta S=2\delta\Phi\cdot(Q_B\Phi+g\Phi*\Phi+g^2\Phi\circ\Phi\circ\Phi)=2\delta\Phi\cdot\delta_B\Phi\ . \tag{6.19}$$

This property is inherited by the gauge-fixed action $\hat{S}[\phi]$ in the following form since $\hat{S}$ is S with ψ set equal to zero:

$$\delta\hat{S}=2(\delta\Phi\cdot\delta_B\Phi)_{\psi=0}=-2\delta\phi\cdot(\delta_B\psi)_{\psi=0}\ , \tag{6.20}$$

with the understanding that $\delta\Phi=-\bar{c}_0\delta\phi$ since ψ is fixed to be zero in $\hat{S}$. Here the dot in the last quantity means the inner product at the $\phi-\psi$ component level (i.e., without $\bar{c}_0$ integration). Now we can prove the above first statement (i), i.e., the on-shell nilpotency of the new BRS transformation $\hat{\delta}_B$. From the definition (6.18), $\hat{\delta}_B\phi$ is identical with the original $\delta_B\phi$ aside from the terms containing at least one factor of ψ:

$$\delta_B\phi=\hat{\delta}_B\phi+\psi^{\exists}f(\phi,\psi)\ . \tag{6.21}$$

The (off-shell) nilpotency of the original BRS transformation δ_B leads to

$$0=(\delta_B)^2\phi=(\hat{\delta}_B)^2\phi+(\delta_B\psi)_{\psi=0}f(\phi,0)+O(\psi)\ .$$

Sine this is an identity in ψ, the ψ-independent parts vanish by themselves. Thus we have an equation

$$(\hat{\delta}_B)^2\phi=-(\delta_B\psi)_{\psi=0}f(\phi,0)\ , \tag{6.22}$$

proving the desired on-shell nilpotency of $\hat{\delta}_B$ since $(\delta_B\psi)_{\psi=0}=0$ is just the equation of motion of the present system $\hat{S}$ as is seen from (6.20):

$$\delta\hat{S}/\delta\phi=-2(\delta_B\psi)_{\psi=0}=0 \; . \tag{6.23}$$

Incidentally this explains why we adopted the action $\hat{S}$ of the form (6.14) for the gauge-fixed system. In fact, originally in I, the action $\hat{S}$ was constructed so as to yield $(\delta_B\psi)_{\psi=0}=0$ as its equation of motion.

Next we prove (ii), i.e., the invariance of $\hat{S}$ under $\hat{\delta}_B\phi$. Again by using (6.20) and (6.21), the change of the action $\hat{S}$ under $\hat{\delta}_B\phi$ is given by

$$\begin{aligned}\hat{\delta}_B\hat{S}&=-2\hat{\delta}_B\phi\cdot(\delta_B\psi)_{\psi=0}\\&=-2(\delta_B\phi)_{\psi=0}\cdot(\delta_B\psi)_{\psi=0} \; ,\end{aligned} \tag{6.24}$$

which can be rewritten in terms of the BRS transformation of the original field Φ, $\delta_B\Phi=\bar{c}_0\delta_B\phi+\delta_B\psi$, as

$$\begin{aligned}\hat{\delta}_B\hat{S}&=-\int d\bar{c}_0(\bar{c}_0\delta_B\phi+\delta_B\psi)\cdot(\bar{c}_0\delta_B\phi+\delta_B\psi)\,|_{\psi=0}\\&=-(\delta_B\Phi\cdot\delta_B\Phi)_{\psi=0} \; .\end{aligned} \tag{6.25}$$

However, the last quantity $\delta_B\Phi\cdot\delta_B\Phi$ vanishes even before setting $\psi=0$. Indeed it follows straightforwardly from the previous identities (5.73), (6.5), and the cyclic symmetries (5.76) that the expression

$$\begin{aligned}\delta_B\Phi\cdot\delta_B\Phi&=(Q_B\Phi+g\Phi*\Phi+g^2\Phi\circ\Phi\circ\Phi)^2\\&=Q_B\Phi\cdot Q_B\Phi+2gQ_B\Phi\cdot(\Phi*\Phi)+g^2[(\Phi*\Phi)\cdot(\Phi*\Phi)+2Q_B\Phi\cdot(\Phi\circ\Phi\circ\Phi)]\\&\quad+2g^3(\Phi*\Phi)\cdot(\Phi\circ\Phi\circ\Phi)+g^4(\Phi\circ\Phi\circ\Phi)\cdot(\Phi\circ\Phi\circ\Phi) \; ,\end{aligned} \tag{6.26}$$

vanishes separately in each order in g: The $O(g^0)$ part is trivial since $Q_B\Phi\cdot Q_B\Phi=\Phi\cdot Q_B{}^2\Phi=0$ by $Q_B{}^2=0$, and the $O(g^1)$ part results from the identity (5.73a) as

$$\begin{aligned}Q_B\Phi\cdot(\Phi*\Phi)&=\tfrac{1}{3}\{[(Q_B\Phi)\Phi\Phi]_3-[\Phi(Q_B\Phi)\Phi]_3+[\Phi\Phi(Q_B\Phi)]_3\}\\&=\tfrac{1}{3}\Phi\cdot[Q_B(\Phi*\Phi)-Q_B\Phi*\Phi+\Phi*Q_B\Phi]=0 \; .\end{aligned} \tag{6.27a}$$

Similarly the $O(g^2)$ part is simply a rewriting of Eq. (5.73b):

$$\begin{aligned}2Q_B\Phi\cdot(\Phi\circ\Phi\circ\Phi)+(\Phi*\Phi)\cdot(\Phi*\Phi)&=2\cdot\tfrac{1}{4}\{[(Q_B\Phi)\Phi\Phi\Phi]_4-[\Phi(Q_B\Phi)\Phi\Phi]_4+[\Phi\Phi(Q_B\Phi)\Phi]_4-[\Phi\Phi\Phi(Q_B\Phi)]_4\}\\&\quad+\tfrac{1}{2}\{-[\Phi(\Phi*\Phi)\Phi]_3+[\Phi\Phi(\Phi*\Phi)]_3\}\\&=\tfrac{1}{2}\Phi\cdot[Q_B(\Phi\circ\Phi\circ\Phi)-(Q_B\Phi\circ\Phi\circ\Phi-\Phi\circ Q_B\Phi\circ\Phi+\Phi\circ\Phi\circ Q_B\Phi)\\&\qquad-(\Phi*\Phi)*\Phi+\Phi*(\Phi*\Phi)]=0 \; .\end{aligned} \tag{6.27b}$$

In quite the same manner, the identities (5.73c) and (5.73d) lead to

$$(\Phi*\Phi)\cdot(\Phi\circ\Phi\circ\Phi)=0 \; , \tag{6.27c}$$

$$(\Phi\circ\Phi\circ\Phi)\cdot(\Phi\circ\Phi\circ\Phi)=0 \; , \tag{6.27d}$$

respectively. Thus (6.26) vanishes and Eq. (6.25) proves the invariance of the gauge-fixed action $\hat{S}$ under the BRS transformation $\hat{\delta}_B$. [It may be interesting to note that the vanishing of (6.26) implies that our gauge-invariant action S in (6.3) with no restriction on the internal ghost number of Φ is invariant under the BRS transformation; i.e., $\delta_B S=2\delta_B\Phi\cdot\delta_B\Phi=0$ (the authors would like to thank K.-I. Kobayashi for pointing out this fact). This should be distinguished from another trivial invariance of S under the local gauge transformation (6.2) with its parameter Λ set equal to $\lambda\Phi$: $\delta(\Lambda=\lambda\Phi)\Phi=-\lambda(Q_B\Phi+2g\Phi*\Phi+3g^2\Phi\circ\Phi\circ\Phi)$.]

Finally in this section let us give the explicit expression of the new BRS transformation $\hat{\delta}_B\phi$ defined in (6.18): From (2.22), (3.54), (5.1), (5.71), (5.68), (6.13), and (6.18), we obtain

$$\hat{\delta}_B\phi=\tilde{Q}_B\phi+g\phi*\phi+g^2\phi\circ\phi\circ\phi \; , \tag{6.28}$$

where the $*$ and ($\circ$ $\circ$) produces are now those at the ϕ-component level and are defined by

$$|(\phi*\phi)(3)\rangle=\int d1\,d2\langle\phi(1)|\langle\phi(2)|w^{(3)}|v(1,2,3)\rangle \; , \tag{6.29a}$$

$$\begin{aligned}|(\phi\circ\phi\circ\phi)(4)\rangle=\int d\sigma_0\int d1\,d2\,d3\langle\phi(1)|\langle\phi(2)|\\\times\langle\phi(3)|u^{(4)}\\\times|v^{(4)}_{\sigma_0}(1,2,3,4)\rangle \; .\end{aligned} \tag{6.29b}$$

Here $|v\rangle$ and $|v^{(4)}_{\sigma_0}\rangle$ are the vertices given in (6.17), $w^{(r)}$ was defined in (3.50) or (3.51) and $u^{(r)}$ is its analogue for the case of the 4-string vertex:

$$u^{(r)}=i\frac{1}{\alpha_r}\sum_{\substack{n\geq 1\\ s}}\left[\frac{1}{n}\delta^{rs}\cos(n\sigma_I^{(r)})-\sum_{m\geq 1}\bar{N}^{(4)rs}_{mn}\cos(m\sigma_I^{(r)})\right]\gamma^{(s)}_{-n} \; .$$

VII. SCATTERING AMPLITUDES AND UNITARITY

It is now an easy matter to calculate string scattering amplitudes based on our gauge-fixed action (6.14). In this section we present an explicit calculation of the tree-level 4-string scattering amplitude for general external string states, and also show that the general N-string tree amplitudes in our theory correctly reproduce the usual dual amplitudes if the external string states are on-shell and physical. The unitarity problem will be discussed lastly.

A. Feynman rules and light-cone diagrams

The canonical quantization procedure may be difficult to apply to our string field theory since the interaction terms are nonlocal with respect to the center-of-mass time $x_{\mu=0}$ of each participating string. However, if we rely on the usual path-integral reasoning,[36] it is easy to read Feynman rules from the action (6.14). For simplicity we confine ourselves to the orientable string with U(N) gauge group henceforth. Then, the string propagator is given by

$$\langle\langle \phi_i^j[Z]\phi_k^l[\tilde{Z}']\rangle\rangle = \left\langle z \left| \frac{-1}{2L} \right| z' \right\rangle \delta_i^l \delta_k^j (2\pi)^{d+1} \times \delta(\alpha+\alpha')\delta(p+p') , \tag{7.1}$$

or equivalently, in bra-ket notation,

$$\langle\langle \, | \phi_i^j(p,\alpha)\rangle \langle \phi_k^l(p',\alpha') | \, \rangle\rangle = -\frac{1}{2L}\delta_i^l\delta_k^j(2\pi)^{d+1} \times \delta(\alpha+\alpha')\delta(p+p') , \tag{7.2}$$

where (i,j) denotes the U(N) matrix index. If we use the usual simplified Feynman rule in (zero-mode) momentum space to take account of the momentum-conservation law in advance, we should drop the momentum-conservation factors

$$(2\pi)^{d+1}\delta(\alpha+\alpha')\delta(p+p')$$

in (7.1) or (7.2). Therefore for the diagrams containing loops, we attach the momentum zero-mode integral

$$\int \frac{d^d p\, d\alpha}{i(2\pi)^{d+1}} \tag{7.3}$$

(as well as the trace with respect to internal modes and matrix index) to each loop. To each 3-string vertex we put either one of the following two factors according to convenience:

$$\begin{aligned} &2gv[Z_1,Z_2,Z_3] \sim 2g\,|v(1,2,3)\rangle , \\ &2gv[\tilde{Z}_1,\tilde{Z}_2,\tilde{Z}_3] \sim 2g\langle v(3,2,1)| , \end{aligned} \tag{7.4}$$

with $|v\rangle$ given in (6.16a). To the 4-string vertex we attach

$$2g^2 v_{\sigma_0}^{(4)}[Z_1,Z_2,Z_3,Z_4] \sim 2g^2\,|v_{\sigma_0}^{(4)}(1,2,3,4)\rangle \tag{7.5}$$

(or its Hermitian conjugate) with $|v_{\sigma_0}^{(4)}\rangle$ given in (6.16b). In the simplified rule, the α- and p-conservation factors in $|v\rangle$ and $|v_{\sigma_0}^{(4)}\rangle$ should be discarded. Note that the factor $\frac{1}{3}$ and $\frac{1}{4}$ appearing in the action $S_{\rm int}=(2g/3)\phi^3+(2g^2/4)\phi^4$ were dropped in the rules (7.4) and (7.5). This is because they always drop out when the operators ϕ^3 and ϕ^4 are contracted with other fields or external legs as we will count the statistical weight explicitly below.

We should note the parallelism of these rules to those in the light-cone gauge string field theory.[21,22] In the latter, the L operator in the propagator (7.1) or (7.2) is replaced by

$$-p_+p_- + L_{\rm trans} . \tag{7.6}$$

Here $L_{\rm trans}$ is exactly the same operator as L if the OSp($d/2$) mode operators $(\alpha_n^\mu,\gamma_n,\bar{\gamma}_n)$ are all restricted to the O($d-2$) transverse one α_n^i alone. Quite the same is true also for 3-string and 4-string vertex operators (7.4) and (7.5) by the formal similarities of vertex structures between ours and light-cone gauge's.

From these observations, we can draw the following important conclusion: All the tree amplitudes in light-cone gauge string-field theory with the p_-'s of all external states set equal to zero, coincide with those in our covariant theory with all OSp($d/2$) operators of external strings reduced to O($d-2$) transverse ones. Some explanation would be necessary. If the p_-'s of all external states are set equal to zero, the p_-'s of all propagators also become zero in the case of *tree* diagrams. Therefore the propagator (7.6) in the light-cone gauge case reduces to $L_{\rm trans}$ and thus to an O($d-2$) version of the OSp($d/2$) symmetric one L of covariant theory. Therefore the calculation of tree amplitudes in both theories becomes quite the same formally. By the symmetries O($d-2$) and OSp($d/2$) in both theories, the amplitudes take the same form if they are written by using invariants under O($d-2$) and OSp($d/2$), respectively. Here we should recall the point mentioned in Sec. V A. Since the two-dimensional fermionic degrees of freedom work as a negative dimension,[44-46,50] the integration over (or equivalently contraction of) internal OSp($d/2$) variables yields the same result as that for O($d-2$) variables. Therefore, the numerical factors appearing in the calculations in both theories coincide with each other.

We think that this argument is sufficient to derive the above conclusion. Nevertheless we present explicit calculations of 4-string scattering amplitudes in the next subsection for completeness. One can convince oneself from those calculations that the above statement is true.

We should explain about the diagrams used in the following calculations. We can draw the usual Feynman diagrams also in this string field theory as shown in Fig. 21, and they are surely simple and helpful to count all the possibilities. They are, however, inconvenient for the explicit calculations of string amplitudes since they do not contain information about the intrinsic direction of each string, for instance. For various other reasons the so-called light-cone diagrams,[20-22] which we use also in our covariant theory, are much more useful.

As a preparation for the next subsection, it is convenient to summarize here the Mandelstam mapping[20] for

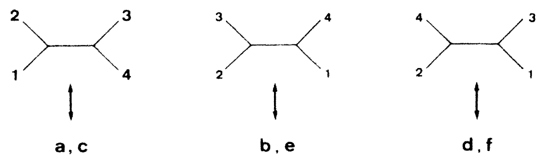


FIG. 21. Feynman diagrams for the 4-string amplitude and their correspondence to the light-cone diagrams in Fig. 22.

the 4-string scattering processes,[22] although we have already mentioned it briefly in Sec. V.

Mandelstam mapping gives a connection between the ρ plane of a light-cone diagram and the upper half complex z plane:

$$\rho(z)=\tau+i\sigma=\sum_{i=1}^{4}\alpha_i\ln(z-Z_i)\ , \qquad (7.7)$$

with string length parameters α_i satisfying $\sum_{i=1}^{4}\alpha_i=0$. Here the Z_i's are Koba-Nielsen variables which are real in the present open-string cases. With a given set of α_i's, each choice of a set of Z_i's corresponds to a light-cone diagram into which (7.7) maps the upper half-plane. However, the variables Z_i are not uniquely specified by the light-cone diagram, since we may combine the conformal transformation (7.7) with a one-to-one conformal transformation of the upper half-plane onto itself. The most general form of such a transformation is given by

$$z\to\frac{Az+B}{Cz+D} \qquad (7.8)$$

with real constants A, B, C, and D, which is called the projective (or Möbius) transformation.[37] [Since the multiplication of a common factor to A, B, C, and D does not change the mapping (7.8), it is conventional to require $AD-BC=1$.] Therefore, if all Z_i's are simultaneously transformed by (7.8), they remain to correspond to the same light-cone diagram as the original Z_i's.

If we define new variables by

$$\begin{aligned}\bar{z}&=\frac{z-Z_4}{z-Z_1}\,\frac{Z_2-Z_1}{Z_2-Z_4}\ ,\\ x&=\frac{Z_3-Z_4}{Z_3-Z_1}\,\frac{Z_2-Z_1}{Z_2-Z_4}\ ,\end{aligned} \qquad (7.9)$$

the Mandelstam mapping (7.7) is rewritten as

$$\rho(z)-T_0=\alpha_2\ln(1-\bar{z})+\alpha_3\ln(\bar{z}-x)+\alpha_4\ln\bar{z}\ , \qquad (7.10)$$

with

$$\begin{aligned}T_0=&-(\alpha_2+\alpha_3)\ln(Z_1-Z_4)\\ &-(\alpha_3+\alpha_4)\ln(Z_2-Z_1)-\alpha_1\ln(Z_2-Z_4)\\ &+\alpha_3\ln(Z_1-Z_3)\ .\end{aligned} \qquad (7.11)$$

By using the arbitrariness of the projective transformation (7.8) we can fix three of the four Z_i's to arbitrary constants. The mapping (7.10) corresponds to a particular choice

$$Z_1=\infty,\ \ Z_2=1,\ \ Z_3=x,\ \ Z_4=0\ , \qquad (7.12)$$

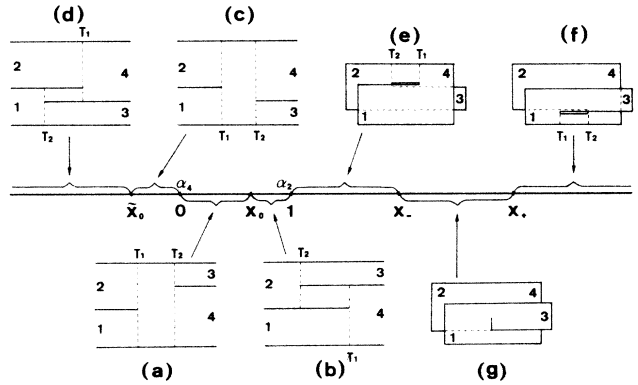


FIG. 22. Light-cone diagrams for the 4-string amplitude with $\alpha_1,\alpha_2>0$ and $\alpha_3,\alpha_4<0$ and the corresponding regions of the Koba-Nielsen variable $x=Z_3$ ($Z_1=\infty, Z_2=1, Z_4=0$).

up to an infinite time (T_0) translation. The correspondence between the value of x and the light-cone diagram now becomes one to one. The interaction points $z_\pm$ are solutions of $d\rho/dz=0$. The corresponding values $\bar{z}_\pm$ of $z_\pm$ are given by

$$\begin{aligned}\bar{z}_\pm=-(2\alpha_1)^{-1}[&\alpha_3+\alpha_4+(\alpha_2+\alpha_4)x\\ &\pm|\alpha_2+\alpha_4|\sqrt{(x-x_+)(x-x_-)}]\ ,\end{aligned} \qquad (7.13a)$$

$$x_\pm=(\alpha_2+\alpha_4)^{-2}[-(\alpha_2\alpha_3+\alpha_1\alpha_4)\pm2(\alpha_1\alpha_2\alpha_3\alpha_4)^{1/2}]\ . \qquad (7.13b)$$

In the following we confine ourselves to the case $\alpha_1,\alpha_2>0$ and $\alpha_3,\alpha_4<0$, in which case real roots $x_\pm$ exist in the region $1<x_-<x_+<\infty$. When $x\notin[x_-,x_+]$, the corresponding light-cone diagram consists of two 3-string vertices with interaction times

$$T_1=\mathrm{Re}\rho(z_-),\ \ T_2=\mathrm{Re}\rho(z_+)\ , \qquad (7.14)$$

and the difference $T=T_2-T_1$ represents the time interval of the propagation of an intermediate string. T is of course a projective-invariant quantity.

Outside the interval $[x_-,x_+]$, we have two points of x realizing $T=0$. We denote the larger one by x_0 which resides in $0<x_0<1$ and the smaller one by $\tilde{x}_0<0$. With points $x_\pm$, 1, x_0, 0, and $\tilde{x}_0$, the region of x outside the interval $[x_-,x_+]$ is divided into six intervals which are mapped into six types of light-cone diagrams as shown in Fig. 22.

On the other hand, when x takes the value inside the interval $[x_-,x_+]$, the two interaction points $\bar{z}_\pm$ in (7.13) become complex such that $z_-=(z_+)^*$. This case corresponds to the light-cone diagram possessing a 4-string vertex as shown in Fig. 22(g).

B. 4-string scattering amplitude

We now calculate the 4-string amplitude explicitly in this covariant theory. For definiteness we take $\alpha_1,\alpha_2>0$ and $\alpha_3,\alpha_4<0$ in the following calculation, though we will see in the next subsection that the on-shell physical amplitudes are independent of the choice of α's, at least, at the tree level.

First let us count the possible types of s-channel diagrams and their statistical weights, in which the grouping of "initial" states $(\varphi(1),\varphi(2))$ and "final" states $(\varphi(3),\varphi(4))$ is fixed. As usual, this is done by counting all the possibilities of how to contract the four external states $\varphi(r)$ $(r=1-4)$ with the second-order Dyson operator $\frac{1}{2}(\frac{2}{3}g\phi^3)(\frac{2}{3}g\phi^3)$. First of all, $\frac{1}{2}$ is canceled by the two possibilities of choosing one operator $\frac{2}{3}g\phi^3$ to be contracted with the initial states. Next, for a chosen operator $\frac{2}{3}g\phi^3$ for the initial side, there are three possibilities which ϕ of ϕ^3 is contracted with $\varphi(1)$, and also for the final side $\frac{2}{3}g\phi^3$ we have a factor 3 corresponding to which ϕ is contracted with $\varphi(3)$. At this stage the Dyson operator looks like $[2g\varphi(1)\phi^2][2g\varphi(3)\phi^2]$, showing that the coupling constant at each vertex becomes $2g$ as announced before. From this stage, there are 2×2 possibilities depending on which ϕ of $2g\varphi(1)\phi^2$ $[2g\varphi(3)\phi^2]$ is contracted with $\varphi(2)$ $[\varphi(4)]$, but they yield four light-cone diagrams different from each other as shown in Fig. 23. Comparing with Fig. 22, we see that diagrams A and B of Fig. 23 are the same as (a) and (c) in Fig. 22. Diagrams C and D in Fig. 23 are obtained from A and B by exchanging 1 and 2, and are missing in Fig. 22. This is the case also for t- and u-channel diagrams; the diagrams with 1 and 2 exchanged are missing. Therefore, the full amplitude is obtained first by summing all the contributions of the diagrams in Fig. 22 and finally by adding its answer and the same one with 1 and 2 exchanged.

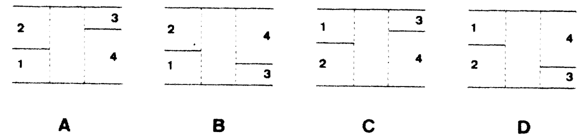


FIG. 23. All the possible s-channel diagrams.

With this understanding, let us start with diagram (a) in Fig. 22. Its contribution to the transition amplitude $\mathscr{T}$ is given by

$$\mathscr{T}_{1a}=(2g)^2\int d5\,d6\,\varphi^\dagger(4)\varphi^\dagger(3)v(435)\left[-\frac{1}{2L}\right]_{56}v^*(1,2,6)\varphi(2)\varphi(1)\ , \tag{7.15}$$

in a rough notation, or, more precisely, by

$$\mathscr{T}_{1a}=\tfrac{1}{2}(2g)^2\mathrm{tr}\langle\varphi(1)|\,\langle\varphi(2)|\,\langle\varphi(3)|\,\langle\varphi(4)|\int_0^\infty d\tau\,|\Delta_\tau(1,2,3,4)\rangle\ , \tag{7.16}$$

$$|\Delta_\tau(1,2,3,4)\rangle=\int d3'd4'd6\langle v(3',4',6)|\,\Omega^{(6)}e^{\tau L^{(6)}}\,|v(1,2,6)\rangle\,|R(3',3)\rangle\,|R(4',4)\rangle \tag{7.17a}$$

$$=\int d5\,d6\langle R(5,6)|\,\Omega^{(6)}e^{\tau L^{(6)}}\,|v(1,2,6)\rangle\,|v(3,4,5)\rangle\ , \tag{7.17b}$$

with $\langle R(1,2)|$ now understood to be the R operator (2.16) with the $\overline{c}_0$ mode omitted. Here we have used the propagator (7.2), the Hermiticity of the ϕ^3 term (6.15b), and also the trick

$$-\frac{1}{L}=\int_0^\infty d\tau\,e^{\tau L}\ . \tag{7.18}$$

The effective 4-string vertex $|\Delta_\tau(1,2,3,4)\rangle$ introduced in (7.17) is quite an analogous quantity to the previous

$$|\Delta(3,4,1,2)\rangle=|\Delta(1,2,3,4)\rangle$$

defined in (5.10) in the nilpotency proof of $(\delta_B^1)^2$ in Sec. V. The only difference is the presence of a time-development operator $e^{\tau L}$ in (7.17) and so the previous $|\Delta(1,2,3,4)\rangle$ is nothing but $|\Delta_\tau(1,2,3,4)\rangle$ at $\tau=0$ (aside from a $\overline{c}_0$-mode factor, of course). Just similar to Eq. (5.20), the effective 4-string vertex $|\Delta_\tau\rangle$ should be proportional to the following vertex $|v_T^{(4)}\rangle$ given in terms of the Neumann function corresponding to the light-cone diagram of Fig. 22(a):

$$|v_T^{(4)}(1,2,3,4)\rangle=e^{F_T^{(4)}(1,2,3,4)}|0\rangle\delta(\alpha,p)\ ,$$

$$F_T^{(4)}(1,2,3,4)=\sum_{r,s=1}^{4}\left[\frac{1}{2}\sum_{m,n\geq 0}\alpha_{-m}^{(r)}\cdot\overline{N}_{mn}^{(4)rs}\alpha_{-n}^{(s)}+i\sum_{m,n\geq 1}\gamma_{-m}^{(r)}\overline{N}_{mn}^{(4)rs}\overline{\gamma}_{-n}^{(s)}\right]\ , \tag{7.19}$$

$$\delta(\alpha,p)=(2\pi)^{d+1}\delta\left[\sum_{r=1}^{4}p_r\right]\delta\left[\sum_{r=1}^{4}\alpha_r\right]\ ,$$

where we have introduced a variable

$$T=|\alpha_6|\tau\ , \tag{7.20}$$

implying the time interval $T_2-T_1=T$ in Fig. 22(a). (A definition of $\overline{N}_{mn}^{(N)rs}$ is given in Appendix A for general N-string light-cone diagrams.) We show this fact generally for an N-string effective vertex $|\Delta^{(N)}\rangle$ in Appendix H. The proportionality factor can be determined by using the formula (5.12) and the form (6.17b) of the 3-string vertex operator $|v(1,2,3)\rangle$ in the same way as in Sec. V, and we find

$$| \Delta_T(1,2,3,4)\rangle = \mu(\alpha_3,\alpha_4,\alpha_5)\mu(\alpha_1,\alpha_2,\alpha_6)e^{T/\alpha_5}[\det(1-\tilde{N}^{55}_{T_2}\tilde{N}^{66}_{T_1})]^{-(d-2)/2} | v_T^{(4)}(1-4)\rangle ,$$
$$(\tilde{N}^{55}_{T_2})_{mn} = (-)^m\sqrt{m}\,\bar{N}^{55}_{mn}\sqrt{n}\,(-)^n e^{-[(m+n)/\alpha_5]T_2} , \tag{7.21}$$
$$(\tilde{N}^{66}_{T_1})_{mn} = \sqrt{m}\,\bar{N}^{66}_{mn}\sqrt{n}\,e^{-[(m+n)/\alpha_6]T_1}, \quad \alpha_5=-\alpha_6=\alpha_1+\alpha_2>0 .$$

Here, to obtain a symmetrical expression, we have calculated this by dividing the time-development operator $e^{\tau L}=e^{(T_2-T_1)L/|\alpha_6|}$ into two parts e^{T_2L/α_5} and e^{T_1L/α_6} and by operating them on the final and initial vertices, respectively. However, of course, the determinant factor $\det(1-\tilde{N}^{55}_{T_2}\tilde{N}^{66}_{T_1})$ depends only on $T=T_2-T_1$ and we may take $T_2=T, T_1=0$ or $T_2=0, T_1=-T$, for instance. The factor $e^{T/\alpha_5}=e^{\tau}$ comes from the constant part $(+1)$ of L in $e^{\tau L}$:

$$L=-\tfrac{1}{2}p^2-\sum_{n=1}^{\infty}[\alpha_{-n}\cdot\alpha_n+n(c_{-n}\bar{c}_n+\bar{c}_{-n}c_n)]+1 . \tag{7.22}$$

Now the amplitude of Fig. 22(a) is found by (7.16), (7.20), and (7.21) to be

$$\mathcal{T}_{1a}=\tfrac{1}{2}(2g)^2\int_0^{\infty}dT\frac{1}{\alpha_5}\det(1-\tilde{N}^{55}_{T_2}\tilde{N}^{66}_{T_1})]^{-(d-2)/2}e^{T/\alpha_5}\mu(\alpha_3,\alpha_4,\alpha_5)\mu(\alpha_1,\alpha_2,\alpha_6)\mathrm{tr}\langle \mathrm{ext}(1,2,3,4)\,|\,v_T^{(4)}(1-4)\rangle , \tag{7.23}$$

with an abbreviation

$$\langle \mathrm{ext}(1,2,3,4)| \equiv \langle\varphi(1)|\langle\varphi(2)|\langle\varphi(3)|\langle\varphi(4)| . \tag{7.24}$$

As anticipated, this result is exactly the same as that in the light-cone gauge string field theory[22] if we simply discard the longitudinal and scalar modes $\alpha^{(r)\mu=\pm}_{-m}$ as well as ghost modes γ_{-m} and $\bar{\gamma}_{-m}$ in $|v_T^{(4)}\rangle$. Indeed we have already cited this form of result for light-cone gauge string field theory in (5.27) or (5.28) in Sec. V for slightly different diagrams.

As mentioned before in (5.31), we have Cremmer-Gervais's identity:[22]

$$\frac{|dT|}{|\alpha_5|}[\det(1-\tilde{N}^{55}_{T_2}\tilde{N}^{66}_{T_1})]^{-12}e^{T/\alpha_5}\exp\left[-\sum_{r=1,2,6}\frac{\tau_0(\alpha_1,\alpha_2,\alpha_6)}{\alpha_r}-\sum_{r=3,4,5}\frac{\tau_0(\alpha_3,\alpha_4,\alpha_5)}{\alpha_r}\right]$$
$$=\left|\frac{\prod_{i=1}^{4}dZ_i}{dV_{abc}}\right|\exp\left[-\sum_{r=1}^{4}\bar{N}^{(4)rr}_{00}\right] \tag{7.25}$$

for the regions $T\geq 0$ in the LHS and $0\leq x\leq x_0$ in the RHS, respectively. [Incidentally, the factors $(-)^m(-)^n$ in the definition (7.21) of $(\tilde{N}^{55}_{T_2})_{mn}$ are missing in the original equation of Cremmer and Gervais, but the above equation is correct since they should have a twist factor necessary for the propagator in their definition.] Therefore the contribution of the diagram Fig. 22(a) with $0\leq T<\infty$ is written in terms of Koba-Nielsen variables as

$$\mathcal{T}_{1a}=\tfrac{1}{2}(2g)^2\int_{0\leq x\leq x_0}\frac{\prod_i dZ_i}{dV_{abc}}\exp\left[-\sum_r\bar{N}^{(4)rr}_{00}\right]\mathrm{tr}\langle \mathrm{ext}(1,2,3,4)\,|\,v_T^{(4)}(1-4)\rangle . \tag{7.26}$$

It is possible to rewrite it into a more familiar looking form. By extracting only the purely zero-mode parts $\tfrac{1}{2}p_r\cdot p_s\bar{N}^{(4)rs}_{00}$ from $e^{F_T^{(4)}}|0\rangle$ in (7.19) and using the expression

$$\bar{N}^{(4)rs}_{00}=(1-\delta_{rs})\ln|Z_r-Z_s|+\delta_{rs}\left[-\sum_{i(\neq r)}\frac{\alpha_i}{\alpha_r}\ln|Z_r-Z_i|+\frac{1}{\alpha_r}\tau_0^{(r)}\right] , \tag{7.27}$$

given in (A12) in Appendix A, we obtain

$$\mathcal{T}_{1a}=(2\pi)^{d+1}\delta\left[\sum_r\alpha_r\right]\delta\left[\sum_r p_r\right]\tfrac{1}{2}(2g)^2$$
$$\times\int_{0\leq x\leq x_0}\frac{\prod_i dZ_i}{dV_{abc}}\prod_{i<j}|Z_i-Z_j|^{p_i\cdot p_j+(\alpha_j/\alpha_i)[1-(1/2)p_i^2]+(\alpha_i/\alpha_j)[1-(1/2)p_j^2]}$$
$$\times\exp\left[\sum_r\frac{\tau_0^{(r)}}{\alpha_r}(\tfrac{1}{2}p_r^2-1)\right]\mathrm{tr}\langle \mathrm{ext}(1,2,3,4)\,|\,e^{F_T'^{(4)}(1,2,3,4)}|0\rangle . \tag{7.28}$$

Here $\tau_0^{(r)}$ denotes the interaction time of string r defined in Appendix A [explicitly, $\tau_0^{(1)}=\tau_0^{(2)}=T_1$ and $\tau_0^{(3)}=\tau_0^{(4)}=T_2$ in this case of Fig. 22(a)], and the prime of $F_T'^{(4)}$ means the omission of $m=n=0$ terms from $\sum \frac{1}{2}\alpha_{-m}^{(r)}\bar{N}_{mn}^{(4)rs}\alpha_{-n}^{(s)}$ in the definition (7.19) of $F_T^{(4)}$.

In quite the same way, we can calculate the contribution $\mathcal{T}_{1b}$ of Fig. 22(b) and obtain the same form of amplitude as (7.28) with the integration region replaced by $x_0\le x\le 1$. Therefore the sum $\mathcal{T}_{1a}+\mathcal{T}_{1b}$ correctly gives the complete s-t dual amplitude with integration over $0\le x\le 1$; indeed, for instance, the amplitude (7.28) with $0\le x\le 1$ just gives the well-known Veneziano amplitude[51] if the four external states are on-shell ground states $(\frac{1}{2}p_r{}^2=1)$, since the last factor $\mathrm{tr}\langle \mathrm{ext}|e^{F_T'^{(4)}}|0\rangle$ becomes simply the Chan-Paton factor[38] $\mathrm{tr}(\lambda_1\lambda_2\lambda_3\lambda_4)$ in such case.

Similarly, the diagrams of Figs. 22(d) and 22(c) give the s-u dual amplitude of the same form as (7.28) with integration region over $-\infty < x\le 0$ and the external states ordered as $\langle \mathrm{ext}(1,2,4,3)|$.

For the case of the t-u amplitude, however, we need the quartic string interaction term for the first time. Figures 22(e) and 22(f), which are constructed by using the usual cubic interaction term twice, give only the amplitude of the same form as (7.28) with regions $1\le x\le x_-$ and $x_+\le x<+\infty$ [and the external states ordered as $\langle \mathrm{ext}(1,3,2,4)|$]. So the region $x_-\le x\le x_+$ is missing for the full t-u dual amplitude to be reproduced correctly. It is just this region that necessitates the existence of a quartic interaction term as has long been known in the light-cone gauge string field theory.[21,22] For completeness, let us calculate the contribution of Fig. 22(g) possessing a quartic interaction vertex and confirm that it actually fills the missing region with correct weight.

First count the weight of Fig. 22(g). There are four possibilities in choosing one ϕ from $S_{\mathrm{int}}^{\mathrm{quartic}}=\frac{2}{4}g^2\phi^4$ to be contracted with one external state, say $\varphi(1)$. Once this is fixed, the rest is unique since the order of external strings are fixed in Fig. 22(g) as $\varphi(1)\varphi(3)\varphi(2)\varphi(4)$. Therefore the amplitude of Fig. 22(g) is given by

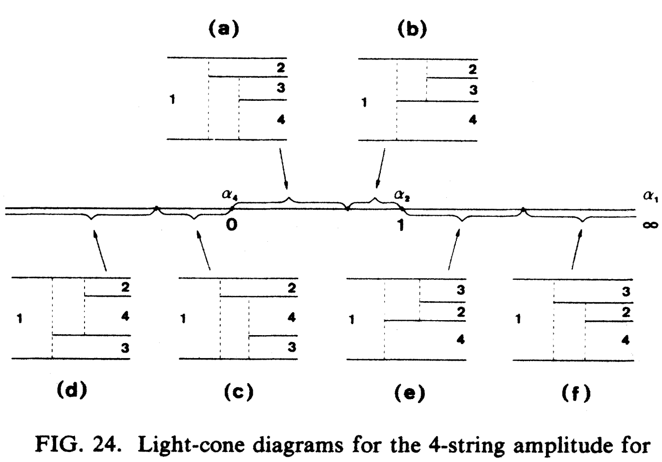


FIG. 24. Light-cone diagrams for the 4-string amplitude for the case $\alpha_1>0$ and $\alpha_2,\alpha_3,\alpha_4<0$ and the corresponding regions of x.

$$\mathcal{T}_{1g}=4(\tfrac{2}{4}g^2)\int_{\sigma_-}^{\sigma_+} d\sigma_0 f(\sigma_0) \times \mathrm{tr}\langle \mathrm{ext}(1,3,2,4)|v_{\sigma_0}^{(4)}(1-4)\rangle \qquad (7.29)$$

from (6.15c) and (6.16b). Here we have pulled the measure $f(\sigma_0)$ out of $|v_{\sigma_0}^{(4)}\rangle$ in (6.16b) and so $|v_{\sigma_0}^{(4)}\rangle$ in this equation should be understood as

$$|v_{\sigma_0}^{(4)}(1-4)\rangle=(2\pi)^{d+1}\delta\left[\sum\alpha_r\right]\delta\left[\sum p_r\right] \times e^{F_{\sigma_0}^{(4)}(1-4)}|0\rangle\ . \qquad (7.30)$$

If we use the measure $f(\sigma_0)$ determined in (4.29), Eq. (7.29) becomes

$$\mathcal{T}_{1g}=4(\tfrac{2}{4}g^2) \times \int_{x_-\le x\le x_+}\frac{\prod_i dZ_i}{dV_{abc}}\exp\left[-\sum_r \bar{N}_{00}^{(4)rr}\right] \times \mathrm{tr}\langle \mathrm{ext}(1,3,2,4)|v_{\sigma_0}^{(4)}(1-4)\rangle\ , \qquad (7.31)$$

which has just the same form as (7.26) with correct weight $4\times\frac{2}{4}g^2=\frac{1}{2}(2g)^2$ and hence fills the missing integration region of t-u amplitude exactly.

Up to now we have assumed $\alpha_1,\alpha_2>0$, $\alpha_3,\alpha_4<0$. We can calculate the amplitudes also for other cases quite similarly. In Fig. 24 we have given light-cone diagrams for the case $\alpha_1>0$, $\alpha_2,\alpha_3,\alpha_4<0$ as an example. (Again they cover only half of the possible diagrams and the other half can be obtained by exchanging 2 and 4 in those diagrams.) For any set of α's we obtain the same form of dual amplitudes as obtained above. The only difference resides in the point that the Neumann functions appearing in those equations are defined graph by graph and depend on the set of α's. Nevertheless we will see in the next subsection that the amplitudes become independent of the α's if the external states are set on the mass shell and are chosen to be physical modes.

As we have seen explicitly, all the above calculations are quite the same as in light-cone gauge string field theory. This property persists in any general N-string tree amplitudes as explained in the previous subsection. Therefore if we know the results in the light-cone gauge string field theory, we can immediately write down our answer; the translation rule is to set $p_-=0$ and to replace the $\mathrm{O}(d-2)$ invariants $\mathbf{p}_r\cdot\mathbf{p}_s$ and $\boldsymbol{\alpha}_m^{(r)}\cdot\boldsymbol{\alpha}_n^{(s)}$ by $\mathrm{OSp}(d/2)$ invariants $p_r\cdot p_s$ and $\alpha_m^{(r)}\cdot\alpha_n^{(s)}+2i\gamma_m^{(r)}\bar{\gamma}_n^{(s)}$. As a matter of fact no one has ever calculated the general N-string amplitudes directly in oscillator language as Cremmer and Gervais performed for the 4-string case.[22] However Mandelstam[20,48] has calculated the amplitude based on the path-integral technique and obtained a general formula for the N-string tree amplitude in the light-cone gauge string field theory. Translating his formula into our language, we find the general N-string tree amplitude in our theory:

$$\mathscr{T}_N=(2\pi)^{d+1}\delta\left[\sum_r \alpha_r\right]\delta\left[\sum_r p_r\right](\tfrac{1}{2})^{N-3}(2g)^{N-2}$$

$$\times\int\frac{\prod_{i=1}^{N} dZ_i}{dV_{abc}}\prod_{i<j}|Z_i-Z_j|^{p_i\cdot p_j+(\alpha_j/\alpha_i)[1-(1/2)p_i{}^2]+(\alpha_i/\alpha_j)[1-(1/2)p_j{}^2]}\exp\left[\sum_r\frac{\tau_0^{(r)}}{\alpha_r}(\tfrac{1}{2}p_r{}^2-1)\right]$$

$$\times\mathrm{tr}\left\langle \mathrm{ext}(1-N)\left|\exp\left[\tfrac{1}{2}\sum_{\substack{m,n\geq 0\\ r,s}}{}' \alpha_{-m}^{(r)}\cdot\bar{N}_{mn}^{rs}\alpha_{-n}^{(s)}+i\sum_{\substack{m,n\geq 1\\ r,s}}\gamma_{-m}^{(r)}\bar{N}_{mn}^{rs}\bar{\gamma}_{-n}^{(s)}\right]\right|0\right\rangle, \tag{7.32}$$

with the prime on $\sum$ denoting the omission of $m=n=0$ terms.

C. α independence of on-shell physical tree amplitudes

The on-shell amplitudes in the light-cone gauge string field theory should be Lorentz invariant, and this Lorentz invariance is actually confirmed in the case of tree amplitudes.[20] This implies in particular that the $\alpha=(2p_+)$ dependence of on-shell tree amplitudes are only through the Lorentz invariant $\alpha_r p_{-s}+p_{-r}\alpha_s+2\mathbf{p}_r\cdot\mathbf{p}_s$ despite the fact that the Neumann functions (or the vertices) depended very nontrivially on the α's. Therefore, if the p_-'s are set equal to zero, the tree amplitudes no longer depend on the α's except for the conservation factor $\delta(\sum\alpha_r)$. From the above-mentioned correspondence between our covariant theory and the light-cone gauge theory, it is quite natural to expect that the on-shell physical tree amplitudes in the covariant theory are also independent of the α's. We show in this subsection that this is really the case, although the proof is not so trivial since the physical modes in our theory are not given by simple transversal modes but by somewhat complicated DiVecchia—Del Guidice—Fubini (DDF) modes.[52]

Now we start with the identification of physical modes in our theory, According to Kugo and Ojima[10,11] the physical states are specified by the subsidiary condition

$$\mathrm{Q}_B||\mathrm{phys}\rangle\rangle=0 \tag{7.33}$$

in this string field theory, where Q_B is the second quantized BRS charge which generates the BRS transformation on our string field ϕ:

$$[\mathrm{Q}_B,\phi]=\hat{\delta}_B\phi=\tilde{Q}_B\phi+g\phi_*\phi+g^2\phi\circ\phi\circ\phi\ . \tag{7.34}$$

In our scattering theory we need only state vector space constructed as a Fock space of asymptotic string states. Then it is generally known[11] that the physical state space defined by (7.33) is given by a Fock space constructed by asymptotic physical *one-string* (or one-particle) creation operators, aside from unimportant zero-norm states. Note that (7.34) yields

$$[\mathrm{Q}_B,\phi^{\mathrm{as}}]=\tilde{Q}_B\phi^{\mathrm{as}} \tag{7.35}$$

for asymptotic string field ϕ^{as}. Therefore the physical states are spanned by one-string creation operators corresponding to the physical modes $|\,\mathrm{phys}\rangle$ defined by

$$\tilde{Q}_B\,|\,\mathrm{phys}\rangle=0\ . \tag{7.36}$$

[Note the difference between $||\mathrm{phys}\rangle\rangle$ in (7.33) and $|\,\mathrm{phys}\rangle$ here. The latter is a one-string wave function while the former is a many-string state.] This Eq. (7.36) is just the physical state condition imposed by Kato and Ogawa in their first quantization formulation.[7] As was proved by them, the physical modes $|\,\mathrm{phys}\rangle$ are spanned by DDF modes alone, again apart from zero-norm modes.

As is well known the DDF operators A_m^i $(i=1,2,\ldots,d-2)$ are constructed as[52]

$$\begin{aligned}A_m^i&=\oint\frac{dz}{2\pi i}z^{m-1}P^i(z)\\&\quad\times\exp\left[-\frac{m}{p_+}\sum_{n\neq 0}\frac{1}{n}\alpha_{n,+}z^{-n}\right],\\P^\mu(z)&=\sum_n\alpha_n^\mu z^{-n}\ .\end{aligned} \tag{7.37}$$

Here and henceforth we use the indices $(\pm,i)$ to specify the components of arbitrary vector v_μ:

$$\begin{aligned}&v_\pm=\frac{1}{\sqrt{2}}(v_{25}\pm v_0)\ ,\\&v_i\ (i=1,2,\ldots,24)\ ;\ \text{transversal components}\ .\end{aligned} \tag{7.38}$$

With these DDF operators (7.37), the wave function $|\,\varphi(1)\rangle$ of a physical one-string state is generally given in the form

$$|\,\varphi(1)=\sum a_{m_1\cdots m_n}^{i_1\cdots i_n}(p)A_{-m_1}^{i_1}A_{-m_2}^{i_2}\cdots A_{-m_n}^{i_n}\,|\,0\rangle\ . \tag{7.39}$$

Here the coefficients $a_{m_1\cdots m_n}^{i_1\cdots i_n}(p)$ specify the polarization and matrix of the state. The on-shell condition $L\,|\,\varphi(1)\rangle=0$ imposes a constraint

$$\tfrac{1}{2}p^2=1-\sum_k m_k \tag{7.40}$$

on the momentum p_μ.

Now that we have identified the wave functions of the on-shell physical states, we turn to our main subject to prove the α independence of on-shell physical amplitudes. For that purpose let us first rewrite the previously obtained general N-string tree amplitude (7.32) by introducing new coefficient N^{rs}_{mn} defined by

$$N^{rs}_{mn}=\bar{N}^{rs}_{mn}\exp(-n\bar{N}^{rr}_{00}-n\bar{N}^{ss}_{00})\ . \tag{7.41}$$

If one notices the relation

$$e^{L^{(r)}\tau_r}f(\alpha^{(r)}_{-n})\,|\,0\rangle=f(\alpha^{(r)}_{-n}e^{-n\tau_r})e^{(1-p_r{}^2/2)\tau_r}\,|\,0\rangle\ , \tag{7.42}$$

it is easy to see that the choice $\tau_r=\bar{N}^{rr}_{00}$ leads to the following expression for the N-string amplitude (7.32):

$$\mathcal{T}_N=(2\pi)^{d+1}\delta\left[\sum_r\alpha_r\right]\delta\left[\sum_r p_r\right]2g^{N-2}\int\frac{\prod_{r=1}^{N}dZ_r}{dV_{abc}}\prod_{r<s}|\,Z_r-Z_s\,|^{p_r\cdot p_s}\mathcal{M}_N\ , \tag{7.43}$$

$$\mathcal{M}_N=\mathrm{tr}\Big\langle \mathrm{ext}(1-N)\Big|\prod_{r<s}|\,Z_r-Z_s\,|^{(\alpha_s/\alpha_r)L^{(r)}+(\alpha_r/\alpha_s)L^{(s)}}\exp\left[-\sum_r\frac{\tau_0^{(r)}}{\alpha_r}L^{(r)}\right]$$
$$\times\exp\left[\tfrac{1}{2}\sum_{\substack{m,n\geq 0\\ r,s}}{}'\,\alpha^{(r)}_{-m}N^{rs}_{mn}\alpha^{(s)}_{-n}+i\sum_{\substack{m,n\geq 1\\ r,s}}\gamma^{(r)}_{-m}N^{rs}_{mn}\bar{\gamma}^{(s)}_{-n}\right]\Big|0\Big\rangle\ . \tag{7.44}$$

If the external string states $\langle\varphi(r)\,|$ in $\langle\mathrm{ext}(1-N)\,|$ are all on-shell and physical, then the $L^{(r)}$'s in (7.44) can be set equal to zero and further the ghost operator parts $i\gamma N\bar{\gamma}$ can be discarded since the DDF operators contain no ghost oscillators. Thus, for on-shell physical amplitude $\mathcal{T}_N$, the matrix element $\mathcal{M}_N$ reduces simply to

$$\mathcal{M}_N=\mathrm{tr}\langle\mathrm{ext}(1-N)\,|\,e^{F'_X}\,|\,0\rangle\ , \tag{7.45}$$

$$F'_X\equiv\tfrac{1}{2}\sum_{\substack{m,n\geq 0\\ r,s}}{}'\,\alpha^{(r)}_{-m}N^{rs}_{mn}\alpha^{(s)}_{-n}\ . \tag{7.46}$$

Now the α dependence of the N-string amplitude $\mathcal{T}_N$ is contained only in N^{rs}_{mn} of (7.46) aside from a trivial conservation factor $\delta(\sum\alpha_r)$. We now show that $\mathcal{M}_N$ is in fact independent of the α_r's. So let us examine how $\mathcal{M}_N$, (7.46), changes when we vary the α_i's keeping their total conservation $\sum_r\alpha_r=0$:

$$\delta\mathcal{M}_N=\mathrm{tr}\Big\langle\mathrm{ext}(1-N)\Big|\tfrac{1}{2}\sum_{\substack{m,n\geq 0\\ r,s}}{}'\,\alpha^{(r)}_{-m}\cdot\alpha^{(s)}_{-n}\delta N^{rs}_{mn}e^{F'_X}\Big|0\Big\rangle\ . \tag{7.47}$$

For this change of the new coefficient δN^{rs}_{mn} we have the following identity whose proof is given in Appendix I:

$$\delta N^{rs}_{mn}=-\sum_{i=1}^{N}\delta\alpha_i\left[\delta_{rs}\frac{1}{\alpha_r}N^{ri}_{m+n,0}+\frac{1}{\alpha_i}\sum_{k=1}^{m-1}(m-k)N^{rs}_{m-k,n}N^{ri}_{k0}+\frac{1}{\alpha_s}\sum_{k=1}^{n-1}(n-k)N^{rs}_{m,n-k}N^{si}_{k0}\right]\ ,$$
$$\sum_{i=1}^{N}\delta\alpha_i=0\ . \tag{7.48}$$

(The same identity was in fact used by Mandelstam in his Lorentz-invariance proof in the light-cone gauge string theory.[53]) Because of (7.48), Eq. (7.47) turns out to be written in the form

$$\delta\mathcal{M}_N=\mathrm{tr}\Big\langle\mathrm{ext}(1-N)\Big|\sum_{\substack{n\geq 1\\ r}}\delta f^r_n\{\tilde{Q}^{(r)}_B,\bar{c}^{(r)}_{-n}\}e^{F'_X}\Big|0\Big\rangle\ . \tag{7.49}$$

This is shown as follows. First note that the BRS transformation of the antighost $\bar{c}^{(r)}_{-n}$ is given by

$$\{\tilde{Q}^{(r)}_B,\bar{c}^{(r)}_{-n}\}=-\tfrac{1}{2}\sum_m\alpha^{(r)}_{-n+m}\cdot\alpha^{(r)}_{-m}+(\text{ghost terms})\ , \tag{7.50}$$

but the ghost terms do not contribute to (7.49) since both the physical external state $\langle\mathrm{ext}\,|$ and $e^{F'_X}\,|\,0\rangle$ contain no ghost modes. Second, the first term on the RHS of (7.50) can be rewritten in front of $e^{F'_X}\,|\,0\rangle$ into the form containing no annihilation operators $\alpha^{(r)}_m$ ($m\geq 1$) by using (7.46); that is, $\sum\delta f^r_n\{\tilde{Q}_B,\bar{c}^{(r)}_{-n}\}e^{F'_X}\,|\,0\rangle$ in (7.49) is evaluated as

$$\sum_{\substack{n\geq 1\\ r}} \delta f_n^r \left[-\tfrac{1}{2}\sum_{m=0}^{n} \alpha_{-n+m}^{(r)}\cdot\alpha_{-m}^{(r)} - \sum_{m=1}^{\infty} \alpha_{-(n+m)}^{(r)}\cdot\alpha_m^{(r)} \right] e^{F'_X}|0\rangle$$

$$= \sum_{\substack{n\geq 1\\ r}} \delta f_n^r \left[-\tfrac{1}{2}\sum_{m=0}^{n} \alpha_{-n+m}^{(r)}\cdot\alpha_{-m}^{(r)} - \sum_{m\geq 1}\sum_{\substack{l\geq 0\\ s}} m N_{ml}^{rs}\alpha_{-(n+m)}^{(r)}\cdot\alpha_{-l}^{(s)} \right] e^{F'_X}|0\rangle$$

$$= \tfrac{1}{2}\sum_{\substack{m,n\geq 0\\ r,s}}{}' \alpha_{-m}^{(r)}\cdot\alpha_{-n}^{(s)} \left[-\delta f_{m+n}^r \delta_{rs} - \sum_{k=1}^{m-1} \delta f_k^r (m-k) N_{m-k,n}^{rs} - \sum_{k=1}^{n-1} \delta f_k^s (n-k) N_{m,n-k}^{rs} \right] e^{F'_X}|0\rangle \ . \tag{7.51}$$

Comparing this with Eq. (7.48), we find that (7.49) actually coincides with (7.47) if we take the coefficients δf_n^r in (7.49) as

$$\delta f_n^r = \sum_{i=1}^{N} \delta\alpha_i \frac{1}{\alpha_r} N_{n0}^{ri} \ . \tag{7.52}$$

Thus we have proved Eq. (7.49). Now recall that the physical states are annihilated by $\tilde{Q}_B$ as was seen in (7.36) and contain no ghost modes. Hence the operation of $\{\tilde{Q}_B^{(r)}, \bar{c}_{-n}^{(r)}\}$ on the physical external state $\langle \mathrm{ext}(1-N)|$ yields zero, and so Eq. (7.49) proves the desired α independence of $\mathcal{M}_N$:

$$\delta\mathcal{M}_N = 0 \ . \tag{7.53}$$

Therefore, in view of (7.43), we see that the general N-string tree amplitudes $\mathcal{T}_N$ are independent of the choice of α_r's except the total conservation factor $\delta(\sum \alpha_r)$ if the external string states are on-shell and physical.

Incidentally, we can now easily understand that our on-shell physical amplitudes $\mathcal{T}_N$ really coincide with those in the light-cone gauge string field theory. Indeed, although the string-length parameters α_r are identified with $2p_+^r$ in the light-cone gauge case, we can now take

$$\alpha_r = 2p_+^r \tag{7.54}$$

also in our covariant theory since the on-shell physical amplitudes are independent of the choice of α_r's. Hence the only apparent difference is that the amplitude $\mathcal{M}_N$ in (7.45) in our case contains the full covariant oscillator $\alpha_{-n}^{(r)\mu}$ in $e^{F'_X}|0\rangle$ and the complicated DDF operators A_{-n}^i in the external physical state $\langle \mathrm{ext}|$, while both of them are simply transverse operators α_{-n}^i in the light-cone gauge case. In order to see that this difference also disappears, we decompose our exponent factor F'_X in (7.46) into the transverse part and others:

$$F'_X = \tfrac{1}{2}\sum_{\substack{m,n\geq 0\\ r,s}}{}' \alpha_{-m}^{(r)\mu} N_{mn}^{rs} \alpha_{-n\mu}^{(s)}$$
$$= \tfrac{1}{2}\sum_{m,n\geq 0}{}' \alpha_{-m}^{(r)i} N_{mn}^{rs} \alpha_{-n}^{(s)i} + \sum_{m,n\geq 1} \alpha_{-m,+}^{(r)} N_{mn}^{rs} \alpha_{-n,-}^{(s)}$$
$$+ \sum_{m\geq 1} \alpha_{-m,+}^{(r)} N_{m0}^{rs} p_-^s + \sum_{m\geq 1} \alpha_{-m,-}^{(r)} N_{m0}^{rs} p_+^s \ . \tag{7.55}$$

The important point is that the last term vanishes when $p_+^r \propto \alpha_r$ as we are choosing now in (7.54), owing to the identity (A20) in Appendix A,

$$\sum_{s=1}^{N} N_{n0}^{rs} p_+^s \propto \sum_{s=1}^{N} e^{-n\bar{N}_{00}^{rr}} \bar{N}_{n0}^{rs} \alpha_s = 0 \ . \tag{7.56}$$

Noticing that the DDF operators A_m^i defined in (7.37) consist of α^i and α_+ oscillators alone, we see that the second and third terms in (7.55) can also be dropped out in the calculation of $\langle \mathrm{ext}|e^{F'_X}|0\rangle$, since the $\alpha_{-m,+}^{(r)}$ oscillators contained in those terms commute with A_m^i and annihilate the bra vacuum $\langle 0|$. Therefore only the transverse mode part survives in (7.55), and then the DDF operators A_m^i in (7.37) also can be replaced simply by α_m^i. Thus we have shown that our on-shell physical amplitudes reduce to those in light-cone gauge:[20]

$$\mathcal{T}_N^{\text{on-shell physical}} = 2\pi\delta\left[\sum_r \alpha_r\right] \mathcal{T}_N^{\text{light-cone gauge}} \ ,$$

$$\mathcal{T}_N^{\text{light-cone gauge}} = (2\pi)^d \delta\left[\sum_r p_r\right] 2g^{N-2} \int \frac{\prod_{r=1}^{N} dZ_r}{dV_{abc}} \prod_{r<s} |Z_r - Z_s|^{p_r\cdot p_s}$$
$$\times \mathrm{tr}\left\langle \mathrm{ext}(A_m^i \to \alpha_m^i) \left| \exp\left[\tfrac{1}{2}\sum_{\substack{m,n\geq 0\\ r,s}}{}' \alpha_{-m}^{(r)i} N_{mn}^{rs} \alpha_{-n}^{(s)i} \right] \right| 0 \right\rangle , \tag{7.57}$$

where the Neumann function N^{rs}_{mn} here is that of the diagram with string length $\alpha_r = 2p^r_+$.

D. Unitarity

If we expand the string field ϕ with respect to its internal modes, ϕ can be viewed as a set of an infinite number of local fields. Most of them are in fact unphysical fields since our string field contains unphysical ghost coordinates $c(\sigma)$ and $\bar{c}(\sigma)$ as well as $X^0(\sigma)$ and $X^{25}(\sigma)$. Those unphysical particles should not contribute to the (on-shell) physical amplitudes by appearing as on-shell intermediate states. This is the problem of physical S-matrix unitarity, or simply unitarity.

As we have calculated explicitly, the on-shell physical amplitudes at the tree level actually satisfy this requirement. For instance, the appearance of $[\det(1-\tilde{N}\tilde{N})]^{-(d-2)/2}$ means that the intermediate one-particle states are essentially only those related with the transverse-mode excitations. Or more directly we have seen that our tree amplitudes coincide with those in (clearly unitary) light-cone gauge string theory.

However, as for this kind of unitarity problems, our covariant theory is guaranteed to be consistent from the beginning even beyond the tree level. This is because our string field theory is constructed so as to possess the BRS symmetry; that is, we have a conserved and nilpotent BRS charge $\mathbf{Q}_B$ at this second quantized level. As Kugo and Ojima have shown quite generally,[11] the subsidiary condition

$$\mathbf{Q}_B \| \text{phys} \rangle\rangle = 0 \tag{7.58}$$

guarantees the physical S-matrix unitarity provided that all the physical one-particle (string) states have positive metric. Based on only the conservation and nilpotency of $\mathbf{Q}_B$, they showed that the unphysical particles can appear only in *zero-norm combinations* in the physical subspace specified by (7.58) and hence as intermediate states in physical particle scatterings. Their argument is valid also for string field theory since a string field is merely an infinite component local field. The above assumption for the positivity of physical one-particle (string) states was proved to hold actually by Kato and Ogawa[7] as was mentioned already in the previous subsection. Therefore the unitarity in the usual sense holds in arbitrary loop order in this theory.

There is another and much more nontrivial problem of unitarity in this theory, which comes from the fact that the string-length parameter α is unphysical. Truly physical states of course contain no information of α and therefore we have to specify how they are connected to our "physical" states with α, in such a way that unitarity holds within the world of the truly physical states. For instance, if we identify the truly physical one-string states with our "physical" ones having a fixed value of α, say $\alpha=\frac{1}{2}$, then the unitarity is clearly violated since the one-string states possessing other values of α are easily created in the scattering and they are unphysical in such a definition.

A very important fact in connection with this is that the tree-level on-shell physical amplitudes are independent of the α's except the total conservation factor $\delta(\sum \alpha_r)$ as was seen in the previous subsection. If this particular property persists in higher loop orders also, the problem of α will be solved consistently as follows: The above property says that the on-shell physical N-string amplitudes $\mathcal{T}_{\text{phys}}$ take the form

$$\mathcal{T}^N_{\text{phys}} = 2\pi\delta\left[\sum_{r=1}^{N} \alpha_r\right] T_N \tag{7.59}$$

with completely α-independent (reduced) amplitude T_N. Then if we go to Fourier conjugate space, say $\tilde{\alpha}$ space, of α parameter, the on-shell physical amplitudes turn out to take the form

$$\mathcal{T}^N_{\text{phys}} = \left[\prod_{r=1}^{N-1} \delta(\tilde{\alpha}_r - \tilde{\alpha}_N)\right] T_N \ . \tag{7.60}$$

This implies that the S matrix is completely local in $\tilde{\alpha}$ space and translation invariant. That is, the scattering takes place only among string states possessing the same $\tilde{\alpha}$ and never produces state with different $\tilde{\alpha}$ from that of initial states. Therefore the unitarity holds on each point in $\tilde{\alpha}$ space separately, and thus we can consistently define physical (one-string) states as those having an arbitrary but fixed value of $\tilde{\alpha}$; for instance, we can define the physical string field as

$$\phi_{\text{phys}}(\tilde{\alpha}=0) = \int d\alpha\, \phi_{\text{phys}}(\alpha)$$

with $\phi_{\text{phys}}(\alpha)$ possessing physical polarizations.

The α independence of on-shell physical amplitudes has not yet been proved beyond the tree level, unfortunately. This problem of α independence is, however, essentially identical with that of Lorentz invariance in light-cone gauge string field theory. So similarly to the Lorentz transformation M_{i-} in the light-cone gauge case,[54] one can try a transformation such as

$$\delta\phi = \left[p^\mu \frac{\partial}{\partial\alpha} - \frac{1}{\alpha}\left\{\tilde{Q}_B, \sum_n \frac{1}{n}\alpha^\mu_{-n}\bar{c}_n\right\}\right]\phi \tag{7.61}$$

to see whether it, or a similar one, is a physical symmetry of the action which leaves the action invariant in the physical subspace: $\delta\hat{S} = \{\mathbf{Q}_B, {}^\exists X\}$. If the transformation of the form like (7.61) is a physical symmetry it is easy to see that it guarantees the α independence of on-shell physical amplitudes at full order level.

In connection with this we find a very encouraging fact that the zero-slope limit discussed in the next section. As will be shown there, our action reduces to a gauge-fixed form of the usual Yang-Mills action. The gauge-invariant part has no explicit α dependence and hence it is exactly $-\frac{1}{4}F_{\mu\nu}{}^2$ if we identify

$$A_\mu(x,\tilde{\alpha}=0) = \int d\alpha\, A_\mu(x,\alpha)$$

with the usual gauge field $A_\mu(x)$. The nontrivial α dependence appears only in the gauge-fixing and Faddeev-Popov ghost terms and therefore our theory in this limit is physically equivalent to the usual Yang-Mills theory without α.

VIII. ZERO-SLOPE LIMIT

We consider the zero-slope[55] limit of our covariant string field theory in this section. That is, we calculate the explicit forms of the gauge-fixed action (6.14) and its BRS transformation (6.28) in the limit that the Regge slope parameter $\alpha'[M^{-2}]$ approaches zero. It should be noticed that we perform this calculation completely off-shell.

Let us define component fields by expanding the string field $|\phi(x,\alpha)\rangle$ with respect to oscillator modes as

$$|\phi(x,\alpha)\rangle = |0\rangle\varphi(x,\alpha)+[\alpha_{-1}^{\mu}|0\rangle A_{\mu}(x,\alpha)+c_{-1}|0\rangle(-i)\bar{C}(x,\alpha)+\bar{c}_{-1}|0\rangle(-)C(x,\alpha)]+\cdots \tag{8.1a}$$

$$= |0\rangle\varphi(x,\alpha)+[a_{-1}^{M}|0\rangle\eta_{MN}A^{N}(x,\alpha)]+\cdots , \tag{8.1b}$$

where $\varphi(x,\alpha)$ is a tachyonic field and $A_{\mu}(x,\alpha)$, $\bar{C}(x,\alpha)$, and $C(x,\alpha)$ are massless fields. All the component fields are matrix valued and satisfy Hermiticity conditions

$$\chi^{\dagger}(x,\alpha)=\chi(x,-\alpha) \quad \text{for } \chi=\varphi, A_{\mu}, \bar{C}, C . \tag{8.2}$$

[Equation (8.2) is easily confirmed by the $|\phi\rangle$ version of string Hermiticity condition (2.15).] In (8.1b) we have used the OSp($d/2$) notation explained in Sec. V A:

$$a_n^M=(\alpha_n^{\mu}, \gamma_n=in\alpha c_n, \bar{\gamma}_n=\alpha^{-1}\bar{c}_n)$$

and the OSp-vector massless field is now defined by

$$A^M(x,\alpha)=(A^{\mu}(x,\alpha), -i\alpha C(x,\alpha), -i\alpha^{-1}\bar{C}(x,\alpha))$$
$$=A^{M\dagger}(x,-\alpha) . \tag{8.3}$$

In the zero-slope limit $\alpha'\to 0$, all the massive component fields become infinitely massive and decouple since $(\alpha')^{-1/2}$ gives the unique mass scale in this theory, and hence only the massless component fields A^M survive.[55] However, not all the effects of massive fields disappear in this limiting theory since they may give finite contributions by propagating inside the Feynman diagrams with external lines of massless fields alone. To examine which types of diagrams can have such finite effects, we denote all the massive fields symbolically by $U(x,\alpha)$ and the massless ones by $A(x,\alpha)$, and rewrite the gauge-fixed string action (6.14) in terms of them in the following schematic form:

$$\hat{S}=\int dx\frac{d\alpha}{2\pi}\left[A\Box A+U\left[\Box-\frac{1}{\alpha'}N\right]U +g\sum_{m=0}^{3}\sum_{n=0}^{\infty}(\alpha')^{(d+2n-6)/4}\partial^n A^m U^{3-m} +g^2\sum_{m=0}^{4}\sum_{n=0}^{\infty}(\alpha')^{(d+n-4)/2}\partial^n A^m U^{4-m}\right], \tag{8.4}$$

where d is the space-time dimension, N is a nonzero number (mode number minus 1) and ∂ denotes derivatives which may operate on either A or U. The form (8.4) is a result of simple dimension counting: We have assigned to A and U the canonical mass dimension $\frac{1}{2}(d-2)$ of a usual bosonic field in d-dimensional space-time, and the coupling constant g is dimensionless, α' having dimension -2. The Yang-Mills theory in d-dimensional space-time has a coupling constant $g_{\rm YM}$ of dimension $-\frac{1}{2}(d-4)$, which hence must be related with the present string coupling constant g by

$$g_{\rm YM}=(\alpha')^{(d-4)/4}g . \tag{8.5}$$

The zero-slope limit is taken in such a way that this Yang-Mills coupling constant $g_{\rm YM}$ is fixed.[55] Therefore we rewrite (8.4) by using $g_{\rm YM}$ as

$$\hat{S}=\int dx\frac{d\alpha}{2\pi}\left[A\Box A+U\left[\Box-\frac{1}{\alpha'}N\right]U +g_{\rm YM}\sum_{m=0}^{3}\sum_{n=0}^{\infty}(\alpha')^{(n-1)/2}\partial^n A^m U^{3-m} +g_{\rm YM}{}^2\sum_{m=0}^{4}\sum_{n=0}^{\infty}(\alpha')^{n/2}\partial^n A^m U^{4-m}\right]. \tag{8.6}$$

Similarly the BRS transformation (6.28) of massless component fields can be written in the form:

$$\hat{\delta}_B A=\partial A+g_{\rm YM}\sum_{m=0}^{2}\sum_{n=0}^{\infty}(\alpha')^{n/2}\partial^n A^m U^{2-m} +g_{\rm YM}{}^2\sum_{m=0}^{3}\sum_{n=0}^{\infty}(\alpha')^{(n+1)/2}\partial^n A^m U^{3-m} . \tag{8.7}$$

We have to count the power of α' for arbitrary Feynman diagrams whose internal lines consist of only massive fields U and external lines of only massless fields A. This omission of massless fields as internal lines is because we are looking for the *quantum* action of massless fields in the zero-slope limit. The Feynman rules are read from the action (8.6). We call the vertices corresponding to the third and fourth terms V_3 and V_4, respectively, which came from cubic and quartic string interaction terms. We immediately notice that the V_3 vertices have a power $(\alpha')^{-1/2}$ while the V_4 vertices have no power, aside from $(\alpha')^{n/2}$ compensating the dimension of derivatives ∂^n. Taking into account also that the massive field propagators carry a power $(\alpha')^1$, we see that the Feynman diagrams with P propagators, n_3 V_3 vertices and n_4 V_4 vertices are proportional to

$$(\alpha')^{(D-n_3+2P)/2} , \tag{8.8}$$

where D is the total number of derivatives contained in the diagrams. Recalling the well-known relation between the numbers of loops (L), propagators (P), and vertices ($V=n_3+n_4$),

$$L=P-(V-1)=P-n_3-n_4+1\ , \tag{8.9}$$

we find that (8.8) becomes

$$(\alpha')^{(D+L+P+n_4-1)/2}\ . \tag{8.10}$$

Therefore the possible Feynman diagrams which contain massive particles inside (i.e., $P\geq 1$) but can give finite contributions in the zero-slope limit $\alpha'\to 0$ are only those satisfying

$$P=1,\quad D=L=n_4=0\ , \tag{8.11}$$

i.e., one-propagator tree diagrams containing no derivatives and no V_4 vertex. Clearly such diagrams have the form shown in Fig. 25 and can be constructed only by using the $A^2\cdot U$-type V_3 vertex twice.[55] All the other diagrams surviving the limit $\alpha'\to 0$ are of course those which are already contained in the action (8.6) and consist of A fields alone.

From a similar power counting we can find the BRS transformation in the zero-slope limit. In this case the relevant Feynman diagrams are obtained from the above considered ones by replacing one of their vertices by the vertex appearing in the RHS of BRS transformation (8.7). So the power of α' of such a diagram is given by

$$(\alpha')^{(D+L+P+n_4)/2}\ . \tag{8.12}$$

This implies that no diagrams containing massive internal lines can contribute in the $\alpha'\to 0$ limit. Therefore the BRS transformation in the zero-slope limit is given by the terms alone which are already contained in (8.7) and consist of A fields only. Notice in particular that the $O(g_{\rm YM}{}^2)$ terms in (8.7) coming from quartic string vertex $V^{(4)}$ do not survive the limit $\alpha'\to 0$.

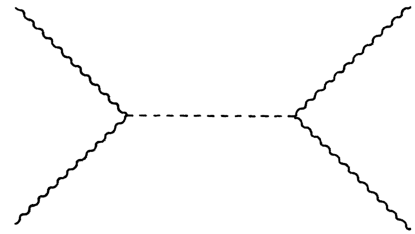


FIG. 25. The only surviving diagrams in the zero-slope limit that contain the massive particle propagator (dotted line) inside. The vertices stand for the massless-massless-massive ones with no derivatives.

Now we are ready to calculate the explicit $\alpha'\to 0$ limiting form of the action $\hat S$ and BRS transformation $\hat\delta_B$ of our covariant string field theory. First we immediately calculate the kinetic term $\phi\cdot L\phi$ of (6.15a) by retaining only the massless field components in (8.1b), i.e., $|\phi\rangle=a_{-1}^M|0\rangle\eta_{MN}A^N(x,\alpha)$:

$$\hat S_0=\phi\cdot L\phi$$
$$=\int dx\frac{d\alpha}{2\pi}{\rm tr}\tfrac{1}{2}A^M(x,-\alpha)\eta_{MN}\Box A^N(x,\alpha)\ . \tag{8.13}$$

In this OSp($d/2$) notation, it is also easy to calculate the cubic interaction term by using (6.15b), (6.16a), (6.17b), and (3.10):

$$\hat S_1=\tfrac{2}{3}g\phi^3|_{\alpha'\to 0}$$
$$=\int\left[\prod_{i=1}^{3}\frac{dp_id\alpha_i}{(2\pi)^{d+1}}\right](2\pi)^{d+1}\delta\left[\sum_i p_i\right]\delta\left[\sum_i\alpha_i\right]\tfrac{2}{3}g_{\rm YM}{\rm tr}\left[\prod_{i=1}^{3}A^{M_i\dagger}(p_i,\alpha_i)\eta_{M_iN_i}\right]$$
$$\times\left\langle 0\left|a_1^{N_3(3)}a_1^{N_2(2)}a_1^{N_1(1)}(\tfrac{1}{2}a_{-1}^{K(r)}\eta_{KL}\overline{N}{}_{11}^{rs}a_{-1}^{L(s)})(\overline{N}{}_1^{t}a_{-1}^{\mu(t)}\,{\bf P}_\mu)\exp\left[-\sum_i\frac{\tau_0}{\alpha_i}\right]\right|0\right\rangle$$
$$=\int\left[\prod_{i=1}^{3}\frac{dp_id\alpha_i}{(2\pi)^{d+1}}\right](2\pi)^{d+1}\delta\left[\sum_i p_i\right]\delta\left[\sum_i\alpha_i\right]$$
$$\times 3\times\tfrac{2}{3}g_{\rm YM}{\rm tr}[A^M(p_1,\alpha_1)\eta_{MN}A^N(p_2,\alpha_2)A_\mu(p_3,\alpha_3)]\frac{(\alpha_1p_2-\alpha_2p_1)^\mu}{-\alpha_3}\ . \tag{8.14}$$

Here we have used the relations $\overline{N}{}_{11}^{rs}=e^{\tau_0/\alpha_r+\tau_0/\alpha_s}$, $\overline{N}{}_1^{r}=(1/\alpha_r)e^{\tau_0/\alpha_r}$, and ${\bf P}^\mu=\alpha_1p_2^\mu-\alpha_2p_1^\mu$ in Eqs. (3.11). By using a relation

$$\alpha_1p_2-\alpha_2p_1=\tfrac{1}{2}[\alpha_3(p_1-p_2)+p_3(\alpha_2-\alpha_1)]\ , \tag{8.15}$$

we can rewrite (8.14) into the x-space representation:

$$\hat S_1=g_{\rm YM}\int dx\left[\prod_{i=1}^{3}\frac{d\alpha_i}{2\pi}\right]2\pi\delta\left[\sum_i\alpha_i\right]{\rm tr}\{-i\partial_\mu A^M(x,\alpha_1)[A^\mu(x,\alpha_2),A^N(x,\alpha_3)]\eta_{MN}$$
$$-(\alpha_1-\alpha_2)/\alpha_3[A^M(x,\alpha_1)\eta_{MN}A^N(x,\alpha_2)]i\partial_\mu A^\mu(x,\alpha_3)\}\ . \tag{8.16}$$

Next we need to calculate the direct $O(g_{\rm YM}{}^2)$ term $\propto g_{\rm YM}{}^2 A^4$ contained in the $\frac{2}{4}g^2\phi^4$ term of the action $\hat{S}$ *and* the diagrams in Fig. 25 with massive particle exchange. If we add both of these contributions as well as diagrams of the form of Fig. 25 with massless particle exchange, it just becomes the 4-string amplitude calculated in the previous section with the $\alpha'\to 0$ limit taken. Since the factor α' appears in the string amplitude always accompanied by momentum variable p_r in the form $(\alpha')^{1/2}p_r$, we need only the 4-string amplitude evaluated at $^{\forall}p_r=0$. This indeed agrees with the previous observation that the relevant diagrams in the $\alpha'\to 0$ limit, i.e., Fig. 25, should contain no derivative factors in the vertices. However, we should note that the 4-string amplitude contains the contributions from massless particle exchange diagrams also, which actually possess ambiguous factors $(p_i+p_j)^2/(p_k+p_l)^2$ if all the momentums are set equal to zero from the start. To avoid this ambiguity, we keep only the variables

$$s=-(p_1+p_2)^2,\quad t=-(p_1+p_4)^2,\quad u=-(p_1+p_3)^2\,, \tag{8.17}$$

nonzero but set all the p_r's in the other places equal to zero; that is, we can take $p_r{}^2=0$ and $\alpha_{-n}^{(r)}\bar{N}\,^{rs}_{n0}p^{(s)}=0$ from the beginning without ambiguity.

Since $p_r{}^2=0$ implies on-shell condition for massless particles, it is convenient to use the expression (7.43) for the 4-string amplitude in previous section, which now reads, for the case of 4 massless external states A(1)—A(4),

$$\mathcal{T}_4^{st}=(2\pi)^{d+1}\delta\left[\sum_{r=1}^{4}\alpha_r\right]\delta\left[\sum_{r=1}^{4}p_r\right]2g^2$$

$$\times\int\frac{\prod_{r=1}^{4}dZ_r}{dV_{abc}}\prod_{r<s}|Z_r-Z_s|^{p_r\cdot p_s}\mathrm{tr}\left\langle A(1)A(2)A(3)A(4)\left|\frac{1}{2!}\left[\sum_{r,s}\tfrac{1}{2}a_{-1}^{M(r)}N^{rs}_{11}\eta_{MN}a_{-1}^{N(s)}\right]^2\right|0\right\rangle\,. \tag{8.18}$$

This ordering of $A(1)$—$A(4)$ corresponds to the s-t amplitude case. We need the expression of N^{rs}_{11} for $r\neq s$ of 4-string light-cone diagrams, which can be easily obtained by using the general formula (A14) as

$$N^{rs}_{11}=\bar{N}\,^{rs}_{11}\exp(-\bar{N}\,^{rr}_{00}-\bar{N}\,^{ss}_{00})=1/(Z_r-Z_s)^2\,. \tag{8.19}$$

By taking the previous gauge choice $Z_1=\infty$, $Z_2=1$, $Z_4=0$ for the projective invariance and using the Veneziano variable $x=Z_3$, (8.18) is evaluated as

$$\mathcal{T}_4^{st}=(2\pi)^{d+1}\delta\left[\sum_r\alpha_r\right]\delta\left[\sum_r p_r\right]2g^2\mathrm{tr}[A^{M_1\dagger}(1)A^{M_2\dagger}(2)A^{M_3\dagger}(3)A^{M_4\dagger}(4)]$$

$$\times\left[(-)^{|2||3|}\eta_{M_1M_3}\eta_{M_2M_4}\int_0^1 dx\,x^{-s/2}(1-x)^{-t/2}+\eta_{M_1M_2}\eta_{M_3M_4}\int_0^1 dx\,x^{-s/2-2}(1-x)^{-t/2}\right.$$

$$\left.+\,\eta_{M_1M_4}\eta_{M_2M_3}\int_0^1 dx\,x^{-s/2}(1-x)^{-t/2-2}\right]\,, \tag{8.20}$$

where $(-)^{|2||3|}$ is the sign factor which becomes -1 when both M_2 and M_3 are ghost indices. There still exists s-u and t-u amplitudes corresponding to the external state ordering $A(1)A(2)A(4)A(3)$ and $A(1)A(3)A(2)A(4)$, respectively, and also the amplitudes obtained by exchanging 1 and 2 from these s-t, s-u, and t-u ones. Since each amplitude has three terms like in (8.20), there exist $(4-1)!\times 3=18$ terms in all. These 18 terms, however, result from various ways of contraction between external four massless states and the following simple effective action with only two terms:

$$\hat{S}'_2=\int\left[\prod_{r=1}^{4}\frac{dp_r d\alpha_r}{(2\pi)^{d+1}}\right](2\pi)^{d+1}\delta\left[\sum_r\alpha_r\right]\delta\left[\sum_r p_r\right]2g_{\rm YM}{}^2\tfrac{1}{4}\mathrm{tr}\left[\prod_{i=1}^{4}A^{M_i}(p_i,\alpha_i)\right]$$

$$\times\left[(-)^{|2||3|}\eta_{M_1M_3}\eta_{M_2M_4}\int_0^1 dx\,x^{-s/2}(1-x)^{-t/2}+2\eta_{M_1M_2}\eta_{M_3M_4}\int_0^1 dx\,x^{-s/2-2}(1-x)^{-t/2}\right]\,. \tag{8.21}$$

Here the $A^M(p,\alpha)$'s are now field operators. In this expression s and t are actually associated with α' as $\alpha's$ and $\alpha't$; therefore the x integrals in (8.21) are evaluated as

$$\int_0^1 dx\, x^{-\alpha's/2}(1-x)^{-\alpha't/2}=\frac{\Gamma\left[-\frac{\alpha'}{2}t+1\right]\Gamma\left[-\frac{\alpha'}{2}s+1\right]}{\Gamma\left[-\frac{\alpha'}{2}(s+t)+2\right]}=1+O(\alpha')\,,$$

$$\int_0^1 dx\, x^{-\alpha's/2-2}(1-x)^{-\alpha't/2}=\frac{\Gamma\left[-\frac{\alpha'}{2}s-1\right]\Gamma\left[-\frac{\alpha'}{2}t+1\right]}{\Gamma\left[-\frac{\alpha'}{2}(s+t)\right]}=-\frac{s+t}{s}+O(\alpha') \tag{8.22}$$

$$=\frac{u}{s}+O(\alpha')\quad(\because s+t+u=0)\,.$$

The action (8.21) still contains the contribution from massless particle exchange diagrams which should be subtracted. The massless contribution to the action is easily calculated by using the previous expression (8.14) of $\hat{S}_1$ for the three-massless interaction term. Recalling that the momenta contracted with external legs are now being set equal to zero and $p_r^2=0$, we calculate it as

$$\hat{S}_2^{\text{massless exchange}}=\int\prod_{r=1}^{4}\left[\frac{dp_r d\alpha_r}{(2\pi)^{d+1}}\right](2\pi)^{d+1}\delta\left[\sum_r\alpha_r\right]\delta\left[\sum_r p_r\right]$$
$$\times\tfrac{1}{2}(2g_{\text{YM}})^2\text{tr}[A^{M_1}(p_1,\alpha_1)\eta_{M_1M_2}A^{M_2}(p_2,\alpha_2)A^{M_3}(p_3,\alpha_3)\eta_{M_3M_4}A^{M_4}(p_4,\alpha_4)]\mathcal{M}\,, \tag{8.23a}$$

where

$$\mathcal{M}=\frac{\alpha_1p_2^\mu-\alpha_2p_1^\mu}{\alpha_1+\alpha_2}\langle A_\mu(-(p_1+p_2),-(\alpha_1+\alpha_2))A_\nu(-(p_3+p_4),-(\alpha_3+\alpha_4))\rangle\frac{\alpha_3p_4^\nu-\alpha_4p_3^\nu}{\alpha_3+\alpha_4}$$
$$=\frac{(\alpha_1p_2-\alpha_2p_1)(\alpha_4p_3-\alpha_3p_4)}{(\alpha_1+\alpha_2)^2(p_1+p_2)^2}$$
$$=-\frac{1}{2}\frac{(\alpha_1\alpha_4+\alpha_2\alpha_3)}{(\alpha_1+\alpha_2)^2}+\frac{u}{2s}\,. \tag{8.23b}$$

So in the genuine $O(g^2)$ action $\hat{S}_2$ given by $\hat{S}_2'-\hat{S}_2^{\text{massless exchange}}$, the momentum-dependent (i.e., nonlocal in x space) parts proportional to u/s exactly cancel between (8.21) and (8.23), and we finally obtain, in x-space representation,

$$\hat{S}_2=\tfrac{1}{2}g_{\text{YM}}{}^2\int dx\left[\prod_{i=1}^{4}\frac{d\alpha_i}{2\pi}\right]2\pi\delta\left[\sum_r\alpha_r\right]\text{tr}\left[\prod_i A^{M_i}(x,\alpha_i)\right]\left[(-)^{|2||3|}\eta_{M_1M_3}\eta_{M_2M_4}+2\eta_{M_1M_2}\eta_{M_3M_4}\frac{\alpha_1\alpha_4+\alpha_2\alpha_3}{(\alpha_1+\alpha_2)^2}\right]. \tag{8.24}$$

Thus the action in the zero-slope limit is given by $\hat{S}_0$ of (8.13) plus $\hat{S}_1$ of (8.16) plus $\hat{S}_2$ of (8.24). However, since it is written in terms of OSp$(d/2)$ vector field A^M, it would be better to rewrite it in a familiar notation in terms of Yang-Mills field A^μ and Faddeev-Popov ghosts C and $\bar{C}$. A straightforward substitution of (8.3) leads to

$$\hat{S}_0=\int dx\frac{d\alpha}{2\pi}\text{tr}[\tfrac{1}{2}A_\mu(-\alpha)\Box A^\mu(\alpha)-i\bar{C}(-\alpha)\Box C(\alpha)]\,,$$

$$\hat{S}_1=g_{\text{YM}}\int dx\frac{d^3\alpha}{(2\pi)^2}\delta\left[\sum_r\alpha_r\right]\text{tr}\left[-i\partial_\mu A_\nu(\alpha_1)[A^\mu(\alpha_2),A^\nu(\alpha_3)]-i\frac{\alpha_1}{\alpha_3}[A_\mu(\alpha_1),A^\mu(\alpha_2)]\partial A(\alpha_3)\right.$$
$$-f_{\alpha_1,\alpha_2}\{\bar{C}(\alpha_1),C(\alpha_2)\}\partial A(\alpha_3)-\partial_\mu\bar{C}(\alpha_1)[A^\mu(\alpha_2),C(\alpha_3)]$$
$$\left.+\frac{\alpha_2-\alpha_3}{\alpha_1}\bar{C}(\alpha_1)[A^\mu(\alpha_2),\partial_\mu C(\alpha_3)]\right],$$

$$\hat{S}_2 = g_{\rm YM}{}^2 \int dx \frac{d^4\alpha}{(2\pi)^3} \delta\left[\sum_r \alpha_r\right]$$
$$\times \mathrm{tr}\left[\tfrac{1}{2} A_\mu(\alpha_1) A_\nu(\alpha_2) A^\mu(\alpha_3) A^\nu(\alpha_4) + \frac{\alpha_1\alpha_4+\alpha_2\alpha_3}{(\alpha_1+\alpha_2)^2} A_\mu(\alpha_1) A^\mu(\alpha_2) A_\nu(\alpha_3) A^\nu(\alpha_4)\right.$$
$$+2i\frac{\alpha_4}{\alpha_2} A_\mu(\alpha_1)\bar{C}(\alpha_2) A^\mu(\alpha_3) C(\alpha_4)$$
$$+2i\frac{\alpha_1\alpha_4+\alpha_2\alpha_3}{(\alpha_1+\alpha_2)^2} A_\mu(\alpha_1) A^\mu(\alpha_2)\left[\frac{\alpha_4}{\alpha_3}\bar{C}(\alpha_3) C(\alpha_4) - \frac{\alpha_3}{\alpha_4} C(\alpha_3)\bar{C}(\alpha_4)\right]$$
$$+2\frac{\alpha_1\alpha_2}{\alpha_3\alpha_4}\left[1-\frac{\alpha_1\alpha_2+\alpha_3\alpha_4}{(\alpha_2+\alpha_3)^2}\right] C(\alpha_1) C(\alpha_2)\bar{C}(\alpha_3)\bar{C}(\alpha_4)$$
$$\left. -\frac{\alpha_2\alpha_4}{\alpha_1\alpha_3}\left[\frac{\alpha_1\alpha_4+\alpha_2\alpha_3}{(\alpha_1+\alpha_2)^2} - \frac{\alpha_1\alpha_2+\alpha_3\alpha_4}{(\alpha_2+\alpha_3)^2}\right]\bar{C}(\alpha_1) C(\alpha_2)\bar{C}(\alpha_3) C(\alpha_4)\right], \quad (8.25)$$

where we have omitted the common argument x of the fields and

$$f_{\alpha_1,\alpha_2} \equiv -(\alpha_1{}^2+\alpha_1\alpha_2+2\alpha_2{}^2)/\alpha_1(\alpha_1+\alpha_2)\ . \quad (8.26)$$

The above expression (8.25) of the zero-slope limit action looks very complicated, and in particular various coefficients in $\hat{S}_1$ and $\hat{S}_2$ have very nontrivial dependence on the "string-length" parameters α. Nevertheless we will see shortly that those complicated α-dependent terms are all absorbed concisely into gauge-fixing and Faddeev-Popov terms and the rest becomes the usual α-independent gauge-invariant Yang-Mills action. For this purpose let us turn to derive the zero-slope limiting form of BRS transformation (6.28). As explained before, the massive fields do not contribute at all in this case and we have only to calculate directly (6.28) by retaining only the massless field components $|\phi\rangle = a_{-1}^M |0\rangle \eta_{MN} A^N(x,\alpha)$.

The $O(g^0)$ part $\tilde{Q}_B\phi$ is immediately calculated since only the $-p_\mu(c_{-1}\alpha_1^\mu+\alpha_{-1}^\mu c_1)$ part of $\tilde{Q}_B$ is relevant here:

$$a_{-1}^M|0\rangle\eta_{MN}(-)^{|M|}\hat{\delta}_B^0 A^N(x,\alpha) = -i\alpha_{-1}^\mu|0\rangle\partial_\mu C(x,\alpha) + ic_{-1}|0\rangle\partial_\mu A^\mu(x,\alpha)\ . \quad (8.27)$$

The $O(g^1)$ part (6.29a) can be calculated relatively simply again by using the OSp$(d/2)$ vector notation. With the help of Eq. (3.51) which says

$$w^{(3)} = \left[c_{-1}^{(1)} e^{\tau_0/\alpha_1} - c_{-1}^{(2)} e^{\tau_0/\alpha_2} + \frac{\alpha_2-\alpha_1}{\alpha_3} c_{-1}^{(3)} e^{\tau_0/\alpha_3}\right] \quad (8.28)$$

for the relevant pieces here and also of $\bar{N}_{11}^{rs} = e^{\tau_0/\alpha_r+\tau_0/\alpha_s}$ $(r\neq s)$, we find

$$\left[\frac{1}{g_{\rm YM}}\right] a_{-1}^{M(3)}|0\rangle_3 \eta_{MN}(-)^{|M|}\hat{\delta}_B^1 A^N(\alpha_3)$$
$$= \prod_{i=1}^2 [A^{M_i}(-\alpha_i)\eta_{M_iN_i}\,{}_i\langle 0|a_1^{N_i}]\, w^{(3)} \sum_{r,s} \tfrac{1}{2} a_{-1}^{K(r)} \eta_{KL} \bar{N}_{11}^{rs} a_{-1}^{L(s)} \exp\left[-\sum_r \frac{\tau_0}{\alpha_r}\right]|0\rangle_{123}$$
$$= \left[-C(-\alpha_1)A^M(-\alpha_2)\eta_{MN}\,{}_{12}\langle 0|a_1^{N(2)} + (-)^{|N|} A^M(-\alpha_1)C(-\alpha_2)\eta_{MN}\,{}_{12}\langle 0|a_1^{N(1)}\right.$$
$$\left. + A^{M_1}(-\alpha_1)A^{M_2}(-\alpha_2)\eta_{M_1N_1}\eta_{M_2N_2}\,{}_{12}\langle 0|a_1^{N_2(2)} a_1^{N_1(1)} \frac{\alpha_2-\alpha_1}{\alpha_3} c_{-1}^{(3)}\right] \sum_{r,s} \tfrac{1}{2} a_{-1}^{K(r)} \eta_{KL} a_{-1}^{L(s)} |0\rangle_{123}$$
$$= -C(-\alpha_1)A^M(-\alpha_2)\eta_{MN} a_{-1}^{N(3)}|0\rangle_3 + (-)^{|N|} A^M(-\alpha_1)C(-\alpha_2)\eta_{MN} a_{-1}^{N(3)}|0\rangle_3$$
$$+\frac{\alpha_2-\alpha_1}{\alpha_3} A^M(-\alpha_1)\eta_{MN} A^N(-\alpha_2) c_{-1}^{(3)}|0\rangle_3\ , \quad (8.29)$$

where we have omitted the common argument x and

$$\int \frac{d\alpha_1 d\alpha_2}{(2\pi)} \delta(\alpha_1+\alpha_2+\alpha_3)$$

in these equations. The $O(g^2)$ part does not survive the $\alpha'\to 0$ limit as explained before and so Eqs. (8.27) and (8.29) give the total BRS transformation in the zero-slope limit. Again by using (8.3) they are rewritten in the expressions for Yang-Mills and Faddeev-Popov ghost fields A_μ, C, and $\bar{C}$, separately:

$$\delta_B A_\mu(\alpha_3)=\partial_\mu C(\alpha_3)+ig_{\rm YM}\int\frac{d\alpha_1 d\alpha_2}{2\pi}\delta\left[\sum_r\alpha_r\right]\times[A_\mu(-\alpha_1),C(-\alpha_2)]\,,$$
$$\delta_B C(\alpha_3)=-\tfrac{1}{2}ig_{\rm YM}\int\frac{d\alpha_1 d\alpha_2}{2\pi}\delta\left[\sum_r\alpha_r\right]\times\{C(-\alpha_1),C(-\alpha_2)\}\,, \tag{8.30}$$
$$\delta_B\bar{C}(\alpha_3)=iB(\alpha_3)\,,$$

where we have multiplied a factor i to make $\delta_B=i\hat{\delta}_B$ coincide with the conventional BRS transformation in Yang-Mills theory and introduced the "Nakanishi-Lautrup field"[56] $B(x,\alpha)$ to denote here

$$B(\alpha_3)=\partial\cdot A(\alpha_3)+g_{\rm YM}\int\frac{d\alpha_1 d\alpha_2}{2\pi}\delta\left[\sum_r\alpha_r\right]\times\left[i\frac{\alpha_1}{\alpha_3}[A_\mu(-\alpha_1),A^\mu(-\alpha_2)]+f_{\alpha_1,\alpha_2}\{\bar{C}(-\alpha_1),C(-\alpha_2)\}\right], \tag{8.31}$$

with f_{α_1,α_2} defined in (8.26).

The BRS transformation (8.30) has very similar form to the usual Yang-Mills one[9-11] and in fact takes exactly the same form as the latter if we perform Fourier transformation with respect to the variable α:

$$\chi(x,\tilde{\alpha})=\int\frac{d\alpha}{2\pi}e^{-i\alpha\tilde{\alpha}}\chi(x,\alpha)\,,\qquad \chi(x,\alpha)=\int d\tilde{\alpha}\,e^{i\alpha\tilde{\alpha}}\chi(x,\tilde{\alpha})\,. \tag{8.32}$$

Indeed in this Fourier conjugate space $\tilde{\alpha}$, (8.30) becomes

$$\delta_B A_\mu(x,\tilde{\alpha})=\partial_\mu C(x,\tilde{\alpha})+ig_{\rm YM}[A_\mu(x,\tilde{\alpha}),C(x,\tilde{\alpha})]=D_\mu C(x,\tilde{\alpha})\,, \tag{8.33a}$$

$$\delta_B C(x,\tilde{\alpha})=-\tfrac{1}{2}ig_{\rm YM}\{C(x,\tilde{\alpha}),C(x,\tilde{\alpha})\}\,, \tag{8.33b}$$

$$\delta_B\bar{C}(x,\tilde{\alpha})=iB(x,\tilde{\alpha})\,. \tag{8.33c}$$

The existence of the $\tilde{\alpha}$ parameter, therefore, implies that the present theory represents an infinite number of copies of usual Yang-Mills theory. We now see that we can consistently identify the usual Yang-Mills field $A_\mu(x)$ with $A_\mu(x,\tilde{\alpha})$ with some fixed value of $\tilde{\alpha}$, e.g., $A_\mu(x,\tilde{\alpha}=0)$. Indeed, with straightforward but a little bit tedious algebraic calculations, we can rewrite the above complicated action (8.25) into the following concise form:

$$\hat{S}=\hat{S}_0+\hat{S}_1+\hat{S}_2=\int dx\,d\tilde{\alpha}\,{\rm tr}[-\tfrac{1}{4}F_{\mu\nu}(x,\tilde{\alpha})F^{\mu\nu}(x,\tilde{\alpha})+i\delta_B(\bar{C}(x,\tilde{\alpha})G(x,\tilde{\alpha}))]\,, \tag{8.34}$$

where $F_{\mu\nu}$ and G are field strength and gauge-fixing function, respectively, given by

$$F_{\mu\nu}(x,\tilde{\alpha})=\partial_\mu A_\nu(x,\tilde{\alpha})-\partial_\nu A_\mu(x,\tilde{\alpha})+ig_{\rm YM}[A_\mu(x,\tilde{\alpha}),A_\nu(x,\tilde{\alpha})]\,, \tag{8.35}$$

$$G(x,\alpha_3)=-\tfrac{1}{2}B(\alpha_3)+\partial\cdot A(\alpha_3)+g_{\rm YM}\int\frac{d\alpha_1 d\alpha_3}{2\pi}\delta\left[\sum_r\alpha_r\right]\left[i\frac{\alpha_1}{\alpha_3}[A_\mu(-\alpha_1),A^\mu(-\alpha_2)]+\tilde{f}_{\alpha_1,\alpha_2}\{\bar{C}(-\alpha_1),C(-\alpha_2)\}\right], \tag{8.36}$$

with $\tilde{f}_{\alpha_1,\alpha_2}$ being any function of α_1 and α_2 satisfying the following relation with the previous f_{α_1,α_2} of (8.26):

$$\tilde{f}_{\alpha_1,\alpha_2}+\tilde{f}_{-(\alpha_1+\alpha_2),\alpha_2}=f_{\alpha_1,\alpha_2}\,. \tag{8.37}$$

A remark would be necessary on the meaning of the $B(x,\alpha)$ field in (8.34). The $B(x,\alpha)$ field here is the Nakanishi-Lautrup field[56] in its proper sense, which is *not* the previously defined dependent variable given by Eq. (8.31) but an independent (auxiliary) field subject to the following BRS transformation law:[9-11]

$$\delta_B\bar{C}(x,\tilde{\alpha})=iB(x,\tilde{\alpha}),\quad \delta_B B(x,\tilde{\alpha})=0\,. \tag{8.38}$$

Notice that the first equation of (8.38) has the same form as (8.33c) but the meaning of $B(x,\tilde{\alpha})$ is different. In fact the previous BRS transformation (8.33) is nilpotent only on-shell just like the string BRS transformation $\hat{\delta}_B$ was, while the BRS transformation here given by (8.33a), (8.33b), and (8.38) is *off-shell* nilpotent. The BRS transformation δ_B in (8.34) is the one in this sense. The previous Eq. (8.31) now holds merely as an *equation of motion* with respect to B-field variation. This is seen from the following more explicit form of the gauge-fixing and Faddeev-Popov part of the action (8.34):

$$S_{\rm GF+FP}=\int dx\,d\tilde{\alpha}\,{\rm tr}\,i\delta_B(\bar{C}(x,\tilde{\alpha})G(x,\tilde{\alpha}))$$
$$=\int dx\,{\rm tr}\left[\tfrac{1}{2}B(-\alpha)B(\alpha)-B(-\alpha)\partial A(\alpha)-g_{\rm YM}B(\alpha_3)\left\{i\frac{\alpha_1}{\alpha_3}[A_\mu(\alpha_1),A^\mu(\alpha_2)]+f_{\alpha_1,\alpha_2}\{\bar{C}(\alpha_1),C(\alpha_2)\}\right\}\right.$$
$$+i\partial^\mu\bar{C}(-\alpha)\cdot D_\mu C(\alpha)+g_{\rm YM}\frac{\alpha_1-\alpha_2}{\alpha_3}\bar{C}(\alpha_3)[A^\mu(\alpha_1),D_\mu C(\alpha_2)]$$
$$\left.+g_{\rm YM}{}^2 f_{\alpha_4,\alpha_1+\alpha_2}\bar{C}(\alpha_3)\bar{C}(\alpha_4)C(\alpha_1)C(\alpha_2)\right]\,. \tag{8.39}$$

Here again we have omitted the obvious α-integration symbols such as

$$\int\frac{d\alpha_1 d\alpha_2 d\alpha_3}{(2\pi)^2}\delta(\alpha_1+\alpha_2+\alpha_3)\,.$$

[Note that $\tilde{f}_{\alpha_1,\alpha_2}$ in G appears in this expression only in the combination $\tilde{f}_{\alpha_1,\alpha_2}+\tilde{f}_{-(\alpha_1+\alpha_2),\alpha_2}$ of (8.37).] If we eliminate the B field by using the equation of motion (8.31) or by path integration over B, we actually recover the previous action (8.25) from the present one (8.34) with expression (8.39).

The action (8.34) gives our final result for the zero-slope limit form of the covariant string field theory. It shows that the gauge-invariant part $-\frac{1}{4}F_{\mu\nu}{}^2(x,\tilde{\alpha})$ just coincides with the usual form of Yang-Mills theory and that all the terms possessing nontrivial α dependence appear only in the gauge-fixing and corresponding Faddeev-Popov terms $S_{\rm GF+FP}$ of (8.39). Since the physical S matrix is independent of the choice of gauge fixing as is well known in the usual Yang-Mills theory,[35,36,11] the present theory has the same physical S matrix even if we choose a gauge local in $\tilde{\alpha}$, e.g., a gauge-fixing function $G(x,\tilde{\alpha})=-\frac{1}{2}B(x,\tilde{\alpha})+\partial\cdot A(x,\tilde{\alpha})$ instead of (8.36). Then the action (8.34) takes the form

$$\hat{S}=\int dx\,d\tilde{\alpha}\,{\rm tr}[-\tfrac{1}{4}F_{\mu\nu}{}^2(x,\tilde{\alpha})+\tfrac{1}{2}B^2(x,\tilde{\alpha})$$
$$-B(x,\tilde{\alpha})\partial\cdot A(x,\tilde{\alpha})$$
$$+i\partial^\mu\bar{C}(x,\tilde{\alpha})D_\mu C(x,\tilde{\alpha})]\,. \tag{8.40}$$

This action clearly describes an infinite (continuous) number of copies of usual Yang-Mills system; the worlds with different values of $\tilde{\alpha}$ are quite independent of one another and realize separately the same physical S matrix as the usual Yang-Mills one. Therefore, as announced before, we can consistently identify our gauge field with a fixed value of $\tilde{\alpha}$, e.g., $A_\mu(x,\tilde{\alpha}=0)$, with the usual Yang-Mills field $A_\mu(x)$. The physical S matrix is unitary on each $\tilde{\alpha}$ separately even in our original zero-slope limit system (8.34). Note that this is a result valid at full quantum level contrary to the tree-amplitude arguments in the previous section.

IX. DISCUSSION

We have presented the covariant string field theory in full detail for the bosonic open-string case. We have established the BRS invariance and its nilpotency for the gauge-fixed action on the one hand, and the gauge invariance and its group law for the gauge-invariant action on the other. It is remarkable that the condition of critical dimensionality $d=26$ was required at *every* order level of coupling constant g to prove the BRS nilpotency and gauge invariance. Also remarkable is the deep connection between *duality* and those invariances of the present covariant theory.

Now there remains no doubt that our string field theory is a consistent and satisfactory one at least at the tree level (or as a classical field theory). We have shown that the on-shell physical amplitudes in our theory correctly reproduce the usual dual amplitudes for general N-string scatterings with arbitrary external states.

A characteristic point to our covariant string field theory is the existence of string-"length" parameter α. Although this parameter is certainly unphysical, it is inevitable in our formulation. We suspect that the presence of the α parameter is essential to any consistent formulation of covariant string field theory. Indeed recall, for instance, the pregeometrical string field theory[4] in which a *natural* generation of string kinetic term was possible thanks to the presence of the α parameter freedom. One may, however, prefer other covariant formulations which are free from such an unphysical parameter α. Actually as such an example there is Witten's formulation of open string.[27,28,57] However, we think that Witten's theory needs further precise confirmations as for its gauge invariance and reproducibility of usual dual amplitudes with such an accuracy as presented in this paper.

On the other hand, the α independence of the tree on-shell physical amplitudes was proved in our formulation. We expect that this α independence holds at any loop order level. Indeed we have shown it is actually the case in the zero-slope limit. The structure of our "Yang-Mills" action (8.34) obtained in that limit is very suggestive; the explicit α-dependent terms are all contained in unphysical gauge-fixing parts. We suspect that essentially the same situation occurs even before taking the zero-slope limit. We are now actually trying to prove it by looking for a physical symmetry of our action under some transformation of ϕ, like (7.61), changing its α parameter and hope to be able to report it in the near future. If such an expectation is the case, the on-shell physical N-string amplitudes of L-loop diagrams have the following particular form of α dependence:

$$2\pi\delta\left[\sum_{r=1}^{N}\alpha_r\right]\cdot\prod_{l=1}^{L}\left[\int_{-\infty}^{+\infty}\frac{d\alpha_l}{2\pi}1\right]\times T(\alpha\text{ indep})\,.$$

[This is understood easily if we take the "Yang-Mills" action of the form (8.40) literally.] The divergent factors $\int_{-\infty}^{+\infty} d\alpha_l/2\pi$ can be factored out unambiguously and absorbed in the overall multiplicative parameter of the action such as $\hbar$ or g. The α-independent part $T(\alpha$ indep) gives the true physical amplitude for which unitarity clearly holds. [Recall that $T(\alpha$ indep) stands for the usual Yang-Mills amplitude for the case of zero-slope limit.]

The closed (bosonic) string field theory was already formulated in our paper II. Now that we have presented the full details of the open-string case here, the reader should be convinced that the closed-string field theory is also correct. We have not tried to construct a string field theory of a mixed system of open and closed strings in this paper. In the light-cone gauge formulation it was necessary to include the closed-string field explicitly in the open-string system.[21] However this is not so clear in the covariant formulation. In fact there are some indications that the closed string is already contained in our pure open-string system. For instance there appears a particular diagram at the one-loop level drawn in Fig. 26 which is absent in the light-cone gauge case in which the string can propagate only in the positive τ direction. The intermediate state of this diagram looks like a closed string, although it must be confirmed by explicit computation whether the closed-string pole is actually generated or not.

There remain certainly various interesting subjects to be studied further. (i) The gauge-fixing procedure should be clarified for the gauge-invariant action. It was pointed out in III that the usual successive gauge-fixing procedure[14] applied to the free-string theory does not work in the case of the interacting theory independently of the detailed form of the string vertex. How is our gauge-fixed action derived from the gauge-invariant one? (ii) The extension to superstring[1] (including heterotic string[2]) is a pressing issue. We expect no serious difficulties in this task. (iii) We have recently proposed a "pregeometrical string field theory"[4] which is completely independent of the space-time metric. The action is the closed-string action[17] consisting of a cubic interaction term alone.[3] We clarified explicitly the mechanism of how the kinetic term is generated by the condensation of the string field. The remaining problem is to find true solutions of the equation of motion $\Phi * \Phi = 0$ (Ref. 4) which corresponds to various possible background metrices. Also missing is the general algorithm to solve such an "algebraic" string field equation. (iv) A much more challenging problem would be the "spontaneous" generation of superstring field theory from the bosonic one in $d=26$. At present only suggestive arguments exist.[5] The more general problem of spontaneous compactification into lower dimensions should also be studied.[58] We can in any case attack them with our machinery of covariant string field theory now.

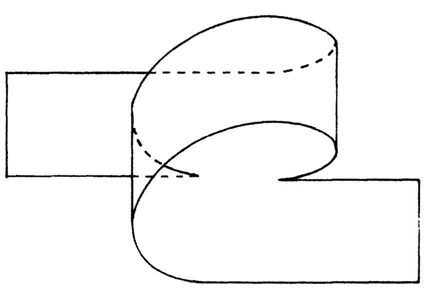


FIG. 26. A candidate diagram which may yield a closed-string pole at the one-loop level in the pure open-string system.

Note added in proof. The authors have recently succeeded in proving the α independence of the on-shell physical amplitudes of any loop order level, as was expected in Sec. VII. Further it is easily shown based on this knowledge that the closed-string pole is actually generated in the pure open-string system at the one-loop level. The details will be reported in a paper in preparation.

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APPENDIX A: PROPERTIES OF THE NEUMANN FUNCTIONS

Consider a Mandelstam mapping[20] from the upper half complex z plane to the string strips in the ρ plane (Fig. 27):

$$\rho(z)=\sum_{i=1}^{N}\alpha_i \ln(z-Z_i)\ , \tag{A1}$$

where α_i and Z_i ($i=1$–N) are real parameters satisfying

$$\sum_{i=1}^{N}\alpha_i=0\ . \tag{A2}$$

In each string strip in the ρ plane we introduce a complex coordinate ζ_r:

$$\zeta_r=\xi_r+i\sigma_r\ \ (\xi_r\le 0,\ 0\le\sigma_r\le\pi)\ , \tag{A3}$$

$$\rho=\alpha_r\zeta_r+\tau_0^{(r)}+i\beta_r\ , \tag{A4}$$

where $\tau_0^{(r)}$ is the interaction "time" for the rth string (cf. Fig. 27) and $\beta_r=\pi\sum_{s=1}^{r-1}\alpha_s$. The interaction time $\tau_0^{(r)}$ is given as

$$\tau_0^{(r)}=\mathrm{Re}\rho(z_0^{(r)})=\sum_{i=1}^{N}\alpha_i \ln|z_0^{(r)}-Z_i| \tag{A5}$$

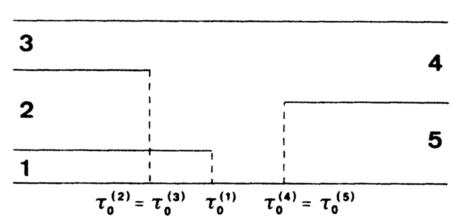

FIG. 27. A typical light-cone diagram to which the upper half complex plane is mapped by Mandelstam mapping (A1).

for a suitable $z_0^{(r)}$ satisfying

$$\frac{d}{dz}\rho(z)\bigg|_{z=z_0}=\sum_{i=1}^{N}\frac{\alpha_i}{z_0-Z_i}=0 . \tag{A6}$$

In the 3-string case of Fig. 2 or the 4-string case of Fig. 6, the solution of (A6) in the upper half-plane is unique and $\tau_0^{(r)}$ is common to all strings.

The (Neumann) functions $\overline{N}{}^{rs}_{nm}$ corresponding to a light-cone diagram like Fig. 27 specified by the parameters (α_i,Z_i) are defined as the Fourier components of the Neumann function $N(\rho,\widetilde{\rho})$ (Refs. 20 and 22):

$$N(\rho_r,\widetilde{\rho}_s)=-\delta_{rs}\left[\sum_{n\geq 1}\frac{2}{n}e^{-n|\xi_r-\widetilde{\xi}_s|}\cos(n\sigma_r)\cos(n\widetilde{\sigma}_s)-2\max(\xi_r,\widetilde{\xi}_s)\right]+2\sum_{n,m\geq 0}\overline{N}{}^{rs}_{nm}e^{n\xi_r+m\widetilde{\xi}_s}\cos(n\sigma_r)\cos(m\widetilde{\sigma}_s)$$

$$=\ln|z-\widetilde{z}|+\ln|z-\widetilde{z}^*|\{=\tfrac{1}{2}\ln[(z-\widetilde{z})(z-\widetilde{z}^*)(z^*-\widetilde{z})(z^*-\widetilde{z}^*)]\} , \tag{A7}$$

where ρ_r and $\widetilde{\rho}_s$ are assumed to lie in the region of the rth and sth string strip, respectively, and

$$\rho_r=\rho(z)=\alpha_r\zeta_r+\tau_0^{(r)}+i\beta_r , \tag{A8a}$$

$$\widetilde{\rho}_s=\rho(\widetilde{z})=\alpha_s\widetilde{\zeta}_s+\tau_0^{(s)}+i\beta_s . \tag{A8b}$$

It is often convenient to rewrite the expansion (A7) as

$$N(\rho_s,\widetilde{\rho}_s)=-\delta_{rs}\left[\sum_{n\geq 1}\frac{1}{2n}(\omega_+^{-n}+\omega_+^{*-n})(\omega_-^{n}+\omega_-^{*n})-2\max(\xi_r,\widetilde{\xi}_s)\right]+\tfrac{1}{2}\sum_{n,m\geq 0}\overline{N}{}^{rs}_{nm}(\omega_r^n+\omega_r^{*n})(\widetilde{\omega}_s^{m}+\widetilde{\omega}_s^{*m}) , \tag{A9}$$

where

$$\omega_r=e^{\zeta_r}=e^{\xi_r+i\sigma_r},\quad \widetilde{\omega}_s=e^{\widetilde{\zeta}_s}=e^{\widetilde{\xi}_s+i\widetilde{\sigma}_s} , \tag{A10}$$

and

$$(\omega_+,\omega_-)=\begin{cases}(\omega_r,\widetilde{\omega}_s) & (\xi_r\geq\widetilde{\xi}_s),\\ (\widetilde{\omega}_s,\omega_r) & (\xi_r\leq\widetilde{\xi}_s) .\end{cases} \tag{A11}$$

Various formulas for the Neumann functions $\overline{N}{}^{rs}_{nm}$ are derived from (A7) and (A1).

1. Integral formulas for $\overline{N}{}^{rs}_{nm}$

We have the following expression for $\overline{N}{}^{rs}_{nm}$:

$$\overline{N}{}^{rs}_{00}=\begin{cases}\ln|Z_r-Z_s| & (r\neq s),\\ -\sum_{i(\neq r)}\dfrac{\alpha_i}{\alpha_r}\ln|Z_r-Z_i|+\dfrac{1}{\alpha_r}\tau_0^{(r)} & (r=s),\end{cases} \tag{A12}$$

$$\overline{N}{}^{rs}_{n0}=\overline{N}{}^{sr}_{0n}=\frac{1}{n}\oint_{Z_r}\frac{dz}{2\pi i}\frac{1}{z-Z_s}e^{-n\zeta_r(z)}\quad(n\geq 1) , \tag{A13}$$

$$\overline{N}{}^{rs}_{nm}=-\frac{1}{nm}\oint_{Z_r}\frac{dz}{2\pi i}\oint_{Z_s}\frac{d\widetilde{z}}{2\pi i}\frac{1}{(z-\widetilde{z})^2}e^{-n\zeta_r(z)-m\widetilde{\zeta}_s(\widetilde{z})}\quad(n,m\geq 1) , \tag{A14}$$

where $\zeta_r(z)$ and $\widetilde{\zeta}_s(\widetilde{z})$ are given by (A8a) and (A8b).

Equation (A12) is derived from (A7) by putting $\widetilde{z}\to Z_s$ (which implies $\widetilde{\xi}_s\to-\infty$) first and then by taking the limit $z\to Z_r$ ($\xi_r\to-\infty$). In the case of $r=s$, it is necessary to substitute the following expression for ξ_r coming from the δ_{rs} term of (A7) before taking the second limit:

$$\xi_r=\frac{1}{\alpha_r}\sum_{i=1}^{N}\alpha_i\ln|z-Z_i|-\frac{1}{\alpha_r}\tau_0^{(r)} , \tag{A15}$$

which is obtained by taking the real part of (A8a).

In order to derive (A13), we first differentiate (A7) [(A9)] with respect to ζ_r:

$$\frac{\partial}{\partial\zeta_r}N(\rho_r,\widetilde{\rho}_s)=\delta_{rs}\left[\tfrac{1}{2}\sum_{n\geq 1}\omega_r^{-n}(\widetilde{\omega}_s^{n}+\widetilde{\omega}_s^{*n})+1\right]+\tfrac{1}{2}\sum_{n,m\geq 0}n\overline{N}{}^{rs}_{nm}\omega_r^n(\widetilde{\omega}_s^{m}+\widetilde{\omega}_s^{*m})$$

$$=\frac{1}{2}\left[\frac{\partial z}{\partial\zeta_r}\right]\left[\frac{1}{z-\widetilde{z}}+\frac{1}{z-\widetilde{z}^*}\right]\quad(\xi_r\geq\widetilde{\xi}_s) . \tag{A16}$$

Here we have assumed that $\xi_r\geq\widetilde{\xi}_s$ and made use of the formula

$$\frac{\partial}{\partial\zeta_r}=\omega_r\frac{\partial}{\partial\omega_r}=\frac{1}{2}\left[\frac{\partial}{\partial\xi_r}-i\frac{\partial}{\partial\sigma_r}\right] . \tag{A17}$$

By taking the limit $\widetilde{z}\to Z_s$ ($\omega_s\to 0$) in (A16), we get

$$\delta_{rs}+\sum_{n\geq 1}n\overline{N}{}^{rs}_{n0}\omega_r^n=\left[\frac{\partial z}{\partial\zeta_r}\right]\frac{1}{z-Z_s} . \tag{A18}$$

Equation (A13) is obtained by performing $\oint d\omega_r\cdot\omega_r^{-n-1}\times$(A18) around the contour enclosing $\omega_r=0$ ($z=Z_r$).

Equation (A14) is similarly obtained by differentiating (A16) with respect to $\widetilde{\zeta}_s$:

$$\frac{\partial}{\partial\zeta_r}\frac{\partial}{\partial\tilde{\zeta}_s}N(\rho_r,\tilde{\rho}_s)=\delta_{rs}\tfrac{1}{2}\sum_{n\geq 1}n\omega_r^{-n}\tilde{\omega}_s^{n}+\tfrac{1}{2}\sum_{n,m\geq 1}nm\overline{N}^{rs}_{nm}\omega_r^n\tilde{\omega}_s^m=\frac{1}{2}\left[\frac{\partial z}{\partial\zeta_r}\right]\left[\frac{\partial\tilde{z}}{\partial\tilde{\zeta}_s}\right]\frac{1}{(z-\tilde{z})^2}, \tag{A19}$$

and then considering

$$\oint d\omega_r\oint d\tilde{\omega}_s\cdot\omega_r^{-n-1}\tilde{\omega}_s^{-m-1}\times(\text{A19}).$$

2. Other properties of $\overline{N}^{rs}_{nm}$

$$\sum_{s=1}^{N}\overline{N}^{rs}_{n0}\alpha_s=\begin{cases}\tau_0^{(r)} & (n=0),\\ 0 & (n\geq 1),\end{cases} \tag{A20}$$

$$\overline{N}^{rs}_{nm}=\left[\frac{\alpha_r}{n}+\frac{\alpha_s}{m}\right]^{-1}\sum_{i=1}^{N}\alpha_i\overline{N}^{ri}_{n0}\overline{N}^{si}_{m0}. \tag{A21}$$

These formulas are easily obtained from (A12)–(A14) and the relation

$$\sum_{i=1}^{N}\frac{\alpha_i}{z-Z_i}=\frac{d\rho(z)}{dz}=\alpha_r\frac{d\zeta_r(z)}{dz}. \tag{A22}$$

3. Singularities at $z_0^{(r)}$

$$\sum_{n\geq 1}\frac{n}{\alpha_r}\overline{N}^{rs}_{n0}e^{n\zeta_r}=\left[\frac{d\rho(z)}{dz}\right]^{-1}\frac{1}{z-Z_s}-\frac{1}{\alpha_r}\delta_{rs}, \tag{A23}$$

$$\sum_{n\geq 1}\frac{n}{\alpha_r}\overline{N}^{rs}_{nm}e^{n\zeta_r}\underset{z\sim z_0^{(r)}}{\sim}\left[\frac{d\rho(z)}{dz}\right]^{-1}\sum_{i=1}^{N}\frac{\alpha_i}{z_0^{(r)}-Z_i}\frac{m}{\alpha_s}\overline{N}^{si}_{m0}\quad(m\geq 1). \tag{A24}$$

Equation (A23) is nothing but (A18). The LHS of (A24) is divergent at $z=z_0^{(r)}$ ($\mathrm{Re}\zeta_r=0$) and is approximated by using (A21) for large n:

$$\overline{N}^{rs}_{nm}\underset{n\to\infty}{\sim}\frac{m}{\alpha_s}\sum_{i=1}^{N}\alpha_i\overline{N}^{ri}_{n0}\overline{N}^{si}_{m0}. \tag{A25}$$

(A25) and (A23) lead to (A24).

4. Projective transformation of $\overline{N}^{rs}_{nm}$

Under the (infinitesimal) projective transformation

$$\delta Z_r=\delta\alpha+\delta\beta Z_r+\delta\gamma Z_r^{\,2}, \tag{A26}$$

the interaction time $\tau_0^{(r)}$ of (A5) and the Neumann functions $\overline{N}^{rs}_{nm}$ change by the following amounts:

$$\delta\tau_0^{(r)}[=\delta\rho(z_0^{(r)})]=\delta\gamma\sum_{i=1}^{N}\alpha_i Z_i, \tag{A27}$$

$$\delta\overline{N}^{rs}_{00}=\delta\beta+\delta\gamma(Z_r+Z_s), \tag{A28}$$

$$\delta\overline{N}^{rs}_{n0}=\delta\gamma\frac{1}{n}\oint_{Z_r}\frac{dz}{2\pi i}e^{-n\zeta_r(z)}\quad(n\geq 1), \tag{A29}$$

$$\delta\overline{N}^{rs}_{nm}=0\quad(n,m\geq 1). \tag{A30}$$

Equations (A28)–(A30) are most easily derived from (A12)–(A14). From the above formulas we see that

$$E_X=\tfrac{1}{2}\sum_{\substack{n,m\geq 0\\ r,s}}\overline{N}^{rs}_{nm}\alpha_{-n}^{(r)}\cdot\alpha_{-m}^{(s)}$$

is invariant under (A26) as long as $\alpha_0^{(r)}$ is conserved, i.e., $\sum_{r=1}^{N}\alpha_0^{(r)}=0$.

APPENDIX B: PROOF OF THE CONNECTION CONDITIONS (3.12), (3.16), and (4.4)

In this appendix we show that the 3-string vertex $|V_X(1,2,3)\rangle$ of (3.9) and $|V_{\text{FP}}(1,2,3)\rangle$ of (3.13) satisfy the connection conditions (3.12) and (3.16), respectively. The following proof applies straightforwardly to the proof of (4.4) for the vertex (4.6) in the 4-string case.

Let us start with the connection condition for $X_\mu^{(r)}$,

$$[\Theta_1X^{(1)}(\sigma_1)+\Theta_2X^{(2)}(\sigma_2)-X^{(3)}(\sigma_3)]\,|V_X(1,2,3)\rangle=0. \tag{B1}$$

Letting the oscillator expansion of $X_\mu^{(r)}(\sigma_r)$ of (2.3),

$$X^{(r)}(\sigma_r)=\frac{1}{\sqrt{\pi}}\left[x_r+i\sum_{n\geq 1}\frac{1}{n}(\alpha_n^{(r)}-\alpha_{-n}^{(r)})\cos(n\sigma_r)\right] \tag{B2}$$

to operate on $|V_X\rangle$ of (3.9), we get

$$\begin{aligned}\sqrt{\pi}X^{(r)}(\sigma_r)\,|V_X(1,2,3)\rangle&=\sqrt{\pi}X^{(r)}(\sigma_r)\left[(2\pi)^d\delta\left[\sum_s p_s\right]\exp(E_X)\,|0\rangle\right]\\&=(2\pi)^d\left[i\delta'\left[\sum_s p_s\right]+\delta\left[\sum_s p_s\right]\left[-i\sum_{n\geq 1}\frac{1}{n}\alpha_{-n}^{(r)}\cos(n\sigma_r)+[X^{(r)}(\sigma_r),E_X]\right]\right]\exp(E_X)\,|0\rangle\\&=(2\pi)^d i\delta'\left[\sum_s p_s\right]e^{E_X}\,|0\rangle+i\left[-\sum_{n\geq 1}\frac{1}{n}\alpha_{-n}^{(r)}\cos(n\sigma_r)+\sum_{\substack{n,m\geq 0\\ s}}\overline{N}^{rs}_{nm}\alpha_{-m}^{(s)}\cos(n\sigma_r)\right]|V_X(1,2,3)\rangle.\end{aligned} \tag{B3}$$

By comparing the coefficient of the oscillator $\alpha_{-n}^{(s)}$ we find that the connection condition (B1) is satisfied if we can show that

$$\sum_s p_s\left[\sum_{r=1,2,}\Theta_r\sum_{m\ge0}\bar N_{m0}^{rs}\cos(m\sigma_r)-\sum_{m\ge0}\bar N_{m0}^{3s}\cos(m\sigma_3)\right]=0 \quad \text{(B4)}$$

and

$$\sum_{r=1,2}\Theta_r\left[-\delta^{rs}\frac{1}{n}\cos(n\sigma_r)+\sum_{m\ge0}\bar N_{mn}^{rs}\cos(m\sigma_r)\right]=-\delta^{3s}\frac{1}{n}\cos(n\sigma_3)+\sum_{m\ge0}\bar N_{mn}^{3s}\cos(m\sigma_3)\quad (n\ge1)\ . \quad \text{(B5)}$$

[The $\delta'(\sum_s p_s)$ part of (B3) automatically satisfies (B1) because $\Theta_1+\Theta_2=1$.] That Eqs. (B4) and (B5) actually hold can be seen by noticing that they are equivalent to the following identity:

$$N(\rho_r,\tilde\rho_s)\,|_{\xi_r=0}=N(\rho_3,\tilde\rho_s)\,|_{\xi_3=0}\quad (r=1 \text{ or } 2)\ , \quad \text{(B6)}$$

for the Neumann function (A7) for three strings with $\alpha_1,\alpha_2>0$ and $\alpha_3<0$ (see Fig. 2). This identity is trivial because ρ_r with $\xi_r=0$ and ρ_3 with $\xi_3=0$ are the same point on the ρ plane.

Next is the connection condition for

$$A_\pm^{(r)}(\sigma_r)=(P^{(r)}\mp X^{(r)'})(\sigma_r)=(1/\sqrt{\pi})\sum_n \alpha_n^{(r)}e^{\pm in\sigma_r}\ .$$

Similarly to (B3) we have

$$A_\pm^{(r)}(\sigma_r)\,|\,V_X(1,2,3)\rangle=\frac{1}{\sqrt{\pi}}\left[\sum_{n\ge0}\alpha_{-n}^{(r)}e^{\mp in\sigma_r}+\sum_{\substack{n,m\ge0\\ s}}n\bar N_{nm}^{rs}\alpha_{-m}^{(s)}e^{\pm in\sigma_r}\right]\times|\,V_X(1,2,3)\rangle\ , \quad \text{(B7)}$$

and the connection condition (3.12) for $O^{(r)}=(1/\alpha_r)P^{(r)}$ and $(1/\alpha_r)X^{(r)'}$ are satisfied if

$$\sum_{r=1,2}\Theta_r\frac{1}{\alpha_r}\left[\delta^{rs}\cos(n\sigma_r)+\sum_{m\ge1}m\bar N_{mn}^{rs}\cos(m\sigma_r)\right]=\frac{1}{\alpha_3}\left[\delta^{3s}\cos(n\sigma_3)+\sum_{m\ge1}m\bar N_{mn}^{3s}\cos(m\sigma_3)\right]\quad(n\ge0) \quad \text{(B8)}$$

and

$$\sum_{r=1,2}\Theta_r\frac{1}{\alpha_r}\left[-\delta^{rs}\sin(n\sigma_r)+\sum_{m\ge1}m\bar N_{mn}^{rs}\sin(m\sigma_r)\right]=\frac{1}{\alpha_3}\left[-\delta^{3s}\sin(n\sigma_3)+\sum_{m\ge1}m\bar N_{mn}^{3s}\sin(m\sigma_3)\right]\quad(n\ge0)\quad \text{(B9)}$$

hold, respectively. [Both in (B8) and (B9) with $n=0$, $\sum_s p_s$ should be multiplied.] Again, (B8) and (B9) are guaranteed by the identities

$$\left.\frac{1}{\alpha_r}\frac{\partial}{\partial\xi_r}N(\rho_r,\tilde\rho_s)\right|_{\xi_r=0}=\left.\frac{1}{\alpha_3}\frac{\partial}{\partial\xi_3}N(\rho_3,\tilde\rho_s)\right|_{\xi_3=0} \quad \text{(B10)}$$

and

$$\left.\frac{1}{\alpha_r}\frac{\partial}{\partial\sigma_r}N(\rho_r,\tilde\rho_s)\right|_{\xi_r=0}=\left.\frac{1}{\alpha_3}\frac{\partial}{\partial\sigma_3}N(\rho_3,\tilde\rho_s)\right|_{\xi_3=0}\ , \quad \text{(B11)}$$

respectively.

The connection condition for $\bar C_\pm^{(r)}=\bar c^{(r)}\mp i\pi_c^{(r)}$, (3.16) with $O^{(r)}=(1/\alpha_r)^2\bar C_\pm^{(r)}$, leads to (B8) and (B9) since

$$\left[\frac{1}{\alpha_r}\right]^2\bar C_\pm^{(r)}(\sigma_r)\,|\,V_{\rm FP}(1,2,3)\rangle=\frac{1}{\alpha_r}\frac{1}{\sqrt{\pi}}\sum_{n=-\infty}^{\infty}\bar\gamma_n^{(r)}e^{\pm in\sigma_r}\delta\left[\sum_s\frac{1}{\alpha_s}\bar c_0^{(s)}\right]\exp(E_{\rm FP})\,|\,0\rangle$$
$$=\frac{1}{\sqrt{\pi}}\frac{1}{\alpha_r}\left[\sum_{n\ge0}\bar\gamma_{-n}^{(r)}e^{\mp in\sigma_r}+\sum_{\substack{n,m\ge0\\ s}}n\bar N_{nm}^{rs}\bar\gamma_{-m}^{(s)}e^{\pm in\sigma_r}\right]|\,V_{\rm FP}(1,2,3)\rangle\ . \quad \text{(B12)}$$

Finally, consider the connection condition for $c^{(r)}$ and $\pi_{\bar c}^{(r)}$. For $\pi_{\bar c}^{(r)}$ we have

$$\alpha_r i\pi_{\bar c}^{(r)}(\sigma_r)\,|\,V_{\rm FP}(1,2,3)\rangle=\frac{1}{\sqrt{\pi}}\left[\alpha_r c_0^{(r)}-i\sum_{n\ge1}\frac{1}{n}(\gamma_n^{(r)}-\gamma_{-n}^{(r)})\cos(n\sigma_r)\right]\delta\left[\sum_s\frac{1}{\alpha_s}\bar c_0^{(s)}\right]\exp(E_{\rm FP})\,|\,0\rangle$$
$$=\frac{1}{\sqrt{\pi}}e^{E_{\rm FP}}\,|\,0\rangle-i\frac{1}{\sqrt{\pi}}\left[-\sum_{n\ge1}\frac{1}{n}\gamma_{-n}^{(r)}\cos(n\sigma_r)+\sum_{\substack{n\ge0\\ m\ge1\\ s}}\bar N_{nm}^{rs}\gamma_{-m}^{(s)}\cos(n\sigma_r)\right]|\,V_{\rm FP}(1,2,3)\rangle\ , \quad \text{(B13)}$$

and hence the condition (3.16) for $O^{(r)}=\alpha_r i\pi_{\bar{c}}^{(r)}$ is satisfied owing to the previous identity (B5) for $X^{(r)}$. As for $c^{(r)}$ it is sufficient to show

$$[\Theta_1 c^{(1)\prime}(\sigma_1)+\Theta_2 c^{(2)\prime}(\sigma_2)-c^{(3)\prime}(\sigma_3)]\,|\,V_{\rm FP}(1,2,3)\rangle=0 \tag{B14}$$

for $c'=(\partial/\partial\sigma)c$. Indeed, then, since $c^{(r)\prime}(\sigma_r)$ vanishes at the string ends, (B14) can be integrated to yield the connection condition (3.16) for $O^{(r)}=\alpha_r c^{(r)}$. The connection condition (B14) leads to the previous equation (B8) for $n\geq 1$ because

$$\begin{aligned} c^{(r)\prime}(\sigma_r)\,|\,V_{\rm FP}(1,2,3)\rangle &= -\frac{1}{\sqrt{\pi}}\frac{1}{\alpha_r}\sum_{n\geq 1}(\gamma_n^{(r)}+\gamma_{-n}^{(r)})\cos(n\sigma_r)\,|\,V_{\rm FP}(1,2,3)\rangle \\ &= -\frac{1}{\sqrt{\pi}}\frac{1}{\alpha_r}\left[\sum_{n\geq 1}\gamma_{-n}^{(r)}\cos(n\sigma_r)+\sum_{\substack{n,m\geq 1\\ s}} n\bar{N}_{nm}^{rs}\gamma_{-m}^{(s)}\cos(n\sigma_r)\right]|\,V_{\rm FP}(1,2,3)\rangle\ . \end{aligned} \tag{B15}$$

APPENDIX C: DERIVATION OF (3.47)

From (3.33) and (3.46)

$$\rho_0-\rho(z)=a(z-z_0)^2+b(z-z_0)^3+c(z-z_0)^4+\cdots\ , \tag{3.33}$$

$$\rho(z')=\rho(z)-a\delta\ , \tag{3.46}$$

we have

$$f^2+\frac{b}{a}f^3+\frac{c}{a}f^4+\cdots=\delta+\epsilon^2+\frac{b}{a}\epsilon^3+\frac{c}{a}\epsilon^4+\cdots\ , \tag{C1}$$

where

$$f\equiv z'-z_0,\quad \epsilon\equiv z-z_0\ . \tag{C2}$$

By expanding f in terms of δ as

$$f=\epsilon+\sum_{n\geq 1}f_n(\epsilon)\delta^n\ , \tag{C3}$$

inserting in (C1), and comparing the coefficients of δ^n ($n=1,2,3$) we get

$$\frac{1}{f_1}=2\epsilon+\frac{3b}{a}\epsilon^2+\frac{4c}{a}\epsilon^3+O(\epsilon^4)\ , \tag{C4}$$

$$\frac{f_2}{(f_1)^2}=-\frac{1}{2\epsilon}-\frac{3b}{4a}-\left[\frac{2c}{a}-\frac{9b^2}{8a^2}\right]\epsilon+O(\epsilon^2)\ , \tag{C5}$$

$$\frac{f_3}{(f_1)^3}=\frac{1}{2\epsilon^2}+\frac{b}{a}\frac{1}{\epsilon}+\left[\frac{2c}{a}-\frac{3b^2}{8a^2}\right]+O(\epsilon)\ . \tag{C6}$$

Equation (3.47b) now follows from (3.35):

$$\left[\frac{d\rho(z)}{dz}\right]^{-1}=-\frac{1}{2a}\frac{1}{\epsilon}+\frac{3b}{4a^2}+\left[\frac{c}{a^2}-\frac{9b^2}{8a^3}\right]\epsilon+O(\epsilon^2)\ , \tag{3.35'}$$

and

$$\begin{aligned}(z'-z)^{-1}=(f-\epsilon)^{-1}&=\left[\sum_{n\geq 1}f_n\delta^n\right]^{-1}\\ &=\frac{1}{f_1}\frac{1}{\delta}-\frac{f_2}{(f_1)^2}+O(\delta)\\ &=O(\epsilon)\frac{1}{\delta}+\left[\frac{1}{2\epsilon}+\frac{3b}{4a}+O(\epsilon)\right]+O(\delta)\ .\end{aligned} \tag{C7}$$

The left-hand side of (3.47a) is rewritten as

$$\left[\frac{d\rho(z)}{dz}\right]^{-1}\frac{dz'}{dz}\frac{1}{(z'-z)^2}\ . \tag{C8}$$

Then from (C2) and (C3) we have

$$\begin{aligned}\frac{dz'}{dz}\frac{1}{(z'-z)^2}&=\left[1+\sum_{n\geq 1}\frac{df_n}{d\epsilon}\delta^n\right]\left[\sum_{n\geq 1}f_n\delta^n\right]^{-2}\\ &=\left[\frac{1}{f_1}\right]^2\frac{1}{\delta^2}+\left[-\frac{d}{d\epsilon}\left[\frac{1}{f_1}\right]-\frac{2}{f_1}\frac{f_2}{(f_1)^2}\right]\frac{1}{\delta}+\left[\frac{d}{d\epsilon}\left[\frac{f_2}{(f_1)^2}\right]+3\left[\frac{f_2}{f_1{}^2}\right]^2-2\frac{f_3}{f_1{}^3}\right]+O(\delta)\ .\end{aligned} \tag{C9}$$

Equation (3.47a) is a consequence of (C4)–(C6), (3.35), and (C9).

APPENDIX D: DERIVATION OF EQ. (3.51)

In this appendix we prove Eq. (3.51), i.e., the equality

$$\frac{1}{\alpha_s}\left[\frac{1}{n}\delta_{rs}\cos(n\sigma_I^{(s)})-\sum_{m\geq 1}\bar{N}^{rs}_{nm}\cos(m\sigma_I^{(s)})\right]=\chi^{sr}\bar{N}^{r}_{n}+\frac{1}{\alpha_s}\sum_{m=1}^{n-1}\bar{N}^{rr}_{n-m,m}\,,$$

$$\chi^{sr}=\delta^{rs}(\alpha_{s-1}-\alpha_{s+1})/\alpha_s+\sum_{t=1}^{3}\epsilon^{srt}$$
$$(\epsilon^{123}=1,\ \alpha_0\equiv\alpha_3,\ \alpha_4\equiv\alpha_1)\ . \quad \text{(D1)}$$

First of all we should notice that the Neumann function defined in (3.11) and appearing here is the one for which a special parameter choice $Z_1=1$, $Z_2=0$, and $Z_3=\infty$ is made in the Mandelstam mapping (A1). To deal with this choice properly, let us first consider the case in which we set $Z_1=1$, $Z_2=0$ but keep $Z_3\gg 1$ finite. Then the Mandelstam mapping (A1) is given by

$$\rho(z)=\alpha_1\ln(z-1)+\alpha_2\ln z+\alpha_3\ln(z-Z_3)\quad (Z_3\gg 1)\ . \quad \text{(D2)}$$

Introducing a new variable $\hat{z}$ by (a projective transformation)

$$\hat{z}\equiv(1-Z_3)z/(z-Z_3)\ , \quad \text{(D3)}$$

the above $\rho(z)$ is rewritten into the form

$$\rho(z)=\hat{\rho}(\hat{z})+T_0\ ,$$
$$\hat{\rho}(\hat{z})=\alpha_1\ln(\hat{z}-1)+\alpha_2\ln\hat{z}\ , \quad \text{(D4)}$$

with a very large "time" translation $T_0=-\alpha_1\ln(-Z_3)-\alpha_2\ln(1-Z_3)$. Now, the RHS of the basic formula (A7) of the Neumann function is written as

$$\ln|z-\tilde{z}|+\ln|z-\tilde{z}^*|=\ln|\hat{z}-\tilde{\hat{z}}|+\ln|\hat{z}-\tilde{\hat{z}}^*|+2\ln|z-Z_3|+2\ln|\tilde{z}-Z_3|-2\ln|Z_3(1-Z_3)|\ . \quad \text{(D5)}$$

On the other hand, Eq. (A7) itself in the limits $z\to Z_3$ ($\xi_r\to-\infty$) and $\tilde{z}\to Z_3$ ($\tilde{\xi}_s\to-\infty$), respectively, leads to

$$\ln|\tilde{z}-Z_3|=\delta_{s3}\tilde{\xi}_s+\sum_{m\geq 0}\bar{N}^{3s}_{0m}e^{m\tilde{\xi}_s}\cos(m\tilde{\sigma}_s)\ ,$$
$$\ln|z-Z_3|=\delta_{r3}\xi_r+\sum_{n\geq 0}\bar{N}^{r3}_{n0}e^{n\xi_r}\cos(n\sigma_r)\ , \quad \text{(D6)}$$

which further yield by taking $\tilde{z}\to 1,0$ ($\tilde{\xi}_{1,2}\to-\infty$) or $z\to 1,0$ ($\xi_{1,2}\to-\infty$)

$$\ln|Z_3(1-Z_3)|=\bar{N}^{13}_{00}+\bar{N}^{23}_{00}\ . \quad \text{(D7)}$$

By substituting (D6) and (D7) into (D5) we find that the basic Equation (A7) or (A9) is now rewritten in the form

$$\begin{aligned}\hat{N}(\hat{\rho}_r,\tilde{\hat{\rho}}_s)&\equiv N(\rho_r,\tilde{\rho}_s)-2[\ln|z-Z_3|+\ln|\tilde{z}-Z_3|-\ln|Z_3(1-Z_3)|]\\&=-\delta_{rs}\left[\sum_{n\geq 1}\frac{1}{2n}(\omega_+^{-n}+\omega_+^{*-n})(\omega_-^{n}+\omega_-^{*n})-2\max(\xi_r,\tilde{\xi}_s)\right]\\&\quad+\tfrac{1}{2}\sum_{n,m\geq 0}\hat{\bar{N}}^{rs}_{nm}(\omega_r^n+\omega_r^{*n})(\tilde{\omega}_s^m+\tilde{\omega}_s^{*m})-2\delta_{r3}\xi_r-2\delta_{s3}\tilde{\xi}_s\\&=\ln|\hat{z}-\tilde{\hat{z}}|+\ln|\hat{z}-\tilde{\hat{z}}^*|\ ,\end{aligned} \quad \text{(D8a)}$$

$$\hat{\bar{N}}^{rs}_{nm}=\bar{N}^{rs}_{nm}-\delta_{n0}\bar{N}^{3s}_{0m}-\delta_{m0}\bar{N}^{r3}_{n0}+\delta_{n0}\delta_{m0}(\bar{N}^{13}_{00}+\bar{N}^{23}_{00})\ . \quad \text{(D8b)}$$

This equation for $\hat{N}$ is easily seen to be in fact *independent* of Z_3 and we can safely take $Z_3\to\infty$ (although not necessary). So Eq. (D8) is regarded as a basic equation for the Neumann function $\hat{N}$ for the parameter choice $Z_1=1$, $Z_2=0$, $Z_3=\infty$, in place of (A7) or (A9). Indeed it is these Fourier components $\hat{\bar{N}}^{rs}_{nm}$ in (D8b) that were given in (3.11). One can easily derive formulas (3.11) from (D8b) by using (A13), (A20), and (A21). The $\hat{\rho}(\hat{z})$ in Eq. (D4) gives the Mandelstam mapping for this parameter choice, and defines the following variables similarly to (A4), (A5), and (A6):

$$\hat{\rho}(\hat{z})=\alpha_1\ln(\hat{z}-1)+\alpha_2\ln\hat{z}=\alpha_r\hat{\xi}_r+\hat{\tau}_0^{(r)}+i\hat{\beta}_r\ , \quad \text{(D9)}$$

$$\hat{\tau}_0^{(r)}=\mathrm{Re}\hat{\rho}(\hat{z}_0^{(r)})=\sum_{i=1}^{2}\alpha_i\ln|\hat{z}_0^{(r)}-Z_i|\ , \quad \text{(D10)}$$

$$\left.\frac{d\hat{\rho}(\hat{z})}{d\hat{z}}\right|_{\hat{z}=\hat{z}_0}=\sum_{i=1}^{2}\frac{\alpha_i}{\hat{z}_0-Z_i}=0\ . \quad \text{(D11)}$$

By (D4), $\hat{\xi}_r$ is identical with the former ξ_r and $\hat{\tau}_0^{(r)}=\tau_0^{(r)}-\mathrm{Re}T_0$, $\hat{\beta}_r=\beta_r-\mathrm{Im}\,T_0$. From (D10) and (D11) we find explicitly

$$\hat{z}_0^{(1)}=\hat{z}_0^{(2)}=\hat{z}_0^{(3)}=-\frac{\alpha_2}{\alpha_3}\ , \quad \text{(D12a)}$$

$$\hat{\tau}_0^{(1)}=\hat{\tau}_0^{(2)}=\hat{\tau}_0^{(3)}=\sum_{i=1}^{3}\alpha_i\ln|\alpha_i|=\tau_0\ . \quad \text{(D12b)}$$

From now we drop the carets since we treat only caretted quantities hereafter.

Now we return to the subject of proving (D1). Taking the limit $\tilde{z}\to Z_s$ ($\tilde{\xi}_s\to-\infty$) in Eq. (D8a) we obtain

$$\ln|z-Z_s| = \delta_{rs}\xi_r - \delta_{r3}\xi_r + \tfrac{1}{2}\sum_{n\geq 0}\bar{N}^{rs}_{n0}(\omega_r^n + \omega_r^{*n}) . \quad \text{(D13)}$$

Operating $\omega_r\partial/\partial\omega_r$ on (D13), we get

$$\omega_r\frac{\partial z}{\partial\omega_r} = (z-Z_s)\left[\delta_{rs} - \delta_{r3} + \sum_{n\geq 1} n\bar{N}^{rs}_{n0}\omega_r^n\right] . \quad \text{(D14)}$$

(Note that $\partial/\partial\omega_r$ is a differentiation with ω_r^* fixed and so $\omega_r\partial/\partial\omega_r\xi_r = \frac{1}{2}$.) While the differentiation $\omega_r\partial/\partial\omega_r$ of (D8a) yields with an abbreviation $\tilde{\Omega}^n_s \equiv (\tilde{\omega}^n_s + \tilde{\omega}^{*n}_s)$

$$-\delta_{rs}\sum_{n\geq 1}\omega_r^n\tilde{\Omega}_s^{-n} + \sum_{n,m\geq 0} n\bar{N}^{rs}_{nm}\omega_r^n\tilde{\Omega}^m_s - 2\delta_{r3} = \left[\frac{1}{z-\tilde{z}} + \frac{1}{z-\tilde{z}^*}\right]\omega_r\frac{\partial z}{\partial\omega_r} , \quad \text{(D15)}$$

when $\xi_r < \tilde{\xi}_s$. Replacing $\omega_r\partial z/\partial\omega_r$ by the RHS of (D14) for $s=2$, we find

$$\delta_{rs}\sum_{n\geq 1}\omega_r^n\tilde{\Omega}_s^{-n} - \sum_{n,m\geq 1} n\bar{N}^{rs}_{nm}\omega_r^n\tilde{\Omega}^m_s = 2\sum_{n\geq 1} n\bar{N}^{rs}_{n0}\omega_r^n - 2\delta_{r3} - \left[\frac{z}{z-\tilde{z}} + \frac{z}{z-\tilde{z}^*}\right]\left[\delta_{r2} - \delta_{r3} + \sum_{n\geq 1} n\bar{N}^{r2}_{n0}\omega_r^n\right] . \quad \text{(D16)}$$

On the other hand, differentiation $\omega_r\partial/\partial\omega_r$ of (D9) leads to

$$\left[\frac{\alpha_1}{z-1} + \frac{\alpha_2}{z}\right]\omega_r\frac{\partial z}{\partial\omega_r} = \alpha_r . \quad \text{(D17)}$$

Substituting (D14) for $s=1$ into this, we obtain

$$\alpha_3 + \frac{\alpha_2}{z} = -\frac{\alpha_r}{\delta_{r1} - \delta_{r3} + \sum_{n\geq 1} n\bar{N}^{r1}_{n0}\omega_r^n} . \quad \text{(D18)}$$

Now, in formula (D16), we put $\tilde{z}$ on the interaction point $z_0^{(s)} = -\alpha_2/\alpha_3$ of (D12a) which corresponds to taking $\tilde{\xi}_s = 0$ $(>\xi_r)$ and $\tilde{\sigma}_s = \sigma_I^{(s)}$. Then, $\tilde{\Omega}^m_s = 2\cos(m\sigma_I^{(s)})$ and (D16) becomes

$$\left[\delta_{rs}\sum_{n\geq 1}\cos(n\sigma_I^{(s)}) - \sum_{n,m\geq 1} n\bar{N}^{rs}_{nm}\cos(m\sigma_I^{(s)})\right]\omega_r^n = \sum_{n\geq 1} n\bar{N}^{rs}_{n0}\omega_r^n - \delta_{r3} - \alpha_3\left[\alpha_3 + \frac{\alpha_2}{z}\right]^{-1}\left[\delta_{r2} - \delta_{r3} + \sum_{n\geq 1} n\bar{N}^{r2}_{n0}\omega_r^n\right] . \quad \text{(D19)}$$

Because of (D18) the last two terms equal

$$-\delta_{r3} + \left[\frac{\alpha_3}{\alpha_r}\right]\left[\delta_{r1} - \delta_{r3} + \sum_{n\geq 1}\bar{N}^{r1}_{n0}n\omega_r^n\right]\left[\delta_{r2} - \delta_{r3} + \sum_{n\geq 1}\bar{N}^{r2}_{n0}n\omega_r^n\right]$$

$$= \frac{\alpha_3}{\alpha_r}\left[\sum_{n\geq 1}[(\delta_{r1} - \delta_{r3})\bar{N}^{r2}_{n0} + (\delta_{r2} - \delta_{r3})\bar{N}^{r1}_{n0}]n\omega_r^n + \sum_{n,m\geq 1} nm\bar{N}^{r1}_{n0}\bar{N}^{r2}_{m0}\omega_r^{n+m}\right] . \quad \text{(D20)}$$

Then, by a little calculation with the help of the formulas given in (3.11),

$$\bar{N}^{rs}_{nm} = -\alpha_1\alpha_2\alpha_3\left[\frac{\alpha_r}{n} + \frac{\alpha_s}{m}\right]^{-1}\bar{N}^r_n\bar{N}^s_m \quad (n,m\geq 1) , \quad \text{(D21)}$$

$$\bar{N}^{rs}_{n0} = -c_s\frac{\alpha_1\alpha_2}{\alpha_s}\bar{N}^r_n \quad (c_1,c_2,c_3) = (1,-1,0) \quad (n\geq 1) , \quad \text{(D22)}$$

the RHS of (D19) is shown to give

$$\sum_{n\geq 1}\left[\left[(\delta_{s2}\alpha_1 - \delta_{s1}\alpha_2) + \frac{\alpha_1\alpha_3}{\alpha_r}(\delta_{r1} - \delta_{r3}) - \frac{\alpha_2\alpha_3}{\alpha_r}(\delta_{r2} - \delta_{r3})\right]\bar{N}^r_n + \sum_{m=1}^{n-1}\bar{N}^{rr}_{n-m,m}\right]n\omega_r^n . \quad \text{(D23)}$$

It is easy to see that the quantity enclosed by the large parentheses here is just α_s times χ^{sr} given in (D1). Therefore, comparing the coefficients of $n\omega_r^n$ of the LHS of (D19) and of (D23), we obtain the desired equation (D1).

APPENDIX E: DIRECT PROOF OF (3.7) IN TERMS OF CREATION-ANNIHILATION OPERATORS

We present in this appendix an alternative proof of the $O(g)$ nilpotency equation (3.7). This proof is a straightforward one by a direct computation of $\sum_{r=1}^{3} Q^{(r)} | V\rangle$ in terms of creation and annihilation operators of oscillator modes and necessarily becomes more lengthy and tedious than the one given in Sec. III. Nevertheless it has an advantage in manifesting how the intercept parameter $\alpha(0)$ is required to be 1 in addition to the condition $d=26$.

For convenience we Fourier transform the 3-string vertex $|V(1,2,3)\rangle$ given in (3.54) into the old c_0 representation:

$$\int d\bar{c}_0^{(1)} d\bar{c}_0^{(2)} d\bar{c}_0^{(3)} \exp\left[\sum_{r=1}^{3} c_0^{(r)} \bar{c}_0^{(r)}\right] |V(1,2,3)\rangle = -\mu(\alpha_1,\alpha_2,\alpha_3)\delta(1,2,3) W e^{F(1,2,3)} |0\rangle_{123} ,$$
$$W \equiv W^{(1)} W^{(2)} W^{(3)} = \frac{1}{3!} \sum_{r,s,t=1}^{3} \epsilon_{rst} W^{(r)} W^{(s)} W^{(t)} ,$$
$$W^{(r)} = c_0^{(r)} + w^{(r)} \quad [w^{(r)} \text{ given in } (3.51)] . \tag{E1}$$

In this representation Q_B and L take the forms

$$Q_B = c_0 L + \frac{\partial}{\partial c_0} M + \tilde{Q}_B ,$$
$$L = -\tfrac{1}{2}p^2 - \sum_{n\geq 1} [\alpha_{-n}\cdot\alpha_n + i(\gamma_{-n}\bar{\gamma}_n - \bar{\gamma}_{-n}\gamma_n)] + \alpha(0) , \tag{E2}$$

where $\alpha(0)$ is the intercept parameter which we will see soon to be fixed to its critical value 1 also by the condition (3.7), $\sum_{r=1}^{3} Q_B^{(r)} | V\rangle = 0$. More precisely we show that the equation

$$Q_B W e^F |0\rangle = 0, \quad Q_B \equiv \sum_{r=1}^{3} Q_B^{(r)} , \tag{E3}$$

holds for each fixed set of values of (p_r,α_r) $(r=1,2,3)$ satisfying conservations

$$\sum_{r=1}^{3} p_r = \sum_{r=1}^{3} \alpha_r = 0$$

if and only if the critical values $\alpha(0)=1$ and $d=26$ are satisfied.

Multiplying e^{-F} by (E3), we can rewrite it equivalently as

$$e^{-F} Q_B W e^F |0\rangle = \left[\tfrac{1}{2} \sum_{r,s,t=1}^{3} \epsilon_{rst} W^{(r)} W^{(s)} \{Q_B^{(F)}, W^{(t)}\} - W Q_B^{(F)}\right] |0\rangle = 0 ,$$
$$Q_B^{(F)} \equiv e^{-F} Q_B e^F = Q_B + [Q_B,F] + \tfrac{1}{2}[[Q_B,F],F] . \tag{E4}$$

Here in the definition of $Q_B^{(F)}$ the terms of higher power in F do not survive since Q_B contains only terms at most quadratic in annihilation operators. By multiplying $W^{(u)}$ $(u=1, 2, \text{or } 3)$ with (E4) and using $(W^{(r)})^2=0$ for each r, we see that the equation

$$W\{Q_B^{(F)}, W^{(r)}\} |0\rangle = 0 \tag{E5}$$

should hold for any r as necessary conditions for (E4). Further, differentiating (E5) with respect to $c_0^{(r)}$ and summing it over $r=1$—3, we obtain

$$\left[\tfrac{1}{2} \sum_{r,s,t=1}^{3} \epsilon_{rst} W^{(r)} W^{(s)} \{Q_B^{(F)}, W^{(t)}\} - W \sum_{r=1}^{3} [L^{(r)}, W^{(r)}]\right] |0\rangle = 0 . \tag{E6}$$

By using (E6), Eq. (E4) is rewritten as

$$W\left[Q_B^{(F)} - \sum_{r=1}^{3} [L^{(r)}, W^{(r)}]\right] |0\rangle = 0 . \tag{E7}$$

As a result, the proof of (E3) or (E4) is reduced to that of (E5) and (E7), which we can write in the forms

$$\{Q_B^{(F)}, W^{(r)}\} |0\rangle \Big|_{c_0^{(s)} = -w^{(s)}\ (s=1,2,3)} = 0 , \tag{E5'}$$

$$\left[Q_B^{(F)} - \sum_{r=1}^{3} [L^{(r)}, W^{(r)}]\right] |0\rangle \Bigg|_{c_0^{(s)} = -w^{(s)}\ (s=1,2,3)} = 0 , \tag{E7'}$$

since

$$W = \prod_{s=1}^{3} W^{(s)} = \prod_{s=1}^{3} (c_0^{(s)} + w^{(s)})$$

is a δ function $\delta(W^{(1)})\delta(W^{(2)})\delta(W^{(3)})$.

After some straightforward algebraic computations by using commutation relations, we find that Eq. (E7') is decomposed into the following four sets of equations, each implying an infinite number of identities among the Fourier components $\bar{N}_{mn}^{rs}$ of the Neumann function:

$$\sum_{\substack{m\geq 1\\ r}} \frac{1}{m} \left\{\frac{1}{\alpha_r} \sum_{n=1}^{m-1} \left[\left[\frac{d-2}{2}\right] n(m-n) - \frac{m^2}{2}\right] \bar{N}_{n,m-n}^{rr} - m w_m^{rr} + \alpha(0) \sum_s w_m^{sr}\right\} \gamma_{-m}^{(r)} = 0 , \tag{E8}$$

$$\sum_{\substack{m\geq 1\\ r,s}} \left[\frac{1}{m} p_r^2 w_m^{rs} - \delta_{rs} \mathbf{P}\cdot\left[2p_r \frac{1}{\alpha_r} \bar{N}_m^r - \frac{1}{\alpha} \mathbf{P} \sum_{n=1}^{m-1} \bar{N}_{m-n,n}^{rr}\right]\right] \gamma_{-m}^{(s)} = 0 , \tag{E9}$$

$$\sum_{\substack{m,n\geq 1\\ r,s}}(1/n\alpha_r\alpha_s)\Bigg[\delta_{rs}\alpha_r(m+n)\bar{N}^{r}_{m+n}+\epsilon_{rs}n\bar{N}^{rs}_{mn}-\alpha_r\alpha_s m\bar{N}^{r}_{m}w^{rs}_{n}+\alpha_r\sum_{k=1}^{n-1}k(n-k)\bar{N}^{rs}_{m,n-k}\bar{N}^{s}_{k}-\alpha_s\sum_{k=1}^{m-1}nk\bar{N}^{r}_{k}\bar{N}^{rs}_{m-k,n}\Bigg]\mathbf{P}\cdot\alpha^{(r)}_{-m}\gamma^{(s)}_{-n}=0\ , \tag{E10}$$

$$\sum_{\substack{m,n,k\geq 1\\ r,s,t}}\frac{1}{k}\Bigg[n\bar{N}^{rs}_{mn}w^{st}_{k}-\frac{\delta_{st}}{\alpha_s}(n+k)\bar{N}^{rs}_{m,n+k}+\frac{1}{2}\frac{\delta_{rs}}{\alpha_r}k\bar{N}^{rt}_{m+n,k}+\frac{1}{\alpha_s}\sum_{l=1}^{n-1}k(n-l)\bar{N}^{rs}_{m,n-l}\bar{N}^{st}_{lk}-\frac{1}{2\alpha_t}\sum_{l=1}^{k-1}m(k-l)\bar{N}^{rt}_{ml}\bar{N}^{st}_{n,k-l}\Bigg]\alpha^{(r)}_{-m}\cdot\alpha^{(s)}_{-n}\gamma^{(t)}_{-k}=0\ . \tag{E11}$$

Here w^{rs}_m is defined from $w^{(r)}$ in (3.51) by the relation

$$w^{(r)}=i\sum_{s=1}^{3}\sum_{m\geq 1}w^{rs}_{m}\gamma^{(s)}_{-m}/m\ ,$$

or, more explicitly,

$$w^{rs}_{m}=\chi^{rs}m\bar{N}^{s}_{m}+\alpha_r{}^{-1}\sum_{n=1}^{m-1}m\bar{N}^{ss}_{m-n,n}\ , \tag{E12}$$

with χ^{rs} given in (3.51), and α in (E9) and ϵ_{rs} in (E10) are defined by

$$\alpha=\alpha_1\alpha_2\alpha_3,\quad \epsilon_{rs}=\sum_{t=1}^{3}\epsilon_{rst}\ . \tag{E13}$$

As a matter of fact Eq. (E7′) contains another type of term proportional to oscillators $\bar{\gamma}^{(r)}_{-m}\gamma^{(s)}_{-n}\gamma^{(t)}_{-k}$, which yields, however, the same equations as (E11). On the other hand, Eq. (E5′) consists of only the terms quadratic in ghost oscillators, which read

$$\sum_{\substack{m,n\geq 1\\ s,t}}\frac{1}{mn}\left[\frac{\delta_{st}}{2\alpha_s}(n-m)w^{rs}_{m+n}+mw^{rs}_{m}w^{st}_{n}+\frac{2\delta_{rs}}{\alpha_r{}^2}mn\bar{N}^{st}_{mn}+\sum_{k=1}^{m-1}n(m+k)w^{rs}_{m-k}\bar{N}^{st}_{kn}\right]\gamma^{(s)}_{-m}\gamma^{(t)}_{-n}=0\quad (\text{for } r=1,2,3)\ . \tag{E14}$$

After all these five equations (E8)–(E11) and (E14) are equivalent to the $O(g)$ nilpotency condition (3.7).

Let us start with the proof of the first equation (E8). We should note that those terms in (E8) come from the renormal ordering of BRS charge at the vertex, and hence *depend* on the values of space-time dimension d and the intercept $\alpha(0)$. For the functions $f_n(x)=\Gamma(nx)/n!\Gamma(nx-n+1)$ given in (3.11) which are related to the components $\bar{N}^{rs}_{mn}$ and $\bar{N}^{r}_{n}$ of the Neumann function we have a generating functional $y(\omega)$ introduced by Mandelstam,[20]

$$y(\omega)\equiv\gamma\sum_{n\geq 1}f_n(\gamma)\omega^n\ , \tag{E15}$$

which satisfies an equation

$$1=e^{y/\gamma}-\omega e^{y}\ . \tag{E16}$$

If we define another function $X(\omega)$ by

$$X(\omega)=\frac{1}{\gamma}\omega\frac{\partial}{\partial\omega}y(\omega)=\sum_{n\geq 1}nf_n(\gamma)\omega^n\ , \tag{E17}$$

then, by differentiating (E16) twice and eliminating y, we find

$$\omega\frac{\partial}{\partial\omega}X-X+(1-2\gamma)X^2+\gamma(1-\gamma)X^3=0\ . \tag{E18}$$

Further the application of an operator

$$\left[X+\frac{1-2\gamma}{3\gamma(1-\gamma)}\right]\omega\frac{\partial}{\partial\omega}-3\omega\frac{\partial}{\partial\omega}X$$

to this yields an equation (at most quadratic in X)

$$(1-2\gamma)\left[\left[\omega\frac{\partial}{\partial\omega}\right]^2X-\omega\frac{\partial}{\partial\omega}X\right]+3\gamma(1-\gamma)\left[X\left[\omega\frac{\partial}{\partial\omega}\right]^2X-3\left[\omega\frac{\partial}{\partial\omega}X\right]^2\right]+2(1-\gamma+\gamma^2)X\omega\frac{\partial}{\partial\omega}X=0\ . \tag{E19}$$

It is easy to see that, by using Eqs. (3.11) and taking $\gamma=-\alpha_{r+1}/\alpha_r$ Eq. (E19) reads in its component form

$$(\alpha_{r-1}-\alpha_{r+1})(m+1)\bar{N}^{r}_{m}+\left[m+\frac{\alpha_{r-1}}{\alpha_{r+1}}-\frac{\alpha_r}{\alpha_{r-1}}\right]\sum_{n=1}^{m-1}\bar{N}^{rr}_{n,m-n}+\sum_{n=1}^{m-1}\frac{(n-m)(12n-m)}{m}\bar{N}^{rr}_{n,m-n}=0\ . \tag{E20}$$

If we use this identity, Eq. (E8) is reduced to the form

$$\sum_{\substack{m\geq 1\\ r}} \frac{1}{m}\left[\frac{1}{\alpha_r}\sum_{n=1}^{m-1}\left[\frac{d-26}{2}\right] n(m-n)\overline{N}^{rr}_{n,m-n} + [\alpha(0)-1]\sum_s w^{sr}_m\right]\gamma^{(r)}_{-m}=0 . \quad \text{(E21)}$$

The LHS corresponds to the quantity (3.49) appearing in the previous proof in Sec. III, and vanishes if and only if the critical conditions $d=26$ and $\alpha(0)=1$ are satisfied as is clearly seen.

Next is the second equation (E9). It actually holds simply by the momentum-conservation condition $\sum_{r=1}^{3}p_r^\mu=0$. It is interesting to note that we could have determined uniquely the ghost front factor W of the vertex $|V\rangle$ from this equation alone if W is assumed to be cubic in ghost operators.

For the proof of the remaining three equations, we need one more relation [(E28) below] which gives a connection between $m\overline{N}^{rs}_{mn}$ and $m\overline{N}^{r}_{m}$. It is derived as follows. Differentiation $\omega_r\partial/\partial\omega_r$ of Eq. (D8a) in the previous appendix yields, when $\xi_r>\tilde{\xi}_s$,

$$\sum_{m,n\geq 0} m\overline{N}^{rs}_{mn}\omega_r^m(\tilde{\omega}_s^n+\tilde{\omega}_s^{*n}) = -\delta_{rs}\left[\frac{\tilde{\omega}_s}{\omega_r-\tilde{\omega}_s}+\frac{\tilde{\omega}_s^*}{\omega_r-\tilde{\omega}_s^*}\right] + \left[\frac{1}{z-\tilde{z}}+\frac{1}{z-\tilde{z}^*}\right] \times\omega_r\frac{\partial z}{\partial\omega_r}+2(\delta_{r3}-\delta_{rs}) . \quad \text{(E22)}$$

Applying an operator $\int_0^{\tilde{\omega}_s}d\tilde{\omega}_s\partial/\partial\tilde{\omega}_s$ to this (i.e., subtraction of the value at $\tilde{\omega}_s=0$), we obtain

$$\sum_{m,n\geq 1} m\overline{N}^{rs}_{mn}\omega_r^m\tilde{\omega}_s^n = -\delta_{rs}\frac{\tilde{\omega}_s}{\omega_r-\tilde{\omega}_s} + \left[\frac{1}{z-\tilde{z}}-\frac{\delta_{s1}}{z-1}-\frac{\delta_{s2}}{z}\right] \times\omega_r\frac{\partial z}{\partial\omega_r} , \quad \text{(E23)}$$

which, by the help of (D14) for $s=2$, is rewritten as

$$\sum_{m,n\geq 1} m\overline{N}^{rs}_{mn}\omega_r^m\tilde{\omega}_s^n = -\delta_{rs}\frac{\tilde{\omega}_s}{\omega_r-\tilde{\omega}_s} + \left[\left[1-\frac{\tilde{z}}{z}\right]^{-1}-\delta_{s1}\left[1-\frac{1}{z}\right]-\delta_{s2}\right] \times\left[\delta_{r2}-\delta_{r3}+\alpha_1\sum_{n\geq 1}n\overline{N}^{r}_{n}\omega_r^n\right] . \quad \text{(E24)}$$

As is well known,[20,22] the relation (E16) enables us to solve (A1) with respect to z:

$$z(\omega_1)=\exp\left[\alpha_1\sum_{n\geq 1}\overline{N}^{1}_{n}\omega_1^n\right] , \quad \text{(E25a)}$$

$$z(\omega_2)=\omega_2\exp\left[\frac{\tau_0}{\alpha_2}+\alpha_1\sum_{n\geq 1}\overline{N}^{2}_{n}\omega_2^n\right] , \quad \text{(E25b)}$$

$$z(\omega_3)=\omega_3^{-1}\exp\left[-\frac{\tau_0}{\alpha_3}+i\pi+\alpha_1\sum_{n\geq 1}\overline{N}^{3}_{n}\omega_3^n\right] . \quad \text{(E25c)}$$

If we introduce X_r $(r=1,2,3)$ by slightly redefining the above X in (E17) as

$$X_r(\omega_r)\equiv\alpha_r\sum_{n\geq 1}n\overline{N}^{r}_{n}\omega_r^n , \quad \text{(E26)}$$

then (E25) can be expressed by X_r as

$$z(\omega_1)=\frac{\alpha_1-\alpha_2X_1}{\alpha_1+\alpha_3X_1},\quad z(\omega_2)=\frac{\alpha_2X_2}{\alpha_2-\alpha_3X_2} ,$$
$$z(\omega_3)=-\frac{\alpha_2}{\alpha_3}-\frac{1}{X_3} . \quad \text{(E27)}$$

Here we have used the first derivative of (E16) to eliminate the exponential dependence on the $\overline{N}^{r}_{n}$'s. Owing to (E27) we can eliminate z and $\tilde{z}$ from (E24) and obtain the desired formula after some calculation:

$$\sum_{m,n\geq 1} m\overline{N}^{rs}_{mn}\omega_r^m\tilde{\omega}_s^n=\delta_{rs}\frac{\omega_r}{\tilde{\omega}_s-\omega_r} - \left[\frac{\delta_{rs}+\chi^{sr}X_s-(\alpha/\alpha_r{}^2\alpha_s)X_rX_s}{\epsilon_{rs}+\alpha_r\alpha_s{}^{-1}X_s-\alpha_5\alpha_r{}^{-1}X_r}\right]X_r . \quad \text{(E28)}$$

Now the third equation (E10) is understood to hold because it is proportional to (E28) if rewritten in the form of a differential equation for the generating functionals of $\overline{N}^{rs}_{mn}$ and $\overline{N}^{r}_{m}$.

The procedure meant by this is as follows: First, we replace the oscillator variables $\mathbb{P}\cdot\alpha^{(r)}_{-m}$ and $\gamma^{(s)}_{-n}/n$ in (E10) formally by the powers ω_r^m and $\tilde{\omega}_s^n$ of the variables ω_r and $\tilde{\omega}_s$ of the generating functionals

$$N(\omega_r,\tilde{\omega}_s)\equiv\sum_{m,n\geq 1}\overline{N}^{rs}_{mn}\omega_r^m\tilde{\omega}_s^n , \quad \text{(E29)}$$

and $X_r(\omega_r)$ [or $X_s(\tilde{\omega}_s)$] of $\overline{N}^{rs}_{mn}$ and $\overline{N}^{r}_{m}$. Then we can perform the summation over the indices m, n, and k in (E10) and rewrite it in terms of those generating functionals. For instance, the first and fourth terms in (E10) are rewritten in the following way:

$$\sum_{m,n\geq 1}(m+n)\overline{N}^{r}_{m+n}x^my^n=\sum_{k\geq 2}k\overline{N}^{r}_{k}x^k\sum_{n=1}^{k-1}\left[\frac{y}{x}\right]^n$$
$$=\sum_{k\geq 1}k\overline{N}^{r}_{k}\frac{x^ky-y^kx}{x-y}$$
$$=(x-y)^{-1}\left[\frac{X_r(x)}{\alpha_r}y-\frac{X_s(y)}{\alpha_s}x\right],$$

$$\sum_{m,n\geq 1}\sum_{k=1}^{n-1} k(n-k)\overline{N}^{rs}_{m,n-k}\overline{N}^{s}_{k}x^m y^n = \left[\sum_{m,n\geq 1} n\overline{N}^{rs}_{mn}x^m y^n\right]\left[\sum_{k\geq 1} k\overline{N}^{s}_{k}y^k\right] = \frac{\partial N(x,y)}{\partial x}\frac{X_s(y)}{\alpha_s} . \quad \text{(E30)}$$

It requires considerable efforts to prove the remaining equations (E11) and (E14). To prove them, we again rewrite them in the form of differential equations for the generating functional of $\overline{N}^{rs}_{mn}$ and $\overline{N}^{r}_{m}$ by replacing formally the oscillators $\alpha^{(r)}_{-m}\alpha^{(s)}_{-n}\gamma^{(t)}_{-k}$ by the variables $\frac{1}{2}k\alpha_t(\omega_r^m\widetilde{\omega}_s^n+\widetilde{\omega}_r^m\omega_s^n)\widetilde{\omega}_t^k$ in (E11), and $\gamma^{(s)}_{-m}\gamma^{(t)}_{-n}$ by

$$mn\alpha_s\alpha_t\tfrac{1}{2}(\omega_s^m\widetilde{\omega}_t^n-\widetilde{\omega}_s^m\omega_t^n)$$

in (E14). Then the above identity (E28) enables us to eliminate $N(\omega_r,\widetilde{\omega}_s)$ from those equations. Since Eq. (E18) holds also for X_r's in (E26), we can further rewrite them by using (E18) in the forms depending only on X_r's without derivatives. The coefficients of X_r's in the resultant equations can be shown to vanish by straightforward calculations.

APPENDIX F: PROOF OF EQS. (4.16) AND (4.17)

First, let us calculate $A=A(z_0)$, $dC/dz=(dC/dz)(z_0)$, and $\overline{C}=\overline{C}(z_0)$ appearing in Eqs. (4.16) and (4.17). They have the meaning of (3.38): For example,

$$A(z_0)=\lim_{z\to z_0}\frac{d\rho(z)}{dz}\{A(\rho)^{(-)}+[A(\rho),E_X]\} , \quad \text{(F1)}$$

where $A(\rho)$ in the RHS is given by

$$A(\rho)=\frac{1}{\alpha_r}\frac{1}{\sqrt{\pi}}\sum_{n=-\infty}^{\infty}\alpha^{(r)}_n e^{n\zeta_r} \quad (\zeta_r=\xi_r+i\sigma_r) \quad \text{(F2)}$$

with arbitrary r. From (4.6) we have

$$[A(\rho),E_X]=\frac{1}{\sqrt{\pi}}\frac{1}{\alpha_r}\sum_{\substack{m\geq 0\\ s}}\left[\sum_{n\geq 1} n\overline{N}^{rs}_{nm}e^{n\zeta_r}\right]\alpha^{(s)}_{-m} . \quad \text{(F3)}$$

[We omit the prefix (4) in the Neumann functions $\overline{N}^{(4)rs}_{nm}$.] Since

$$d\rho(z)/dz=O(z-z_0) \quad (z\sim z_0) ,$$

$$A(\rho)^{(-)}\equiv(\sqrt{\pi}\alpha_r)^{-1}\sum_{n\geq 0}\alpha^{(r)}_{-n}e^{-n\zeta_r}$$

in (F1) does not contribute, and we have, from (F3), (A23), and (A24),

$$A(z_0)=\frac{1}{\sqrt{\pi}}\sum_r\frac{\alpha_r}{z_0-Z_r}\left[\frac{p_r}{\alpha_r}+\sum_{\substack{n\geq 1\\ s}}\frac{n}{\alpha_s}\overline{N}^{sr}_{n0}\alpha^{(s)}_{-n}\right] . \quad \text{(F4)}$$

Similarly from

$$\frac{dC}{dz}(z_0)=\lim_{z\to z_0}\frac{d\rho(z)}{dz}\left[\frac{dC(\rho)^{(-)}}{d\rho}+\left[\frac{dC(\rho)}{d\rho},E_{\rm FP}\right]\right] , \quad \text{(F5)}$$

$$\overline{C}(z_0)=\lim_{z\to z_0}\frac{d\rho(z)}{dz}\{\overline{C}(\rho)^{(-)}+[\overline{C}(\rho),E_{\rm FP}]\} , \quad \text{(F6)}$$

with

$$C(\rho)=\alpha_r\frac{1}{\sqrt{\pi}}\sum_{n=-\infty}^{\infty}c^{(r)}_n e^{n\zeta_r} , \quad \text{(F7)}$$

$$\frac{dC(\rho)}{d\rho}=\frac{1}{\alpha_r}\frac{dC}{d\zeta_r}=-i\frac{1}{\sqrt{\pi}}\frac{1}{\alpha_r}\sum_{n\neq 0}\gamma^{(r)}_n e^{n\zeta_r} , \quad \text{(F8)}$$

$$\overline{C}(\rho)=\frac{1}{\sqrt{\pi}}\left[\frac{1}{\alpha_r}\right]^2\sum_{n=-\infty}^{\infty}\overline{c}^{(r)}_n e^{n\zeta_r} = \frac{1}{\sqrt{\pi}}\frac{1}{\alpha_r}\sum_{n=-\infty}^{\infty}\overline{\gamma}^{(r)}_n e^{n\zeta_r} , \quad \text{(F9)}$$

and $E_{\rm FP}$ given by (4.6), we have

$$\frac{dC}{dz}(z_0)=-i\frac{1}{\sqrt{\pi}}\sum_r\frac{\alpha_r}{z_0-Z_r}\sum_{\substack{n\geq 1\\ s}}\frac{n}{\alpha_s}\overline{N}^{sr}_{n0}\gamma^{(s)}_{-n} , \quad \text{(F10)}$$

$$\overline{C}(z_0)=\frac{1}{\sqrt{\pi}}\sum_r\frac{\alpha_r}{z_0-Z_r}\left[\frac{\overline{\gamma}^{(r)}_0}{\alpha_r}+\sum_{\substack{n\geq 1\\ s}}\frac{n}{\alpha_s}\overline{N}^{sr}_{n0}\overline{\gamma}^{(s)}_{-n}\right] . \quad \text{(F11)}$$

Likewise, $A^*=A(z_0^*)$, $dC^*/dz=(dC/dz)(z_0^*)$, and $\overline{C}^*=\overline{C}(z_0^*)$ are obtained by replacing z_0 by z_0^* in (F4), (F10), and (F11), respectively. Note that the above formulas are valid for both the 3-string and 4-string vertices. (In the 3-string case we have $z_0=z_0^*$.)

From (F4), (F10), and (F11) the LHS of (4.16) becomes

$$\frac{1}{a}\left[A^2-2\frac{dC}{dz}\overline{C}\right]-\frac{1}{a^*}\left[A^{*2}-2\frac{dC^*}{dz}\overline{C}^*\right]=\frac{4i}{\pi}\left[\frac{1}{2}\sum_{\substack{n,m\geq 0\\ r,s}}M^{rs}_{nm}\alpha^{(r)}_{-n}\cdot\alpha^{(s)}_{-m}+i\sum_{\substack{n\geq 1,m\geq 0\\ r,s}}M^{rs}_{nm}\gamma^{(r)}_{-n}\overline{\gamma}^{(s)}_{-m}\right] , \quad \text{(F12)}$$

where M^{rs}_{nm} is given by

$$M^{rs}_{nm}=\sum_{i,j}{\rm Im}\left[\frac{1}{a}\frac{\alpha_i}{z_0-Z_i}\frac{\alpha_j}{z_0-Z_j}\right]\frac{n}{\alpha_r}\overline{N}^{ri}_{n0}\frac{m}{\alpha_s}\overline{N}^{sj}_{m0} \quad (n,m\geq 1) , \quad \text{(F13)}$$

$$M^{rs}_{n0}=M^{sr}_{0n}=\sum_i{\rm Im}\left[\frac{1}{a}\frac{\alpha_i}{z_0-Z_i}\frac{\alpha_s}{z_0-Z_s}\right]\frac{n}{\alpha_r}\overline{N}^{ri}_{n0}\frac{1}{\alpha_s} \quad (n\geq 1) , \quad \text{(F14)}$$

$$M_{00}^{rs}=\text{Im}\left[\frac{1}{a}\frac{\alpha_r}{z_0-Z_r}\frac{\alpha_s}{z_0-Z_s}\right]\frac{1}{\alpha_r\alpha_s} . \tag{F15}$$

For fixed values of α_r, the Neumann functions $\overline{N}\,^{rs}_{nm}$ defined by (A7) depend on Z_{1-4} through the Mandelstam mapping (A1). In the following we take a special parametrization of Z_r by the σ coordinate σ_0 of the interaction point lying on a fixed time τ_0:

$$Z_r=Z_r(\sigma_0) . \tag{F16}$$

That is, when σ_0 varies over the range $\sigma_-\leq\sigma_0\leq\sigma_+$, Z_r of (F16) realize

$$\sigma_0=\text{Im}\sum_{r=1}^{4}\alpha_r\ln(z_0-Z_r)$$

with

$$\tau_0=\text{Re}\sum_{r=1}^{4}\alpha_r\ln(z_0-Z_r)$$

unchanged. For such a parametrization of (F16), $(d/d\sigma_0)Z_r(\sigma_0)$ is given by

$$\frac{d}{d\sigma_0}Z_r(\sigma_0)=\text{Im}\left[\frac{1}{a}\frac{1}{z_0-Z_r}\right] \tag{F17}$$

with a defined in (3.34). Indeed, when (F17) holds, we have

$$\begin{aligned}\frac{d}{d\sigma_0}\sum_r\alpha_r\ln(z_0-Z_r)&=\sum_r\frac{\alpha_r}{z_0-Z_r}\left[\frac{dz_0}{d\sigma_0}-\frac{dZ_r}{d\sigma_0}\right]\\&=-\sum_r\frac{\alpha_r}{z_0-Z_r}\text{Im}\left[\frac{1}{a}\frac{1}{z_0-Z_r}\right]\\&=-\frac{1}{2i}\left[\frac{1}{a}\sum_r\frac{\alpha_r}{(z_0-Z_r)^2}+\frac{1}{a^*}\frac{1}{z_0-z_0^*}\sum_r\alpha_r\left[\frac{1}{z_0-Z_r}-\frac{1}{z_0^*-Z_r}\right]\right]\\&=i\end{aligned} \tag{F18}$$

by using Eq. (3.32) and the definition of a in (3.34). This implies

$$\frac{d}{d\sigma_0}\sigma_0=1,\quad\frac{d}{d\sigma_0}\tau_0=0 , \tag{F19}$$

as is required by $\sum_r\alpha_r\ln(z_0-Z_r)=\tau_0+i\sigma_0$. Now the Neumann function $\overline{N}\,^{rs}_{nm}$ is a function of σ_0, and (4.16) is proved if we can show that

$$M_{nm}^{rs}=\frac{d}{d\sigma_0}\overline{N}\,^{rs}_{nm}\quad(n,m\geq0) . \tag{F20}$$

For $\overline{N}\,^{rs}_{00}$ (F20) is shown by differentiating (A12) and using (F17) and (3.32). For $\overline{N}\,^{rs}_{n0}$ $(n\geq1)$, the integral representation (A13) is useful. We have

$$\begin{aligned}\frac{d}{d\sigma_0}\overline{N}\,^{rs}_{n0}&=\frac{1}{n}\oint_{Z_r}\frac{dz}{2\pi i}\left[-\frac{d}{dz}\left[\frac{1}{z-Z_s}\right]\frac{dZ_s}{d\sigma_0}+\frac{n}{z-Z_s}\sum_i\frac{\alpha_i/\alpha_r}{z-Z_i}\frac{dZ_i}{d\sigma_0}\right]e^{-n\zeta_r(z)}\\&=\oint_{Z_r}\frac{dz}{2\pi i}\frac{1}{z-Z_s}\sum_i\frac{\alpha_i/\alpha_r}{z-Z_i}\left[\frac{dZ_i}{d\sigma_0}-\frac{dZ_s}{d\sigma_0}\right]e^{-n\zeta_r(z)} ,\end{aligned} \tag{F21}$$

where use has been made of $\alpha_r(d/dz)\zeta_r(z)=\sum_i\alpha_i(z-Z_i)^{-1}$. Then, from

$$\frac{dZ_i}{d\sigma_0}-\frac{dZ_s}{d\sigma_0}=[(z-Z_s)-(z-Z_i)]\text{Im}\left[\frac{1}{a}\frac{1}{(z_0-Z_i)(z_0-Z_s)}\right] , \tag{F22}$$

(F21) is further rewritten as

$$\sum_i \mathrm{Im}\left[\frac{1}{a}\frac{1}{(z_0-Z_i)(z_0-Z_s)}\right]\frac{\alpha_i}{\alpha_r}\oint_{Z_r}\frac{dz}{2\pi i}\frac{1}{z-Z_i}e^{-n\zeta_r(z)}-\sum_i \mathrm{Im}\left[\frac{1}{a}\frac{\alpha_i}{z_0-Z_i}\frac{1}{z_0-Z_s}\right]\frac{1}{\alpha_r}$$
$$\times\oint_{Z_r}\frac{dz}{2\pi i}\frac{1}{z-Z_s}e^{-n\zeta_r(z)}=\sum_i \mathrm{Im}\left[\frac{1}{a}\frac{\alpha_i}{(z_0-Z_i)(z_0-Z_s)}\right]\frac{n}{\alpha_r}\bar{N}^{ri}_{n0}=M^{rs}_{n0}\ , \qquad \text{(F23)}$$

where the second integral drops out again owing to (3.32). Equation (F20) for $n,m\geq 1$ is shown from (A21) and (F20) with $m=0$. Thus, we have finished the proof of (4.16).

In order to show (4.17), first recall that $C=C(z)$ is given by

$$C(z)=C(\rho)^{(-)}+[C(\rho),E_{\mathrm{FP}}]$$
$$=C(\rho)^{(-)}-i\frac{1}{\sqrt{\pi}}\sum_{\substack{n\geq 0,m\geq 1\\ s}}\bar{N}^{rs}_{nm}e^{n\zeta_r}\gamma^{(s)}_{-m}\ , \qquad \text{(F24)}$$

where $C(\rho)^{(-)}$ is the creation and zero-made operator part of (F7). $C(z)$ given in (F24) is analytic around $z=z_0$ and has an expansion

$$C(z)=c_0+c_1(z-z_0)+\tfrac{1}{2}c_2(z-z_0)^2+\cdots\ , \qquad \text{(F25)}$$

where

$$c_0=C(z_0),\quad c_1=\frac{dC}{dz}(z_0)\ ,\qquad c_2=\frac{d^2C}{dz^2}(z_0)\ . \qquad \text{(F26)}$$

Then, from (3.35) we have

$$\frac{dC(z)}{d\rho}=c_1\frac{dz}{d\rho}+c_2(z-z_0)\frac{dz}{d\rho}+O(z-z_0)$$
$$=\left[-\frac{1}{2a}\frac{1}{z-z_0}+\frac{3b}{4a^2}\right]c_1-\frac{1}{2a}c_2+O(z-z_0)\ , \qquad \text{(F27)}$$

from which we obtain the formula

$$c_2=\frac{d^2C}{dz^2}(z_0)=\lim_{z\to z_0}\left[-2a\frac{dC(z)}{d\rho}+\left[-\frac{1}{z-z_0}+\frac{3b}{2a}\right]c_1\right]\ . \qquad \text{(F28)}$$

On the other hand, from (F24) we have

$$\frac{d}{d\sigma_0}C(z_0)=\lim_{z\to z_0}\left[\frac{dC(z)}{d\rho}\frac{d\rho_0}{d\sigma_0}+\left[-i\frac{1}{\sqrt{\pi}}\right]\sum_{\substack{n\geq 0,m\geq 1\\ s}}\left[\frac{d}{d\sigma_0}\bar{N}^{rs}_{nm}\right]e^{n\zeta_r}\gamma^{(s)}_{-m}\right]\ , \qquad \text{(F29)}$$

where the limiting procedure is necessary because each term on the RHS separately diverges at $z=z_0$. On the RHS of (F29), $d\rho_0/d\sigma_0=i$ because

$$\rho_0=\tau_0+i\sigma_0+\mathrm{const} \qquad \text{(F30)}$$

(see Fig. 6), and from (F20) we have

$$\sum_{\substack{n\geq 0,m\geq 1\\ s}}\left[\frac{d}{d\sigma_0}\bar{N}^{rs}_{nm}\right]e^{n\zeta_r}\gamma^{(s)}_{-m}=\sum_{i,j}\mathrm{Im}\left[\frac{1}{a}\frac{\alpha_i}{z_0-Z_i}\frac{\alpha_j}{z_0-Z_j}\right]\left[\frac{1}{\alpha_r}\delta_{ir}+\sum_{n\geq 1}\frac{n}{\alpha_r}\bar{N}^{ri}_{n0}e^{n\zeta_r}\right]\sum_{\substack{m\geq 1\\ s}}\frac{m}{\alpha_s}\bar{N}^{sj}_{m0}\gamma^{(s)}_{-m}\ . \qquad \text{(F31)}$$

From (A23) and (3.35) we have the Laurent expansion

$$\frac{1}{\alpha_r}\delta_{ir}+\sum_{n\geq 1}\frac{n}{\alpha_r}\bar{N}^{ri}_{n0}e^{n\zeta_r}=\left[\frac{d\rho(z)}{dz}\right]^{-1}\frac{1}{z-Z_i}$$
$$=\frac{1}{2a}\left[-\frac{1}{z-z_0}+\frac{3b}{2a}\right]\frac{1}{z_0-Z_i}+\frac{1}{2a}\frac{1}{(z_0-Z_i)^2}+O(z-z_0)\ . \qquad \text{(F32)}$$

Then, by making use of the formulas

$$\sum_i \mathrm{Im}\left[\frac{1}{a}\frac{\alpha_i}{z_0-Z_i}\frac{\alpha_j}{z_0-Z_j}\right]\frac{1}{z_0-Z_i}=-i\frac{\alpha_j}{z_0-Z_j}\ , \qquad \text{(F33)}$$

$$\frac{1}{a}\sum_i \mathrm{Im}\left[\frac{1}{a}\frac{\alpha_i}{z_0-Z_i}\frac{\alpha_j}{z_0-Z_j}\right]\frac{1}{(z_0-Z_i)^2}=i\frac{3b}{2a^2}\frac{\alpha_j}{z_0-Z_j}-\frac{i}{a^*}\frac{1}{z_0-z_0^*}\frac{\alpha_j}{z_0^*-Z_j}\ , \tag{F34}$$

and (F30), (F29) is rewritten as

$$\frac{d}{d\sigma_0}C(z_0)=i\lim_{z\to z_0}\left[\frac{dC(z)}{d\rho}-\frac{1}{2a}\left[-\frac{1}{z-z_0}+\frac{3b}{2a}\right]\frac{dC}{dz}(z_0)-\frac{1}{2a^*}\frac{1}{z_0-z_0^*}\frac{dC}{dz}(z_0^*)+\frac{3b}{4a^2}\frac{dC}{dz}(z_0)\right]\ . \tag{F35}$$

Equation (4.17) follows from (F35) and (F28).

APPENDIX G: PROOF OF (4.20)

We show in this appendix that Eq. (4.20) actually holds:

$$\frac{d}{d\sigma_0}\ln f(\sigma_0)=\mathrm{Im}\left[\frac{9b^2}{2a^3}-\frac{6c}{a^2}\right]\ . \tag{G1}$$

First of all we note that this equation is invariant under projective transformation (4.26):

$$\delta Z_r=\alpha+\beta Z_r+\gamma Z_r{}^2\quad (r=1\text{--}4)\ . \tag{G2}$$

Indeed the invariance of the LHS, or equivalently of $f(\sigma_0)$ in (4.29)

$$f(\sigma_0)=\left|\frac{\prod_{r=1}^{4}(dZ_r e^{-\bar N_{00}^{rr}})}{dV_{abc}d\sigma_0}\right| \tag{G3}$$

is clear from the invariances of dV_{abc} and $dZ_r e^{-\bar N_{00}^{rr}}$ [see (A28)]. The invariance of the RHS of (G1) is seen as follows. The equation

$$\left.\frac{d\rho}{dz}\right|_{z=z_0}=\sum_{i=1}^{4}\frac{\alpha_i}{z_0-Z_i}=0 \tag{G4}$$

determining the interaction point z_0 says that z_0 also receives the same projective transformation as (G2), and thus z_0-Z_r transforms as

$$\delta(z_0-Z_r)=(\beta+2\gamma z_0)(z_0-Z_r)-\gamma(z_0-Z_r)^2\ . \tag{G5}$$

Accordingly a, b, and c, defined by

$$a=\frac{1}{2}\sum_{r=1}^{4}\frac{\alpha_r}{(z_0-Z_r)^2},\quad b=-\frac{1}{3}\sum_{r=1}^{4}\frac{\alpha_r}{(z_0-Z_r)^3}\ ,\quad c=\frac{1}{4}\sum_{r=1}^{4}\frac{\alpha_r}{(z_0-Z_r)^4}\ , \tag{G6}$$

transform as

$$\delta a=-2(\beta+2\gamma z_0)a\ ,\quad \delta b=-3(\beta+2\gamma z_0)b-2\gamma a\ ,\quad \delta c=-4(\beta+2\gamma z_0)c-3\gamma b\ , \tag{G7}$$

from which one can easily verify the projective invariance of the quantity $9b^2/2a^3-6c/a^2$.

Now since we have checked the projective invariance, it is enough to prove (G1) by taking a particular gauge; we take $Z_1=1$, $Z_2=x$, $Z_3=0$, and $Z_4=\infty$, then Mandelstam mapping becomes

$$\rho(z)=\alpha_1\ln(z-1)+\alpha_2\ln(z-x)+\alpha_3\ln z\ , \tag{G8}$$

and Eq. (G4) now reads

$$\sum_r\frac{\alpha_r}{z_0-Z_r}=\frac{\alpha_1}{z_0-1}+\frac{\alpha_2}{z_0-x}+\frac{\alpha_3}{z_0}=0\ , \tag{G9}$$

or equivalently

$$\alpha_4 z_0{}^2+[(\alpha_2+\alpha_3)+(\alpha_1+\alpha_3)x]z_0-\alpha_3 x=0\ , \tag{G10}$$

and determines the interaction point z_0 as in (7.13):

$$z_0=-(\alpha_{23}+\alpha_{13}x+i\alpha_{13}\Delta)/2\alpha_4\ , \tag{G11}$$

$$\alpha_{13}\Delta\equiv\{-[\alpha_{13}{}^2x^2+2x(\alpha_1\alpha_2+\alpha_3\alpha_4)+\alpha_{23}{}^2]\}^{1/2}\ . \tag{G12}$$

Here we have used a convenient notation $\alpha_{ij}\equiv\alpha_i+\alpha_j$.

In this gauge (G3) becomes, with the help of Eq. (A12),

$$f(\sigma_0)=\left|\frac{dx}{d\sigma_0}\right||x-1|^{A_{12}}|x|^{A_{23}}e^{-B\tau_0}\quad\left[A_{ij}\equiv\frac{\alpha_j}{\alpha_i}+\frac{\alpha_i}{\alpha_j},\ B=\sum_{r=1}^{4}\frac{1}{\alpha_r}\right]\ , \tag{G13}$$

where τ_0 and σ_0 are the real and imaginary parts of $\rho(z_0)$ and hence we have

$$\rho(z_0)=\tau_0+i\sigma_0\overset{d/dx}{\Longrightarrow}\frac{\alpha_2}{x-z_0}=\frac{d\tau_0}{dx}+i\frac{d\sigma_0}{dx}\ . \tag{G14}$$

With (G13) and (G14), the LHS of the desired equation (G1) is given by

$$\begin{aligned}\frac{d}{d\sigma_0}\ln f(\sigma_0)&=\left[\frac{d^2x}{d\sigma_0{}^2}\Big/\frac{dx}{d\sigma_0}\right]+\frac{dx}{d\sigma_0}\left[\frac{A_{12}}{x-1}+\frac{A_{23}}{x}\right]-B\frac{d\tau_0}{d\sigma_0}\\&=-\frac{\mathrm{Im}[(1-z_0^{\prime *})(x-z_0)^2]}{\alpha_2(\mathrm{Im}z_0)^2}+\frac{|x-z_0|^2}{\alpha_2\mathrm{Im}z_0}\left[\frac{A_{12}}{x-1}+\frac{A_{23}}{x}\right]-B\frac{\mathrm{Re}(x-z_0)}{\mathrm{Im}z_0}\ ,\end{aligned} \tag{G15}$$

hereafter the prime denotes d/dx.

On the other hand, the RHS of (G1) can be estimated in this gauge as follows. First, differentiating (G9) a few times with respect to x, we obtain

$$-\sum_{r=1}^{3}\frac{\alpha_r}{(z_0-Z_r)^2}(z_0'-\delta_{r2})=0\,,$$
$$-2\sum_{r=1}^{3}\frac{\alpha_r}{(z_0-Z_r)^3}(z_0'-\delta_{r2})^2+\sum_{r=1}^{3}\frac{\alpha_r}{(z_0-Z_r)^2}z_0''=0\,, \tag{G16}$$
$$6\sum_{r=1}^{3}\frac{\alpha_r}{(z_0-Z_r)^4}(z_0'-\delta_{r2})^3-6\sum_{r=1}^{3}\frac{\alpha_r}{(z_0-Z_r)^3}(z_0'-\delta_{r2})z_0''+\sum_{r=1}^{3}\frac{\alpha_r}{(z_0-Z_r)^2}z_0'''=0\,,$$

which give the following relations for a, b, and c defined by (G6):

$$2az_0'=\alpha_2/(z_0-x)^2\,,$$
$$\frac{3b(z_0')^2}{a}+\frac{2z_0'(2z_0'-1)}{z_0-x}+z_0''=0\,, \tag{G17}$$
$$\frac{12c(z_0')^4}{a}+\frac{6(z_0')^2(-3z_0'^2+3z_0'-1)}{(z_0-x)^2}+\frac{6z_0'z_0''(1-z_0')}{z_0-x}+z_0'z_0'''-3(z_0'')^2=0\,.$$

By using these we find

$$\frac{9b^2}{2a^3}-\frac{6c}{a^2}=\frac{1}{\alpha_2}\left\{\frac{(z_0-x)^2}{z_0'}\left[\frac{z_0'''}{z_0'}-2\left[\frac{z_0''}{z_0'}\right]^2\right]+2(z_0-x)\left[\frac{z_0''}{z_0'}\right]\left[1+\frac{1}{z_0'}\right]-2\left[z_0'-1+\frac{1}{z_0'}\right]\right\}. \tag{G18}$$

We can further eliminate z_0'' and z_0''' from this expression by using

$$\frac{z_0''}{z_0'}=\frac{-2i(\alpha_4z_0'+\alpha_{13})}{\alpha_{13}\Delta},\quad \frac{z_0'''}{z_0'}=\frac{-6(2\alpha_4z_0'+\alpha_{13})(\alpha_4z_0'+\alpha_{13})}{(\alpha_{13}\Delta)^2}\,, \tag{G19}$$

which follow from Eq. (G10) with differentiation $(d/dx)^2$ and $(d/dx)^3$. Thus the problem to show (G1) is reduced by using (G11), (G15), (G18), and (G19) to prove the equality between

$$-\alpha_2\alpha_4(\mathrm{Im}z_0)\mathrm{Im}\left[\frac{9b^2}{2a^3}-\frac{6c}{a^2}\right]-\frac{\alpha_4\mathrm{Im}[(1-z_0'^*)(x-z_0)^2]}{\mathrm{Im}z_0}-\alpha_2\alpha_4B\,\mathrm{Re}(x-z_0)$$
$$=\mathrm{Im}\left[\frac{(x-z_0)^2}{\alpha_{13}\Delta}\left[2\alpha_4^2(1-z_0'-z_0'^*)-\alpha_4\alpha_{13}+\frac{\alpha_{13}^2}{z_0'}\right]+i(x-z_0)\left[2\alpha_4z_0'-\alpha_2(2+\alpha_4B)+\frac{2\alpha_{13}}{z_0'}\right]-\alpha_{13}\Delta\left[z_0'+\frac{1}{z_0'}\right]\right] \tag{G20}$$

and

$$-\alpha_4|x-z_0|^2\left[\frac{A_{12}}{x-1}+\frac{A_{23}}{x}\right]=\alpha_2[A_{12}x+A_{23}(x-1)]\,. \tag{G21}$$

Now (G21) is a very simple quantity linear in x, and so we need to rewrite (G20) further. Substituting $z_0'+z_0'^*=-\alpha_{13}/\alpha_4$ in the first term and collecting the terms with the same power in z_0' we rewrite it as

$$(\text{G20})=\mathrm{Im}\left\{\frac{\alpha_{13}}{z_0'}\frac{(x+i\Delta-z_0)^2}{\Delta}+2i\alpha_4z_0'\left[x-z_0+i\frac{\alpha_{13}\Delta}{2\alpha_4}\right]\right.$$
$$\left.+\alpha_{13}\Delta\alpha_4(\alpha_4-\alpha_2)\left[\left[\frac{x-z_0}{\alpha_{13}\Delta}-\frac{i\alpha_2(2+\alpha_4B)}{2\alpha_4(\alpha_4-\alpha_2)}\right]^2+\frac{1}{4}\left[\frac{\alpha_2(2+\alpha_4B)}{\alpha_4(\alpha_4-\alpha_2)}\right]^2\right]\right\}. \tag{G22}$$

The last term can be thrown away since it is real. An important fact there is that $z_0'\Delta$ has a constant (i.e., x-independent) absolute value; indeed from (G10) and (G11) we have

$$z_0'\Delta=-i\left[z_0-\frac{\alpha_3}{\alpha_{13}}\right]$$
$$=\frac{i\alpha_{13}}{2\alpha_4}\left[x+i\Delta+\frac{\alpha_1\alpha_2+\alpha_3\alpha_4}{\alpha_{13}{}^2}\right], \quad \text{(G23)}$$

$$|z_0'\Delta|^2=\frac{\alpha_1\alpha_2\alpha_3\alpha_4}{\alpha_4{}^2\alpha_{13}{}^2}. \quad \text{(G24)}$$

Because of this fact and the form (G23) of $z_0'\Delta$, it is now easy to see that the imaginary part of the first term of (G22) as well as of the second and third terms is actually linear in x, as (G21) is. By substituting (G11), (G12), and (G23), one can check that (G22) actually coincides with (G21).

APPENDIX H: EVALUATION OF THE GENERAL N-STRING EFFECTIVE VERTEX

In this appendix we evaluate the general N-string effective vertex $|\Delta^{(N)}(1,2,\dots,N)\rangle$ aside from the measure by using the path-integral method, i.e., we give a derivation of the exponent factor bilinear in bosonic and ghost creation operators in the N-string amplitude (7.32). Equation (7.19) will be shown as a special case of the 4-string effective vertex $|\Delta_\tau(1,2,3,4)\rangle$. Actually the N-string tree amplitude in the light-cone gauge theory was given by Mandelstam[20] by using the path-integral technique, and the following argument is an application of his method to the covariant theory.

The general N-string effective vertex $|\Delta^{(N)}(1,2,\dots,N)\rangle$ is defined as a generalization of the $N=4$ case, (7.17): The N-string tree amplitude is written in terms of it as

$$\mathcal{T}_N=(2\pi)^{d+1}\delta\left[\sum\alpha\right]\delta\left[\sum p\right](\tfrac{1}{2})^{N-3}(2g)^{N-2}\text{tr}\langle\varphi(1)|\langle\varphi(2)|\cdots\langle\varphi(N)|\int_0^\infty[d\tau]\,|\Delta^{(n)}(1,2,\dots,N)\rangle, \quad \text{(H1)}$$

where

$$[d\tau]=\prod_{i=1}^{N-3}d\tau_i$$

and each of the parameters τ_i comes from (7.18) applied to each propagator of the intermediate string. Therefore $|\Delta^{(N)}(1,2,\dots,N)\rangle$ generally takes the form

$$|\Delta^{(N)}\rangle=\int\left[\prod\langle R(i,j)|\right]\prod(\Omega e^{\tau L})\left[\prod|v\rangle\right]. \quad \text{(H2)}$$

The meaning of this expression will be well illustrated by taking an explicit example, Fig. 28, for which (H2) is given by

$$|\Delta^{(N)}\rangle_{\text{Fig. 28}}=\int d6\,d7\,d8\,d9\langle R(9,8)|\langle R(6,7)|\Omega^{(6)}\Omega^{(8)}$$
$$\times e^{\tau_1L^{(6)}}e^{\tau_2L^{(8)}}|v(1,7,8)\rangle$$
$$\times|v(2,3,6)\rangle|v(4,5,9)\rangle.$$

As was explained in Sec. V A, the 3- and 4- string vertices, the propagator L, and the reflection operator $|R(1,2)\rangle$ have the OSp symmetry. Therefore the calculation of $|\Delta^{(N)}\rangle$ may be performed in a completely OSp-invariant manner and hence it is convenient to use an OSp-

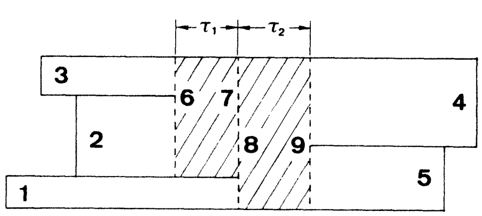

FIG. 28. A typical light-cone diagram in which the shaded region corresponds to the effective vertex $|\Delta^{(N)}\rangle$.

covariant notation here:

$$P^M(\sigma)=\frac{1}{\sqrt{\pi}}\sum_{n=-\infty}^{\infty}a_n^M\cos(n\sigma)$$
$$=\frac{1}{\sqrt{\pi}}\sum_{n\geq0}p_n^M\cos(n\sigma)$$
$$=\left[P^\mu(\sigma),-\alpha c'(\sigma),\frac{1}{\alpha}\bar{c}(\sigma)\right],$$
$$a_n^M=(\alpha_n^\mu,\gamma_n,\bar{\gamma}_n)\quad(n\neq0),$$
$$a_0^M=(\alpha_0^\mu=p^\mu,0,0)=p_0^M\quad(n=0), \quad \text{(H3)}$$
$$p_n^M=a_n^M+a_{-n}^M\quad(n>0).$$

Similarly to this $P^M(\sigma)$ we have an OSp extension $X^M(\sigma)$ of the coordinate variable $X^\mu(\sigma)$, by which the Lagrangian of first quantized theory is given by $\frac{1}{2}\partial_aX^M\partial_bX^N\eta_{MN}\eta^{ab}$.

In terms of the path-integral language, our task to evaluate the vertex $|\Delta^{(N)}\rangle$ is equivalent to obtaining the Neumann function defined in a region corresponding to the vertex, which is a shaded region, for example, in Fig. 28. We shall consider, instead of only the shaded region, the whole region of Fig. 28, whose boundaries are specified by large but finite values of ξ_r, i.e., ξ_r^0. With these ξ_r^0, we rewrite $|\Delta^{(N)}\rangle$ into two factors,

$$e^{+\xi^0L}(e^{-\xi^0L}|\Delta^{(N)}\rangle) \quad \text{(H4)}$$

with $\xi^0L\equiv\sum_{r=1}^N\xi_r^0L^{(r)}$ and evaluate first the latter factor $e^{-\xi^0L}|\Delta^{(N)}\rangle$ by the path-integral technique and then multiply the former factor to take the limit $\xi_r^0\to-\infty$.

Denoting by N_c the Neumann function of the finite region of Fig. 28 bounded by ξ_r^0, we obtain through the usual path-integral procedure the following expression for the second factor of (H4):

$$e^{-\xi^0 L}|\Delta^{(N)}\rangle \sim \exp\left[\frac{1}{4\pi}\sum_{r,s}\int_0^{\pi} d\sigma_r d\tilde{\sigma}_s P^M(\sigma_r)\eta_{MN}P^N(\tilde{\sigma}_s)N_c(\sigma_r,\xi_r^0;\tilde{\sigma}_s,\tilde{\xi}_s^0)\right] . \tag{H5}$$

The Neumann function given in (A7) is the one corresponding to the infinite region. However, the Neumann function N_c here is a different one, which satisfies the boundary conditions that the normal derivatives of N_c to the boundaries $\xi_r = \xi_r^0$ as well as to the horizontal boundaries of Fig. 28 vanish. This condition can be satisfied by adding a homogeneous solution N_0 of the Laplace equation to the Neumann function N of (A7). An appropriate homogeneous solution N_0 is

$$N_0(\sigma_r,\xi_r;\tilde{\sigma}_s,\tilde{\xi}_s) = -\delta_{rs}\sum_{n\geq 1}\frac{2}{n}e^{-n|2\xi_r^0-\xi_r-\tilde{\xi}_s|}\cos(n\sigma_r)\cos(n\tilde{\sigma}_s)$$
$$+2\sum_{n,m\geq 0}\bar{N}_{nm}^{rs}[(e^{n\xi_r}+e^{n(2\xi_r^0-\xi_r)}-\delta_{n0})(e^{m\tilde{\xi}_s}+e^{m(2\xi_s^0-\tilde{\xi}_s)}-\delta_{m0})-e^{n\xi_r}e^{m\tilde{\xi}_s}]\cos(n\sigma_r)\cos(m\tilde{\sigma}_s) , \tag{H6}$$

and the Neumann function N_c over the finite region is given by $N_c = N + N_0$. Note here that N_0 goes to zero in the limit $\xi_r^0 \to -\infty$ and thus N_c recovers the Neumann function N over the infinite region.

Precisely speaking, $N_c = N + N_0$ does not give a truly correct Neumann function. N_0 represents merely a first reflection wave from the boundary ξ_r^0 and we need an infinite sequence of reflection waves. Fortunately, however, $N_c = N + N_0$ already gives a correct answer if the limit $\xi_r^0 \to -\infty$ is taken later. Thus N_c has an expression at the boundary $\xi_r = \xi_r^0$:

$$N_c(\sigma_r,\xi_r^0;\tilde{\sigma}_s,\tilde{\xi}_s^0) = -\delta_{rs}\sum_{n\geq 1}\frac{4}{n}\cos(n\sigma_r)\cos(n\tilde{\sigma}_s)+\delta_{rs}2\xi_r^0$$
$$+2\sum_{n,m\geq 0}(2-\delta_{n0})(2-\delta_{m0})\bar{N}_{nm}^{rs}e^{n\xi_r^0+m\tilde{\xi}_s^0}\cos(n\sigma_r)\cos(m\tilde{\sigma}_s) . \tag{H7}$$

Substituting this into (H5) we find

$$e^{-\xi^0 L}|\Delta^{(N)}\rangle \sim \exp\left[-\frac{1}{4}\sum_{\substack{n\geq 1\\ r}}\frac{1}{n}p_n^{(r)M}\eta_{MN}p_n^{(r)N}+\sum_r \xi_r^0\frac{{p_r}^2}{2}+\frac{1}{2}\sum_{\substack{n,m\geq 0\\ r,s}}\bar{N}_{nm}^{rs}e^{n\xi_r^0+m\tilde{\xi}_s^0}p_n^{(r)M}\eta_{MN}p_m^{(s)N}\right] . \tag{H8}$$

From Eq. (H3) we have

$$a_m^M = \frac{1}{2}\left[p_m^M + 2m\frac{\partial}{p_{mM}}\right] \quad (m\geq 1) \tag{H9}$$

and thus

$$a_m^M \exp\left[-\frac{1}{4}\sum_{\substack{n\geq 1\\ r}}\frac{1}{n}p_n^{(r)M}\eta_{MN}p_n^{(r)N}\right] = 0 \quad (m\geq 1) . \tag{H10}$$

Therefore the first terms in the exponent in (H8) are found to give a functional expression of the ground states of oscillating modes of all the N external strings:

$$\exp\left[-\frac{1}{4}\sum_{\substack{n\geq 1\\ r}}\frac{1}{n}p_n^{(r)M}\eta_{MN}p_n^{(r)N}\right] = |0\rangle_{1-N} . \tag{H11}$$

Now we can rewrite Eq. (H8) into

$$e^{-\xi^0 L}|\Delta^{(N)}\rangle \sim \exp\left[\sum_r \xi_r^0\frac{{p_r}^2}{2}+\frac{1}{2}\sum_{\substack{n,m\geq 0\\ r,s}}\bar{N}_{nm}^{rs}e^{n\xi_r^0+m\tilde{\xi}_s^0}p_n^{(r)N}\eta_{NM}p_m^{(s)M}\right]|0\rangle_{1-N} . \tag{H12}$$

In order to obtain $|\Delta^{(N)}\rangle$ we have to operate

$$\exp\left[\sum_r \xi_r^0 L^{(r)}\right]$$

on (H12) from the left. As a result the term $\sum_r \xi_r^0({p_r}^2/2)$ in (H12) disappears and the oscillators $a_n^{(r)}$ in $p_n^{(r)}$ are replaced by $a_n^{(r)}e^{n\xi_r^0}$:

$$|\Delta^{(N)}\rangle \sim \exp\left[\frac{1}{2}\sum_{\substack{n,m\geq 0\\ r,s}} \bar{N}^{rs}_{nm}[(e^{2n\xi_r^0}-\delta_{n0})a_n^{(r)N}+a_{-n}^{(r)N}]\eta_{NM}[(e^{2m\tilde{\xi}_s^0}-\delta_{m0})a_m^{(s)M}+a_{-m}^{(s)M}]\right]|0\rangle_{1-N} . \tag{H13}$$

Taking the limit of $\xi_r^0\to-\infty$, we finally arrive at the desired expression of the oscillator part of Eq. (7.32), since all the annihilation operators drop out:

$$|\Delta^{(N)}\rangle \sim \exp\left[\frac{1}{2}\sum_{\substack{n,m\geq 0\\ r,s}} \bar{N}^{rs}_{nm}a_{-n}^{(r)N}\eta_{NM}a_{-m}^{(s)M}\right]|0\rangle_{1-N} . \tag{H14}$$

APPENDIX I: PROOF OF EQ. (7.48)

In the Mandelstam mapping (A1), let us consider an infinitesimal change of the parameters α_i $(i=1-N)$ which preserves the constraint $\sum_{i=1}^N\alpha_i=0$:

$$\alpha_i\to\alpha_i+\delta\alpha_i \quad \left[\sum_{i=1}^N\delta\alpha_i=0\right] . \tag{I1}$$

Equation (7.48) is a consequence of the fact that the Neumann function $N(\rho_r,\tilde{\rho}_s)$ $(=\ln|z-\tilde{z}|+\ln|z-\tilde{z}^*|)$ of (A7) is invariant under the variation (I1) since it changes neither z nor $\tilde{z}$. In order to calculate $\delta N(\rho_r,\tilde{\rho}_s)$ we need an expression for $\delta\xi_r$. Under the variation (I1) we have, for $\tau_0^{(r)}$ of (A5),

$$\delta\tau_0^{(r)}=\mathrm{Re}[(\rho+\delta\rho)(z_0^{(r)}+\delta z_0^{(r)})-\rho(z_0^{(r)})]=\mathrm{Re}\left[\frac{d\rho}{dz}(z_0^{(r)})\delta z_0^{(r)}+\delta\rho(z_0^{(r)})\right]=\mathrm{Re}\ \delta\rho(z_0^{(r)})=\sum_{i=1}^N\delta\alpha_i\ln|z_0^{(r)}-Z_i| , \tag{I2}$$

where use has been made of (A6). Then from

$$\alpha_r\xi_r+\tau_0^{(r)}=\mathrm{Re}\ \rho(z)=\sum_{i=1}^N\alpha_i\ln|z-Z_i| \tag{I3}$$

we get

$$\delta\xi_r=\frac{1}{\alpha_r}\left[\sum_{i=1}^N\delta\alpha_i\ln|z-Z_i|-\delta\alpha_r\xi_r-\delta\tau_0^{(r)}\right]=\frac{1}{\alpha_r}\sum_{i=1}^N\left[\delta\alpha_i-\frac{\alpha_i}{\alpha_r}\delta\alpha_r\right](\ln|z-Z_i|-\ln|z_0^{(r)}-Z_i|) . \tag{I4}$$

Equation (I4) can be further rewritten by making use of the formulas obtained by putting $\tilde{z}\to Z_s$ $(\tilde{\xi}_s\to-\infty)$ in (A7):

$$\ln|z-Z_s|=\delta_{rs}\xi_r+\sum_{n\geq 0}\bar{N}^{rs}_{n0}e^{n\xi_r}\cos(n\sigma_r),\quad \ln|z_0^{(r)}-Z_s|=\sum_{n\geq 0}\bar{N}^{rs}_{n0}\cos(n\sigma_I^{(r)}) , \tag{I5}$$

where $\sigma_I^{(r)}$ is the σ_r coordinate of the interaction point of the rth string. We have

$$\delta\xi_r\Big|_{\sigma_r=\sigma_I^{(r)}}=\frac{1}{\alpha_r}\sum_{i=1}^N\delta\alpha_i\sum_{n\geq 1}\bar{N}^{ri}_{n0}(e^{n\xi_r}-1)\cos(n\sigma_I^{(r)}) , \tag{I6}$$

where use has been made of the property (A20).

Now, let us calculate $\delta N(\rho_r,\tilde{\rho}_s)$ at $\sigma_r=\sigma_I^{(r)}$ and $\tilde{\sigma}_s=\sigma_I^{(s)}$:

$$0=\delta N(\rho_r,\tilde{\rho}_s)$$

$$=2\delta_{rs}\left[\sum_{n\geq 1} e^{-n|\xi_r-\tilde{\xi}_s|}\cos(n\sigma_I^{(r)})\cos(n\sigma_I^{(s)})\frac{1}{\alpha_r}\sum_i \delta\alpha_i \sum_{k\geq 1}\bar{N}^{ri}_{k0}|e^{k\xi_r}-e^{k\tilde{\xi}_s}|\cos(k\sigma_I^{(r)}) + \frac{1}{\alpha_r}\sum_i\delta\alpha_i\sum_{k\geq 1}\bar{N}^{ri}_{k0}\{\exp[k\max(\xi_r,\tilde{\xi}_s)]-1\}\cos(k\sigma_I^{(r)})\right]$$

$$+2\sum_{n,m\geq 0}\bar{N}^{rs}_{nm}\left[n\frac{1}{\alpha_r}\sum_i\delta\alpha_i\sum_{k\geq 1}\bar{N}^{ri}_{k0}(e^{k\xi_r}-1)\cos(k\sigma_I^{(r)}) + m\frac{1}{\alpha_s}\sum_i\delta\alpha_i\sum_{k\geq 1}\bar{N}^{si}_{k0}(e^{k\tilde{\xi}_s}-1)\cos(k\sigma_I^{(s)})\right]e^{n\xi_r+m\tilde{\xi}_s}\cos(n\sigma_I^{(r)})\cos(m\sigma_I^{(s)})$$

$$+2\sum_{n,m\geq 0}\delta\bar{N}^{rs}_{nm}e^{n\xi_r+m\tilde{\xi}_s}\cos(n\sigma_I^{(r)})\cos(m\sigma_I^{(s)})+(\delta\sigma \text{ terms})\ , \tag{I7}$$

where ($\delta\sigma$ terms) represents terms containing $\delta\sigma_r$ or $\delta\tilde{\sigma}_s$. It is enough to consider the case $\sigma_I^{(r)},\sigma_I^{(s)}=0$ or π. Then these ($\delta\sigma$ terms) drop out since they are proportional to $\sin(n\sigma_I^{(r)})$ or $\sin(m\sigma_I^{(s)})$. Further, the first term in the large parentheses $2\delta_{rs}(\dots)$ in (I7) is rewritten as

$$\sum_{n\geq 1}e^{-n|\xi_r-\tilde{\xi}_s|}\frac{1}{\alpha_r}\sum_i\delta\alpha_i\sum_{k\geq 1}\bar{N}^{ri}_{k0}|e^{k\xi_r}-e^{k\tilde{\xi}_s}|\cos(k\sigma_I^{(r)})=\frac{1}{\alpha_r}\sum_i\delta\alpha_i\left[{\sum_{n,m\geq 0}}'\bar{N}^{ri}_{n+m,0}e^{n\xi_r+m\tilde{\xi}_s}\cos[(n+m)\sigma_I^{(r)}] - \sum_{k\geq 1}\bar{N}^{ri}_{k0}\exp[k\max(\xi_r,\tilde{\xi}_s)]\cos(k\sigma_I^{(r)})\right]\ , \tag{I8}$$

where $\sum'$ denotes the summation excluding $n=m=0$. Then, by comparing the coefficient of $e^{n\xi_r+m\tilde{\xi}_s}$ in (I7) we obtain the following formulas:

$$\delta\bar{N}^{rs}_{00}=\delta_{rs}\frac{1}{\alpha_r}\sum_i\delta\alpha_i\sum_{k\geq 1}\bar{N}^{ri}_{k0}\cos(k\sigma_I^{(r)})\ , \tag{I9}$$

$$\delta\bar{N}^{rs}_{nm}-\bar{N}^{rs}_{nm}(n\delta\bar{N}^{rr}_{00}+m\delta\bar{N}^{ss}_{00})+\sum_i\delta\alpha_i\left[\delta_{rs}\frac{1}{\alpha_r}\bar{N}^{ri}_{n+m,0}+\frac{1}{\alpha_r}\sum_{k=1}^{n-1}(n-k)\bar{N}^{rs}_{n-k,m}\bar{N}^{ri}_{k0} + \frac{1}{\alpha_s}\sum_{k=1}^{m-1}(m-k)\bar{N}^{rs}_{n,m-k}\bar{N}^{si}_{k0}\right]=0\quad(n+m\geq 1)\ . \tag{I10}$$

Equation (7.48) is an immediate consequence of (I10) and the defintion (7.41).

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