Anomaly constraints and new U(1) gauge bosons

S. M. Barr
Brookhaven National Laboratory, Upton, New York 11973

B. Bednarz and C. Benesh
Department of Physics, University of Washington, Seattle, Washington 98195
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Just as the hypercharge assignments in the standard model follow from anomaly freedom, so there are anomaly constraints on any possible extra U(1) charges. We show that one can always distinguish these from the constraints from grand unification. We discuss various predictions when the standard model is unified but the new U(1) is not, when all the groups are unified, and when none of them are. New low-energy U(1) groups imply new low-energy fermions. We show how information about new Z bosons can tell us about such fermions.

I. INTRODUCTION

Many authors have considered the phenomenology of extra Z bosons.[1-3] A main reason to find the existence of such particles plausible is that they arise naturally in the context of grand unified theories.[2] SO(10) theories (which many would regard as the simplest unified theories), for example, predict one extract Z boson.[3] Superstring theories also, because they are based on gauge groups whose rank is larger than 4, give rise to extra Z bosons.[4] Of course even in such theories the mass of these particles is an open question but could well be comparable to that of the ordinary Z. Some previous papers have focused on the constraints on the couplings of such extra Z bosons (let us henceforth call them Z' bosons) that come from grand unification.[2] Here we investigate the constraints imposed from the weaker condition of anomaly cancellation. There are at least three good reasons for doing this. (1) Extra Z bosons could exist even if $SU(3)\times SU(2)\times U(1)$ is not unified. New gauge groups have been proposed by theorists for many reasons: as family or "horizontal" symmetries, as technicolor groups in technicolor models or in composite Higgs models or "hypercolor" groups in preon models, and, in supersymmetric models, as the group needed for Fayet-Iliopoulos supersymmetry breaking. Other possibilities are that in Kaluza-Klein models the isometry group of the extra dimensions has rank >4, or that extra gauge forces may be dynamically generated in, say, preon models. The point is that in *any* model extra U(1) groups, whatever their origin may be, must satisfy anomaly constraints. So, while the authors admit a prejudice towards grand unification, the generality of the present approach commends it to our attention. (2) Even if grand unification is true it is important to see to what extent the predictions that follow from it that are derived in Ref. 2 are strictly tests of unification and to what extent they merely follow from anomaly cancellation. To take a parallel case, the $SU(3)\times SU(2)_L\times U(1)_Y$ charge assignments in the standard model are beautifully explained by grand unification and constitute in our view a strong argument in its favor. However, given the $SU(3)\times SU(2)_L$ representation of the known fermions (and assuming there are no others) the $U(1)_Y$ charge assignments follow uniquely from anomaly freedom. In a sense, the hypercharge values are not as strong a piece of evidence for unification as they may at first appear. (Though of course from a mathematical point of view unification may be said to "explain" the anomaly cancellation.)

This leads us to the third point. (3) Even though there are good reasons to believe that $SU(3)\times SU(2)\times U(1)$ are probably unified in a simple group, there is no compelling argument that an additional group would have to be unified with them. In Sec. III we therefore consider such groups as $SU(5)\times U(1)$ [which could arise from the breaking of even larger groups like $SU(5)\times G$ or $SO(10)\times G$, etc.].

II. THE ν-SCATTERING PARAMETERS

What we would like to achieve is a determination of the charges of the known fermions under the extra gauge group U(1)'. It turns out that from ν-scattering data one can in principle (though it is very difficult in practice) entirely determine these charges. [If there are several extra U(1)' factors this remains so if one of the U(1)' bosons contributes most to the deviation from the standard model.] There are some fortuitous circumstances that allow us to do this as we shall see below.

Let us call the U(1)' charges of $\binom{u}{d}_L$, $\binom{\nu}{e^-}_L$, $\bar{u}_L$, $\bar{d}_L$, and e_L^+, respectively, a, b, c, d, and e. [We assume throughout that the U(1)' charges are family independent.] In ν scattering the effective four-Fermi interactions depend essentially on six quantities $\epsilon_L(u)$, $\epsilon_R(u)$, $\epsilon_L(d)$, $\epsilon_R(d)$, g_V^e, and g_A^e. These are defined[5] as

$$J_\mu^{\rm had}=\sum_{i=u,d}[\epsilon_L(i)\bar{q}_i\gamma_\mu(1-\gamma_5)q_i+\epsilon_R(i)\bar{q}_i\gamma_\mu(1+\gamma_5)q_i]+\cdots\ ,$$
$$J_\mu^{\rm lept}=\bar{e}\gamma_\mu(g_V^e+g_A^e\gamma_5)e+\cdots\ , \tag{1}$$

where $J_\mu^{\rm had}$ and $J_\mu^{\rm lept}$ are the hadronic and leptonic weak neutral currents. The standard model predicts definite values for these six parameters in terms only of $\sin^2\theta_W$, which can be measured independently by measuring α and $M_W{}^2$. If there are one or more extra U(1) factors then there will be deviations from these predicted values. We consider the case where one of the Z' bosons dominates or where there is only one. Then we can write down the following expressions for the ν-scattering parameters:

$$\begin{aligned}
\epsilon_L(u)&=(\tfrac{1}{2}-\tfrac{2}{3}x)(\rho+2\epsilon')+a(\epsilon'+2\epsilon'') ,\\
\epsilon_R(u)&=(-\tfrac{2}{3}x)(\rho+2\epsilon')-c(\epsilon'+2\epsilon'') ,\\
\epsilon_L(d)&=(-\tfrac{1}{2}+\tfrac{1}{3}x)(\rho+2\epsilon')+a(\epsilon'+2\epsilon'') ,\\
\epsilon_R(d)&=(-\tfrac{1}{3}x)(\rho+2\epsilon')-d(\epsilon'+2\epsilon'') ,\\
g_V^e&=(-\tfrac{1}{2}+2x)(\rho+2\epsilon')+(b-e)(\epsilon'+2\epsilon'') ,\\
g_A^e&=(-\tfrac{1}{2})(\rho+2\epsilon')+(b+e)(\epsilon'+2\epsilon'') .
\end{aligned} \tag{2}$$

Here $x=\sin^2\theta_W$, and ρ, ϵ', and ϵ'' are parameters that depend on the mass of the Z' and its mixing with Z. (See Ref. 2 for details.) Note the very useful fact that we can invert these equations to get a through e. There is an overall constant of proportionality which is the same in each case and is equal to $(\epsilon'+2\epsilon'')$; we absorb this into a redefined a, b, c, d, and e:

$$\begin{aligned}
a&=\frac{\frac{1}{2}-\frac{1}{3}x}{1-x}\epsilon_L(u)+\frac{\frac{1}{2}-\frac{2}{3}x}{1-x}\epsilon_L(d) ,\\
b&=\frac{\frac{1}{2}-x}{1-x}[\epsilon_L(u)-\epsilon_L(d)]+\tfrac{1}{2}(g_V^e+g_A^e) ,\\
c&=-\frac{\frac{2}{3}x}{1-x}[\epsilon_L(u)-\epsilon_L(d)]-\epsilon_R(u) ,\\
d&=\frac{\frac{1}{3}x}{1-x}[\epsilon_L(u)-\epsilon_L(d)]-\epsilon_R(d) ,\\
e&=\frac{x}{1-x}[\epsilon_L(u)-\epsilon_L(d)]-\tfrac{1}{2}(g_V^e-g_A^e) .
\end{aligned} \tag{3}$$

The fortuitous circumstances mentioned above are, first, that there are enough (six) parameters which are measurable in principle to enable us to solve for the charges, and, second, that the unknown mass and mixings of the Z' drop out of the resulting expressions. Even in the case of *several* U(1)′ factors Eqs. (2) and (3) still hold if $a,\ldots,e$ are interpreted as the suitable weighted charges of the several U(1)′ factors. (See Ref. 2.)

III. SU(5)×$\tilde{\rm U}$(1), $G\times\tilde{\rm U}$(1), $G\supset$SU(5)

Here we denote the U(1) factor $\tilde{\rm U}(1)$, since the extra U(1) at low energy which we call U(1)′ can be a linear combination of $\tilde{\rm U}(1)$ and $\mathrm{U}(1)_Y$, the weak hypercharge group. Since the $\tilde{\rm U}(1)$ charge must commute with the SU(5) (or, more generally, G), it is clear that with the standard embedding of the standard-model group

$$\begin{aligned}
\tilde a=\tilde c=\tilde e&=\text{charge of } \mathbf{10}_L\equiv Q_{10} ,\\
\tilde b=\tilde d&=\text{charge of } \mathbf{5}_L^*\equiv Q_{5^*} .
\end{aligned} \tag{4}$$

The tildes mean that we are referring to the $\tilde{\rm U}(1)$ [rather than U(1)′] charges. If we write $Q'=\tilde Q+t(Y/2)$, where Q' is the U(1)′ charge and $\tilde Q$ is the $\tilde{\rm U}(1)$ charge, we have

$$\begin{aligned}
a&=Q_{10}+\tfrac{1}{6}t ,\\
b&=Q_{5^*}-\tfrac{1}{2}t ,\\
c&=Q_{10}-\tfrac{2}{3}t ,\\
d&=Q_{5^*}+\tfrac{1}{3}t ,\\
e&=Q_{10}+t .
\end{aligned} \tag{5}$$

There are several combinations of parameters that are useful in discussing tests of grand unification. These were first introduced in Ref. 2. They are

$$\begin{aligned}
R&\equiv 2\epsilon_L(u)-\epsilon_R(u)+2\epsilon_L(d) ,\\
S&\equiv \epsilon_L(u)+2\epsilon_R(u)+\epsilon_L(d)+5\epsilon_R(d) ,\\
T&\equiv (1+\tfrac{2}{3}x)\epsilon_L(u)+2(1-x)\epsilon_R(u)+(1-\tfrac{8}{3}x)\epsilon_L(d) ,\\
V&\equiv 2\epsilon_L(u)+2\epsilon_R(u)+\epsilon_L(d)+\epsilon_R(d)+g_V^e ,\\
W&\equiv \epsilon_L(d)-\epsilon_R(d)-g_A^e .
\end{aligned} \tag{6}$$

Or in terms of a, b, c, d, and e these are

$$\begin{aligned}
R&\propto 5Q_{10} ,\\
S&\propto 2a-2c-5d ,\\
T&\propto 2(1-x)(a-c) ,\\
V&\propto 3a+b-2c-d-e ,\\
W&\propto a-b+d-e ,
\end{aligned} \tag{7}$$

with the same constant of proportionality in each case. In the case of grand unification of the standard model where Eq. (5) holds we get

$$\begin{aligned}
R&\propto 5Q_{10} ,\\
S&\propto -5Q_{5^*} ,\\
T&\propto \tfrac{5}{3}(1-x)t ,\\
V&=0 ,\\
W&=0 .
\end{aligned} \tag{8}$$

Note that $V=W=0$ is a prediction of grand unification. In certain cases, it turns out that $t=0$ (at least at tree level) (Ref. 6) and hence $T=0$. (See Appendix A.) For larger groups than SU(6) or SO(10), or when $\tilde{\rm U}(1)$ is not unified, there is no reason why t must vanish. Hence $T=0$ is *not* a prediction in general of grand unification. [In Ref. 2 this was erroneously asserted because the possibility that $\tilde{\rm U}(1)$ could sometimes mix with $\mathrm{U}(1)_Y$ was not appreciated.] In SO(10) models however T *does* vanish, which is significant as we shall see later. Furthermore, as emphasized in Ref. 2, the ratio

$$R/S=-Q_{10}/Q_{5^*} \quad (9)$$

gives us valuable information about the unified symmetry. If, for example, $\tilde{U}(1)$ is unified with $SU(3)\times SU(2)\times U(1)_Y$ in a simple group then R/S tends to take on characteristic values. In the case of SO(10) we get

$$R/S=1/3 \; . \quad (10)$$

(And also, as noted above, $T=0$.) If we have an $SU(N)$, $N\geq 6$ unified model, and if $SU(3)\times SU(2)\times U(1)_Y$ is embedded in the standard way, and if the e_L^+, u_L, d_L, and $\bar{u}_L$ are in a $\psi^{(\alpha\beta)}$ and ν_L, $\bar{e}_L^-$, and $\bar{d}_L$ are in a ψ_α, then

$$R/S=2 \; . \quad (11)$$

[This is because $\tilde{U}(1)$ will act as the SU(5) "quinticity." This particular prediction is valid no matter how many Z''s there are even if all of them contribute comparably to deviations from the standard model.] There are other interesting special values as well.

The question we are interested in here is what happens in the general case where $\tilde{U}(1)$ is *not* unified with SU(5) in some larger simple group. Are there any predictions for R/S that come just from the anomaly-freedom constraints? The answer is yes. There are three anomaly conditions that the $\tilde{U}(1)$ charges must satisfy in $SU(5)\times\tilde{U}(1)$:

$$\begin{aligned}&\text{(i)}\;\; SU(5)^2\times\tilde{U}(1)\text{ anomaly}=0\;,\\&\text{(ii)}\;\; \tilde{U}(1)^3\text{ anomaly}=0\;,\\&\text{(iii)}\;\; \text{gravity}\times\tilde{U}(1)\text{ anomaly}=0\;.\end{aligned} \quad (12)$$

If the only fermions around are the usual families consisting of $\mathbf{10}_L+\mathbf{5}_L^*$ then there is no (nontrivial) solution for the $\tilde{U}(1)$ charges. [We are assuming here, and throughout, that the charges of the extra U(1)'s are *independent of family*.] There is, of course, the solution $Q_{10}=Q_{5^*}=0$. This gives $R=S$, $V=W=0$. (T need not vanish.) We will call such a solution a "trivial solution." If we are to get a nontrivial solution we must have new light fermions, and the predictions for R/S will depend on what new light fermions we assume to exist. If there are too many such fermions, of course, then anomaly freedom will not give us any predictions. The new fermions must themselves have no $SU(5)^3$ anomaly. There are some simple choices: namely, real representations, and pairs of representations consisting of a complex representation plus its conjugate. A particular case of a real representation is a singlet of SU(5) which we might think of as an antineutrino ($\bar{\nu}_L$). We will consider various simple possibilities below.

A. All new fermions in *r*-dimensional representations

If r is a complex representation then there must be an equal number of r and r^*, so this means that the new fermions would be in conjugate pairs ($r+r^*$). If r is real then there can be any number of r and $r=r^*$. If we denote the $\tilde{U}(1)$ charges of the r by Q_i ($i=1,\ldots,n$) and of the r^* by Q_i' ($i=1,\ldots,n$), then Eq. (13) becomes

$$\begin{aligned}&\text{(i)}\;\; n_f(3Q_{10}+Q_{5^*})+I(r)\sum_{i=1}^{n}(Q_i+Q_i')\;,\\&\text{(ii)}\;\; n_f(10Q_{10}^3+5Q_{5^*}^3)+r\sum_{i=1}^{n}(Q_i^3+Q_i'^3)\;,\\&\text{(iii)}\;\; n_f(10Q_{10}+5Q_{5^*})+r\sum_{i=1}^{n}(Q_i+Q_i')\;.\end{aligned} \quad (13)$$

Here $I(r)$ is the index of the SU(5) representation of dimension r [i.e., $I(r)=\mathrm{tr}_r\lambda^2/\mathrm{tr}_5\lambda^2$], and n_f is the number of families. Subtracting $I(r)$ times Eq. (13)(iii) from r times Eq. (13)(i) gives

$$R/S=-Q_{10}/Q_{5^*}=\left[\frac{r-5I(r)}{3r-10I(r)}\right]. \quad (14)$$

This result applies no matter how many extra U(1) factors there are or how they mix (even if several of the Z''s contribute significantly). Notice that R/S will be non-negative unless $1/5<I(r)/r<3/10$. In fact there are no irreducible SU(5) representations with $I(r)/r$ in this range. [For increasingly large representations $I(r)/r$ increases without bound.] Thus for the case we are considering R/S is always non-negative.

Let us consider the special case where all Q_i are equal ($Q_i=Q$ all i) and where all the Q_i' are equal ($Q_i'=Q'$ all i). Then there are three homogeneous equations for four unknowns Q_{10}, Q_{5^*}, Q, and Q'. We can use the two linear equations (13)(i) and (13)(iii) to solve for ($Q+Q'$) and Q_{10} in terms of Q_{5^*}. Then substituting into the cubic equation (13)(ii), one gets a *quadratic* (not a cubic) for Q. The fact that we get a quadratic means that there are not always real roots. In fact there are only real roots if $I(r)/r\lesssim 0.47$. It turns out that there are only a finite number of SU(5) representations for which this bound is satisfied. These are listed below:

r	$I(r)$	R/S
1	0	$\frac{1}{3}$
5	1	0
10	3	∞
15	7	$\frac{4}{5}$
24	10	$\frac{13}{14}$

(15)

B. Real representation of SU(5)

1. *One real representation r of SU(5)*

Here there is no solution as there are three homogeneous equations and three charges (Q_{10}, Q_{5^*}, and Q_r). (Note that since the equations are homogeneous the overall normalization of the charges is irrelevant. Hence there are really only two unknowns.)

2. *One real representation r of SU(5) per family*

Each family is now $\mathbf{10}+\mathbf{5}^*+r$. Since we always assume family-independent charges, there are again three charges: Q_{10}, Q_{5^*}, and Q_r. And again, there is no solution in general. However, for $r=1$ there is the "SO(10)-like" solution $Q_{10}=1$, $Q_{5^*}=-3$, $Q_1=5$. This gives $R/S=\frac{1}{3}$, $V=W=0$. Since U(1)′ may have an admixture of hypercharge, T need not vanish. For an actual SO(10) model, however, as shown in Appendix A, we do have $T=0$ (at least at the tree level).[6] Thus a measurement of T is enough to distinguish this case from SO(10).

3. *Two real representations of SU(5)* $(r+r')$

There are an infinity of real representations of SU(5) to consider, of course. Nevertheless some statements are possible. Generally, when r and r' are both very large, the anomaly equations have only one real root. (Since there is a cubic equation involved there can be one, two, or three real roots.) Furthermore, for this root

$$R/S \simeq +(\tfrac{1}{2})^{1/3} . \tag{16}$$

This is easily seen. The equations are

$$\begin{aligned} &\text{(i)}\ \ 0=9Q_{10}+3Q_{5^*}+I(r)Q_r+I(r')Q_{r'} ,\\ &\text{(ii)}\ \ 0=30Q_{10}{}^3+15Q_{5^*}{}^3+rQ_r{}^3+r'Q_{r'}{}^3 ,\\ &\text{(iii)}\ \ 0=30Q_{10}+15Q_{5^*}+rQ_r+r'Q_{r'} . \end{aligned} \tag{17}$$

Solving the linear equations [(17)(i) and (17)(iii)] tells us that if $r,r'>N$ (N large) then typically Q_r are $Q_{r'}\lesssim 1/N$. The cubic then tells us that $0=15(2Q_{10}{}^3+Q_{5^*}{}^3)+O(1/N^2)$. So that $-Q_{10}/Q_{5^*}=R/S\simeq(\frac{1}{2})^{1/3}$. [If it should happen that $I(r)/r-I(r')/r'\lesssim r^{-2/3}$ then it will *not* be true that Q_r and $Q_{r^*}\lesssim 1/N$ and our argument will not apply. However, as we go to higher representations r very rapidly increases and it is quite unlikely that this condition is satisfied for more than a few representations, if ever.]

Now if one or both of r and r' are small then we can get more than one root. However, we have found only one case where there is a root that gives $R/S<0$. That is $r+r'=\mathbf{75}+\mathbf{200}$. This gives three roots, for one of which $R/S<0$. In *all other cases* as far as we know

$$R/S>0 \tag{18}$$

(see Table I).

TABLE I. The possible values of R/S for the case of the extra fermions being a pair of real SU(5) representations $r+r'$. There is only one case where $R/S<0$. We have only shown the smallest representations.

r' \ r	1	24	75	200
1	$\frac{1}{3}$	0.80	0.50	0.47
24		$\frac{13}{14}$	1.02	0.82,0.0015,1.91
75			X	0.803,4.82,−0.132
200				X

TABLE II. The possible values of R/S for the case of the extra fermions being a pair of real SU(5) representations *per family*. There are only two cases where $R/S<0$.

r' \ r	1	24	75	200
1	$\frac{1}{3}$	$\frac{1}{3}$,0.794,3.78	$\frac{1}{3}$	$\frac{1}{3}$
24		$\frac{13}{14}$	0.795,−2.74,−0.365	0.794
75			X	0.794
200				X

4. *Two real representations* $r+r'$ *per family*

Each family is taken to be $\mathbf{10}+\mathbf{5}^*+r+r'$. The preceding discussion applies here as well. That is, for large r and r' we get $R/S\simeq+(\frac{1}{2})^{1/3}$ and for all cases one finds $R/S>0$ with (as far as we know) only a single exception. For $r+r'=\mathbf{24}+\mathbf{75}$ we find that two of the three roots give $R/S<0$ (see Table II). We see in Table I that R/S approaches $+(\frac{1}{2})^{1/3}=0.793\ldots$ quite rapidly indeed even for small representations. So that in fact $R/S=\frac{1}{3}$ or 0.79 except in four special cases.

For more than two real representations there is no prediction for R/S: it can be positive, negative, or zero. However, when there is a prediction due to the anomaly conditions, we have found that R/S is virtually always >0. The reason for this is rather simple. There is a tendency for the **10** and the **5*** to have U(1)′ charges of opposite sign so that their anomalies will tend to cancel rather than add. It is not surprising therefore that in the cases where $\widetilde{\text{U}}(1)$ is unified, as in SO(10) or SU(N), $N\geq 6$, we find also $R/S>0$ [see Eqs. (10) and (11)].

IV. $SU(3)\times SU(2)\times U(1)_Y\times U(1)'$

In this case there are six anomaly equations to satisfy:

$$\begin{aligned} &\text{(i)}\ \ SU(3)^2\times U(1)'\ \text{anomaly}=0 ,\\ &\text{(ii)}\ \ SU(2)^2\times U(1)'\ \text{anomaly}=0 ,\\ &\text{(iii)}\ \ U(1)_Y^2\times U(1)'\ \text{anomaly}=0 ,\\ &\text{(iv)}\ \ \text{gravity}\times U(1)'\ \text{anomaly}=0 ,\\ &\text{(v)}\ \ U(1)_Y\times U(1)'^2\ \text{anomaly}=0 ,\\ &\text{(vi)}\ \ U(1)'^3\ \text{anomaly}=0 . \end{aligned} \tag{19}$$

One possibility that always solves the equations is a U(1)′ which *acting on the fermions* is the same as $U(1)_Y$. [Of course we assume that the fermions are anomaly-free under $SU(3)\times SU(2)\times U(1)_Y$.] Furthermore it is not hard to see that to any set of U(1)′ fermion charges which satisfy Eq. (19) we can add an arbitrary constant times hypercharge (Y) and still have a solution. Thus without loss of generality we can add a seventh condition, namely

$$\text{(vii)}\ \ \operatorname*{tr}_{\text{fermions}} Y\cdot Y'=0 , \tag{19}$$

where Y' is the charge of U(1)$'$, with the understanding that for any solutions to Eq. (19)(i)–(19)(vii) we can always take a linear combination with hypercharge.

Altogether, then, we have five conditions linear in Y', one quadratic, and one cubic. As before, to get any nontrivial solution we must assume that there are some extra light fermions. We will for the most part consider cases where there are some additional fermions *per family*, and where their charges are independent of family. We must add fermions that are already anomaly-free under $SU(3)\times SU(2)\times U(1)_Y$. The simplest cases are those of real representations and complex-conjugate pairs.

A. One conjugate pair of fermions ($r+r^*$), plus any number N, of $SU(3)\times SU(2)\times U(1)_Y$ singlets, $\bar{\nu}_L$, per family

We will show that there are always two classes of solutions (corresponding to the two roots of the quadratic [Eq. (19)(v)]) which we will call the "ordinary" solution and the "peculiar" solution. The "ordinary" solution for $N=0$ gives the trivial prediction $R=S=V=W=0$; and for $N=1$ or $N\geq 3$ gives the "SO(10)-like" solution $R/S=\frac{1}{3}$, $V=W=0$. [As noted above, the SO(10)-like solutions can be distinguished from actual SO(10) by the fact that $T=0$ for SO(10), while T need not vanish if there is no SO(10).] For $N=2$ one can have either the SO(10)-like or the trivial solution. The "peculiar" solutions give different predictions for every choice of r. To obtain these peculiar solutions requires solving a quadratic and cubic simultaneously which is rather messy and generally gives bizarre irrational charges. We have little to say about these solutions and concentrate on the "ordinary" ones. There is one case though where the "peculiar" solution is easy to obtain. If r is one of the "known" representations of $SU(3)\times SU(2)\times U(1)_y$, like $(3,2,\frac{1}{6})$ or $(3^*,1,-\frac{2}{3})$, etc., then the "peculiar" solution is just obtained from the "ordinary" one by interchanging the U(1)$'$ charges of r and its known counterpart with the same $SU(3)\times SU(2)\times U(1)_y$ quantum numbers. We call such a solution a "flipped" solution. For example, if $r=(3,2,\frac{1}{6})$ the flipped solution interchanges r and $\binom{u}{d}_L$.

1. N=0

If we denote the charges of $\binom{u}{d}_L$, $\binom{\nu}{e^-}_L$, $(\bar{u})_L$, $(\bar{d})_L$, $(e^+)_L$, r_L, and r_L^* by a, b, c, d, e, g, and g', respectively, then the equations are

$$\begin{aligned}
&\text{(i)}\ \ 0=2a+c+d+I_3(r)(g+g')\ ,\\
&\text{(ii)}\ \ 0=3a+b+I_2(r)(g+g')\ ,\\
&\text{(iii)}\ \ 0=a+3b+8c+2d+6e+6Y(r)^2(g+g')\ ,\\
&\text{(iv)}\ \ 0=6a+2b+3c+3d+e+d(r)(g+g')\ ,\\
&\text{(v)}\ \ 0=a^2-b^2-2c^2+d^2+e^2+Y(r)(g^2-g'^2)\ ,\\
&\text{(vi)}\ \ 0=6a^3+2b^3+3c^3+3d^3+e^3+d(r)(g^3+g'^3)\ ,\\
&\text{(vii)}\ \ 0=a-b-2c+d+e+d(r)Y(r)(g-g')\ ,
\end{aligned}\tag{20}$$

where $I_3(r)$, etc., have obvious definitions. Now, there are seven homogeneous equations and seven unknowns (really six since the overall normalization of the charges is arbitrary). Thus we expect no solutions. However, there *is* a solution. Consider

$$\begin{aligned}
(a,b,c,d,e)&=t(\tfrac{1}{6},-\tfrac{1}{2},-\tfrac{2}{3},\tfrac{1}{3},1)\\
&=t\left[\frac{Y}{2}\right],\\
(g,g')&=s(1,-1)\ .
\end{aligned}\tag{21}$$

This obviously satisfies the six anomaly conditions. The remaining condition, orthogonality to Y [Eq. (20)(vii)], fixes the ratio s/t. This is what we call the ordinary solution. If r is the same as one of the known representations of $SU(3)\times SU(2)\times U(1)_Y$ then there is a "flipped" solution as well. Otherwise there is no "peculiar" solution in this case. In Appendix B we show that there are no other solutions than those given here. Note, as we said above, that we get $R=S=V=W=0$.

2. N=1

Let us suppose there is exactly *one* $SU(3)\times SU(2)\times U(1)_Y$ singlet (which we will call henceforth $\bar{\nu}_L$) in addition to $r+r^*$, per family. Denote the U(1)$'$ charge of $\bar{\nu}_L$ by f. Then Eqs. (20) are modified by adding f to the right-hand side of Eq. (20)(iv) and f^3 to the right-hand side of Eq. (20)(vi). Now there are eight unknowns so we expect one-parameter solution (the parameter being the normalization). In fact there is a *two*-parameter solution. That solution is

$$\begin{aligned}
(a,b,c,d,e,f)&=t(\tfrac{1}{6},-\tfrac{1}{2},-\tfrac{2}{3},\tfrac{1}{3},1,0)\\
&\quad+v(1,-3,1,-3,1,5)\ ,\\
(g,g')&=s(1,-1)\ ,
\end{aligned}\tag{22}$$

where s is fixed in terms of t and v by the orthogonality condition Eq. (20)(vii). The numbers multiplying t are the hypercharges, while those multiplying v are the U(1)$'$ charges of $SU(3)\times SU(2)\times U(1)_Y\times U(1)'\subset SO(10)$. This is the "ordinary" solution, which gives $R/S=\frac{1}{3}$, $V=W=0$, T arbitrary. The "peculiar" solutions on the other hand have only one parameter (which is the normalization) as expected. But they are quite messy (involving irrational charges) and have to be found case by case. In Appendix B we show that there are no other solutions than these.

3. N=2

Here we have the extra fermions ($r+r^*+\bar{\nu}_L+\bar{\nu}_L$) per family. If we call the U(1)$'$ charges of the two $\bar{\nu}_L$, f_1 and f_2, then we modify Eq. (20) by adding f_1+f_2 to Eq. (20)(iv) and $f_1{}^3+f_2{}^3$ to Eq. (20)(vi).

Now we have nine unknowns and expect two-parameter solutions where one of these parameters is the overall normalization. And indeed the following is a two-parameter solution:

$$(a,b,c,d,e)=t(\tfrac{1}{6},-\tfrac{1}{2},-\tfrac{2}{3},\tfrac{1}{3},1)\ ,$$
$$(g,g')=s(1,-1)\ , \qquad (23)$$
$$(f_1,f_2)=v(1,-1)\ ,$$

where s is fixed in terms of v and t by the orthogonality condition, as before. (Another two-parameter ordinary solution has f_1 or $f_2=0$ and is just equivalent to the $N=1$ case.) There are also (messy) two-parameter "peculiar" solutions. The "ordinary" solutions just given are unique (see Appendix B) and give the "trivial" solution (or the degenerate solution with f_1 or $f_2=0$, the same result as $N=1$).

4. $N\geq 3$

Now we have N $\bar{\nu}_L$ with U(1)′ charges f_i ($i=1,\ldots,N\geq 3$). Equation (20) (iv) has an extra term $\sum_{i=1}^{N} f_i$ on the right-hand side and Eq. (20) (vi) has an extra term $\sum_{i=1}^{N}(f_i)^3$. There should be N-parameter solutions. Consider

$$\left[a,b,c,d,e,\sum_{i=1}^{N} f_i\right]=t(\tfrac{1}{6},-\tfrac{1}{2},-\tfrac{2}{3},\tfrac{1}{3},1,0)+v(1,-3,1,-3,1,5)\ ,$$
$$(g,g')=s(1,-1)\ , \qquad (24)$$
$$(f_2,f_3,\ldots,f_N)\ \text{solutions of}\ \sum_{i}^{N}(f_i)^3=\left[\sum_{i}^{N} f_i\right]^3 .$$

If we call $\sum_{i=1}^{N} f_i=\sigma$ then the equations become just the same as for $N=1$ with f replaced by σ. So obviously Eq. (24) is a solution since Eq. (22) is. Equation (24) is an N-parameter solution: s is fixed by the orthogonality equation, which leaves t, v, and f_3 through f_N. This is the unique (see Appendix B) "ordinary" solution, and gives the SO(10)-like solution. There are also N-parameter families of "peculiar" solutions.

B. Extra fermions (per family) are $\bar{\nu}_L$ [SU(3)×SU(2)×U(1)$_Y$ singlets]

The reasoning is very similar to that in the previous case (Appendix B). We will simply state the result. In the case IV A we solved the five linear conditions and substituted into the quadratic. This gave two roots corresponding to the "ordinary" and "peculiar" solutions. Here however when we solve the five linear conditions and substitute into the quadratic, we find the quadratic is identically satisfied. Hence there is only an "ordinary" family of solutions; for *N=0* there is no solution, for *N=1* there is a one-parameter solution:

$$(a,b,c,d,e,f)=v(1,-3,1,-3,1,5)\ . \qquad (25)$$

This gives the "SO(10)-like" result.

For *N=2* there is a one-parameter solution:

$$(a,b,c,d,e)=(0,0,0,0,0)\ , \qquad (26)$$
$$(f_1,f_2)=s(1,-1)\ .$$

This gives the "trivial" solution $R=S=V=W=0$, T arbitrary. There is also a one-parameter solution with f_1 or $f_2=0$ and the remaining charges as in the $N=1$ case. This gives the same "SO(10)-like" result as $N=1$.

For *N=3* there is a two-parameter solution:

$$(a,b,c,d,e,f_1)=v(1,-3,1,-3,1,5)\ , \qquad (27)$$
$$(f_2,f_3)=s(1,-1)\ .$$

There are also the solutions from permitting the f_i. This gives the SO(10)-like result.

For $N\geq 4$ there is an $(N-1)$-parameter solution:

$$\left[a,b,c,d,e,\sigma=\sum_{i=1}^{N} f_i\right]=v(1,-3,1,-3,1,5)\ , \qquad (28)$$
$$\sum_{i=1}^{N}(f_i)^3=\left[\sum_{i=1}^{N} f_i\right]^3 .$$

The $N-1$ parameters may be taken to be $f_2,\ldots,-f_N$ which determine f_1 through the second expression of Eq. (28), and determine v through $v=\frac{1}{5}\sum_{i=1}^{N} f_i$. This gives the SO(10)-like solution.

C. Two conjugate pairs $(r_1+r_1^*)+(r_2+r_2^*)$ per family

Here we expect two-parameter solutions. If we call the U(1)′ charges of the extra fermions (g,g') and (h,h') then a two-parameter solution is

$$(a,b,c,d,e)=t(\tfrac{1}{6},-\tfrac{1}{2},-\tfrac{2}{3},+\tfrac{1}{3},1)\ ,$$
$$(g,g')=s(1,-1)\ , \qquad (29)$$
$$(h,h')=w(1,-1)\ ,$$

with one parameter fixed by the orthogonality condition. This gives the trivial result $R=S=V=W=0$, T arbitrary. This however is *not* the unique solution. In general there are other two-parameter families of solutions. They would be messy and would have to be found case by case, which we have not done. (If r_1 and r_2 are in the same representations as known fermions then some of these other solutions can be found by "flipping.")

1. Two conjugate pairs plus $\bar{\nu}_L$ per family

Here there is an obvious three-parameter "SO(10)-like" solution. However other solutions exist as well.

D. One real representation r per family

Here one expects no solution as there are only six parameters (*five*, without the normalization) and *seven* equations. If $r=\bar{\nu}_L$ then there is the SO(10)-like solution of course. This is a consequence of the group theory of SO(10). Amazingly, however, there is another simple case with a solution, and it has no apparent group-theoretical reason to exist.

If $r=(1,3,0)$ representation of $SU(3)\times SU(2)\times U(1)_Y$ then there is a solution (where g is the charge of r)

$$(a,b,c,d,e,g)=v\left(1,\tfrac{13}{9},-\tfrac{2}{3},-\tfrac{4}{3},\tfrac{4}{9},-\tfrac{10}{9}\right)\ . \qquad (30)$$

We have found no other cases with solutions.

E. All extra fermions are color singlets

If all the extra fermions are color singlets then the $SU(3)^2\times U(1)'$ anomaly tells us that

$$2a+c+d=0\ . \qquad (31)$$

This result clearly does not depend on how many extra fermions there are, or whether they are associated with families, or how many Z' bosons there are or how they mix.

F. All extra fermions are $SU(2)_L$ singlets

Here the $SU(2)^2\times U(1)'$ anomaly tells us

$$3a+b=0\ . \qquad (32)$$

The same comments apply as in Sec. IV E.

V. SUMMARY OF RESULTS

In the Introduction we mentioned three motivations for looking at the anomaly constraints on extra Z' bosons. We will discuss our results under these three headings.

(1) One object was to see whether the tests of grand unification in ν scattering given in Ref. 2 really tested grand unification or only anomaly freedom. The signature of grand unification is that $V=W=0$. And if $G\times\tilde{U}(1)$ is further unified then the ratio R/S takes on characteristic values, such as $R/S=1/3$ for SO(10) and $R/S=2$ for many SU(N) models. We have found a number of cases with no grand unification where, nevertheless, $V=W=0$ and where $R/S=1/3$. These so-called "SO(10)-like" solutions, however, can be distinguished from SO(10) by the fact that in SO(10) we have $T=0$ whereas anomaly freedom alone does not constrain T. The cases with this "SO(10)-like" result are (1) the extra fermions per family are all $\bar{\nu}_L$, (2) the "ordinary" solution when the extra fermions per family are $[r+r^*+N(\bar{\nu}_L)]$, and (3) a particular family of solutions when the extra fermions per family are $(r_1+r_1^*+r_2+r_2^*+\nu_L)$.

We have found no nonunified cases where anomaly freedom alone predicts $V=W=0$ and $R/S\neq 1/3$. It is possible, then, always to separate the predictions of grand unification from those of anomaly freedom. Grand unification predicts $V=W=0$ and either $R/S\neq 1/3$ or $R/S=1/3$, $T=0$. In some cases anomaly freedom alone can predict $V=W=0$, $R/S=1/3$, T arbitrary.

(2) We are also interested in the case where $SU(3)\times SU(2)\times U(1)_Y$ is grand unified (since there is evidence for that) but the extra U(1) is not, and whether any values of R/S are preferred because of anomaly freedom in such a case. We have indeed found some predictions.

If $SU(3)\times SU(2)\times U(1)_Y\times U(1)'\subset SO(10)\times U(1)'$ then it is easy to see that $Q_{10}=Q_{5^*}$ so that

$$R/S=-1\ .$$

An example of this is furnished by SO(10) (or E_6, etc.) unification where the extra U(1)' is part of a family group.

If we have SU(5) unification then we have found various interesting cases. For example from Eq. (15) we see that certain simple cases lead to R/S taking one of the values 0, ∞, $\frac{1}{3}$, $\frac{4}{5}$, and $\frac{13}{14}$. More generally, we have found that if the U(1)' charges are "family independent" (i.e., the same for repeated representations) then virtually always when there is a prediction for R/S it comes out to be *positive*. (Two isolated exceptions were noted in Secs. III B 3 and III B 4.)

(3) Finally we are interested in what predictions could be made if there is no grand unification, just on the basis of anomaly freedom. In general, there is the interesting, and potentially useful, fact that the predictions for a, b, c, d, and e (when there are predictions) are correlated with the number and type of extra light fermions. [A rather amazing example is provided by the case where, in each family, there is a (1,3,0) of $SU(3)\times SU(2)\times U(1)_Y$. This has the solution given in Eq. (30).] So that, if there are extra light Z bosons, a measurement of deviations from the standard model in theory could tell us a great deal about the spectrum of undiscovered light fermions.

A useful general prediction is that if all extra light fermions are color singlets $2a+b+c=0$ and if they are all $SU(2)_L$ singlets then $3a+b=0$ [see Eqs. (31) and (32)].

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APPENDIX A

To see how the charge of the extra U(1) [U(1)'], which we denote Y', can have an admixture of hypercharge Y, let us look at an example. Consider an SU(7) model where

$$SU(7)\to SU(5)\times SU(2)\to SU(3)\times SU(2)\times U(1)_Y\times U(1)_2$$
$$\to SU(3)\times SU(2)\times U(1)_{\tilde{Y}}$$
$$\to SU(3)\times U(1)\ .$$

Let the generator of $U(1)_y$ be

$$\frac{Y}{2}=\tfrac{1}{6}\mathrm{diag}(3,3,-2,-2,-2,0,0)\ ,$$

and the generator of $U(1)_2$ be

$$X=\mathrm{diag}(0,0,0,0,0,1,-1)\ .$$

Superlarge breaking can be done by, among other things, Higgs-boson vacuum expectation values (VEV's) like $\langle\Omega_{\rm adj}\rangle=\alpha Y+\beta X$, and $\langle H^{67}\rangle\neq 0$. Let the further breaking of $U(1)_Y\times U(1)_2$ down to $U(1)_{\tilde{Y}}$ be done by, say, $\langle h^{612}\rangle\neq 0$. Then it is easy to see that

$$\frac{\tilde{Y}}{2}=\left[\frac{Y}{2}-X\right]$$

is the unbroken "hypercharge." That is, $\widetilde{Y}$ is the generator corresponding to the ordinary Z boson. If the known fermions are in $\psi_L^{(\alpha\beta)}$ and $\psi_{\alpha L}$ $(\alpha,\beta=1,\ldots,5)$ then $\widetilde{Y}$ acting on these particles will give the usual hypercharges. It is not hard to see that the charge to which the Z' couples is

$$Y' \propto 12\left[\frac{Y}{2}\right]+5X=12\left[\frac{\widetilde{Y}}{2}\right]+17X .$$

[Since $\widetilde{Y}$ and Y' are generators of the simple group SU(7), we have $\mathrm{tr}_{\mathrm{rep}}\widetilde{Y}Y'=0$.] Notice that Y' contains an admixture of $\widetilde{Y}$ which is hypercharge. So even if all interactions are unified in a simple group like SU(7) the U(1)$'$ charges (a,b,c,d,e) can have an admixture of hypercharge. If there is no unification then of course it is obvious that Y' can mix with hypercharge.

There are some cases where Y' cannot have an admixture of hypercharge. Consider SU(6), for example. If we embed $SU(3)\times SU(2)\times U(1)_Y\times U(1)'$ in SU(6), the generators of $U(1)_Y$ and U(1)$'$ will be

$$\frac{Y}{2}=\tfrac{1}{6}\mathrm{diag}(3,3,-2,-2,-2,0) ,$$
$$X=\mathrm{diag}(-1,-1,-1,-1,-1,5) .$$

Now, if the ordinary fermions are to have the correct hypercharge assignments, we can see that the hypercharge (the charge to which the ordinary Z boson couples) must be exactly $Y/2$ with no admixture of X. Hence the charge of the extra U(1) is just X with no admixture of Y. This is obviously a peculiarity of SU(6) among all the SU(N) groups.

The same holds also of SO(10). For if we embed $SU(3)\times SU(2)\times U(1)_Y\times U(1)'$ in SO(10) the charge X of U(1)$'$ will be 1, -3, and 5 on the SU(5) representations **10**, **5***, and **1** in the spinor of SO(10). If the hypercharge assignments of the usual fermions (which we take to be in the **16**) are to come out right, then the hypercharge has no admixture of X, and hence the charge of the extra U(1) can have no admixture of Y.

Thus for both SU(6) and SO(10) there is the prediction $T=0$. But in general, in unified models and in nonunified models, T is unconstrained.

APPENDIX B

We prove the assertions in Sec. IV B, about the uniqueness of the solutions. Consider the case of the extra fermions (per family) being $r_L+r_L^*+n(\bar{\nu}_L)$. Denote the U(1)$'$ charges of the r_L,r_L^* by g,g', and of the neutrinos by f_i, $i=1,\ldots,N$. Call $\sum_i f_i\equiv\sigma$. Then the five linear equations in (20) depend on a, b, c, d, e, g, g', and σ. Thus we can solve for a, c, d, e, and $(g-g')$ in terms of b, σ, and $(g+g')\equiv\Sigma$. Substituting into the quadratic gives two solutions for Σ in terms of b and σ. It turns out that one of these is always $\Sigma=0$ corresponding to the "ordinary" solution with $g=g'$. (This can be checked explicitly.) So the quadratic is of the form $\Sigma[\Sigma-f(b,\sigma)]=0$ where $f(b,\sigma)$ is linear in b and σ. Let us consider the ordinary $(\Sigma=0)$ solution. Substituting $\Sigma=0$ into the expressions for a, c, d, e, and $(g-g')$, we get these in terms only of b and σ. Substituting them into the cubic equation one gets an equation of the form

$$0=C(b,\sigma)+\sum_{i=1}^{N}(f_i)^3 , \tag{B1}$$

where C is a homogeneous cubic in b and σ. Now suppose $N=1$. Then Eq. (B1) reduces to (since then $\sigma=f_1$)

$$0=C(b,\sigma)+\sigma^3 . \tag{B2}$$

Since we know that Eq. (22) is a *two*-parameter solution (where the two parameters can be chosen to be b and σ), it follows that (B2) has solution for any b and σ in some range of values. This implies that

$$C(b,\sigma)=-\sigma^3 \tag{B3}$$

identically. Thus the cubic just gives, for $N=1$, $0=0$ and is redundant.

Now if $N=2$ the only thing to change is the second term in (B1). Thus Eq. (B3) still must be true. Hence the cubic equation now is

$$0=-\sigma^3+(f_1)^3+(f_2)^3 ,$$
$$\sigma\equiv f_1+f_2 . \tag{B4}$$

This gives

$$3f_1f_2(f_1+f_2)=0 .$$

The roots $f_1=0$ and $f_2=0$ are equivalent to the $N=1$ case. So the new case is

$$\sigma=(f_1+f_2)=0$$

and all the equations become equivalent to the $N=0$ case. This is just solution (23) in the text. So we have found that there are no other solutions than in the text.

If $N=3$ the cubic equation is

$$0=(f_1+f_2+f_3)^3-({f_1}^3+{f_2}^3+{f_3}^3) . \tag{B5}$$

This is a quadratic equation for f_1 in terms of f_2 and f_3 and so has at most two roots. They are $f_1=-f_2$ and $f_1=-f_3$. In the degenerate case $f_2=-f_3$, f_1 is undetermined. These three cases are clearly just equivalent under interchanging the f_i, so consider without loss of generality $f_1=-f_2$. Then $\sigma=f_3$ and $\sum_i(f_i)^3=(f_3)^3=\sigma^3$ and the equations collapse to the $N=1$ case. Thus we have found all of the three-parameter solutions (the parameters are b, σ, and f) and they are just those discussed in case (A3) in the text [Eq. (24)].

For $N\geq4$ there should be N-parameter solutions. The cubic is now

$$0=\left[\sum_i^N f_i\right]^3-\sum_i^N(f_i)^3 . \tag{B6}$$

We can solve the resultant quadratic for f_1 in terms of $f_2, f_3, \ldots, f_N$. (There are two roots.) Then $\sigma = \sum_i^N f_i = \sigma\,(f_2, f_3, \ldots, f_N)$ is a definite function of $(N-1)$ parameters. Since (B6) says that $\sum_i^N (f_i)^3 = \sigma^3$, all the equations now only depend on the f_i through σ and give the same solutions as the $N=1$ case. These are two-parameter solutions in terms of b and σ. But σ itself depends on $f_2, \ldots, f_N$. Thus there are N free parameters $b, f_2, \ldots, f_N$. The predictions for (a,b,c,d,e) are clearly exactly the same as for the $N=1$ case. That is, the predictions are SO(10)-like. So the result is that the "ordinary" solutions discussed in the text are unique.

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