Path integrals in parametrized theories: The free relativistic particle

James B. Hartle
Department of Physics, University of California, Santa Barbara, California 93106

Karel V. Kuchař
Department of Physics, University of Utah, Salt Lake City, Utah 84112

(Received 7 July 1986)

The connection between the canonical and the path-integral formulations of the quantum mechanics of a free relativistic particle is discussed as a model of theories in which time is one of the dynamical variables (parametrized theories). Several features central to the canonical formulation, such as the choice of Hilbert space, are reflected in the measure of the sum over paths and especially in the range of integration.

I. INTRODUCTION

Hamiltonian quantum mechanics and Feynman's sum over histories method provide two alternative approaches by which to construct the quantum mechanics of a physical system. In the canonical approach one identifies a preferred time, isolates the physical degrees of freedom, constructs a Hilbert space, establishes commutation relations, and calculates dynamics by solving the Schrödinger equation. In the path-integral approach one identifies the histories (paths), supplies an action for each, chooses a measure for the sum, and calculates transition amplitudes as sums over histories. Given a classical system, there is often considerable choice in the steps along the road to quantum mechanics via either approach: A different choice of time, Hilbert space, unitarily inequivalent commutation relations, or operator ordering, for example, will lead to different canonical quantum theories. Typically some of these choices seem more natural than others, but perhaps that is only because we are more familiar with them. In the path-integral approach the possibilities for choice are even more obvious. Even classically there are many different ways of parametrizing a history and many different physically equivalent actions. Certainly there are many measures one can use to define an integral.

Given a particular canonical formulation of a system's quantum mechanics, the question naturally arises as to whether one can directly write down the corresponding sum over histories formulation. In particular, what are the choices of histories, action, measure, etc., which correspond to a given selection of time, Hilbert space, commutation relations, etc.? For simple theories, whose actions are quadratic functions of the velocities with constant coefficients, this is not a big problem. In these cases, there is an essentially unique natural choice in each approach, and the choices correspond to each other. This is the situation in nonrelativistic particle quantum mechanics and familiar gauge field theories. For more complicated theories or theories expressed in a more complicated form, however, the issue becomes significant. This is especially the case for parametrized theories in which time—the variable so central to the canonical theory—is one of the dynamical variables. The action is then invariant under reparametrizations of the time and as a consequence sums over histories involve a nontrivial measure analogous to that arising from the invariances of gauge theories. The form of the measure which corresponds to a given canonical choice of time is, however, much more restricted than that which might have been imagined in analogy with gauge theories or guessed from invariance arguments. There is thus a close connection between the particular form of canonical quantum mechanics and the corresponding form of the measure in the sum over histories.

The simplest example in which this feature emerges is nonrelativistic particle quantum mechanics rewritten as a parametrized theory. We have analyzed the connection between canonical and path-integral quantum mechanics for such systems in a previous paper.[1] However, the most intriguing and important example of a parametrized theory is general relativity. The action for general relativity is invariant not only under reparametrizations of a given foliation of spacetime but also under a switch to an entirely different foliation. The central problem in formulating a canonical theory of gravity is the choice of the variable to play the role of time. One expects that any such choice must be reflected in the measure for the corresponding sum over histories. The situation is characteristically different from that of a nonrelativistic particle. For example, one possible action for general relativity (obtained from the Arnowitt-Deser-Misner (ADM) action by eliminating the lapse function) is not quadratic in the analogs of the velocities but rather has a square-root form reminiscent of a relativistic particle:[2]

$$S[g_{ab},N^a]=\int_R dt \int_\Sigma d^3x\,(gR\,(U_{ab}U^{ab}-U^2))^{1/2}\ , \tag{1.1}$$

where

$$U_{ab}\equiv\dot{g}_{ab}-N_{a\,|\,b}-N_{b\,|\,a}\ . \tag{1.2}$$

Here, g_{ab} is the spatial metric on the spacelike hypersurface Σ labeled by the time t, $g=\det(g_{ab})$ is its determinant, and R is the spatial scalar curvature on Σ. A vertical bar denotes a spatial covariant derivative, and a dot a time derivative. The shift vector N^a was left in the action while the lapse function N has been eliminated by using the Hamiltonian constraint.[3]

The transition between the canonical quantum mechanics of general relativity and its sum over history formulation has been extensively studied.[4–12] Unfortunately, such analysis is far from being transparent due to the complexity of the theory and to its infinites. By and large these works have concentrated on the local form of the measure and not on its global properties such as the ranges of integration. Considerable insight into the issues which arise in the general relativity can be obtained by looking at simpler reparametrization invariant systems which also have a "square-root" action. The simplest of these is the free relativistic particle in flat space. The transition between canonical and sum over histories quantum mechanics is clear-cut for the relativistic particle because it is a finite-dimensional, indeed trivial, system whose quantum mechanics is well understood. We shall study this transition in this paper.

The quantum mechanics of the free relativistic particle has been investigated both in the single-particle version of this theory[13] and in the many-particle theory.[14] Both versions can be given canonical and sum over histories formulations.[13–15] We shall concentrate here on the transition between the two. To obtain the simplest example we shall consider the theory of the relativistic particle in its single-particle form. At the expense of some manifest relativistic invariance, this theory gives a very clean example of the connection between Hamiltonian and sum over histories quantum mechanics for a theory whose action is not quadratic in the momenta. We shall comment on the connection with the many-particle theory in the conclusion.

The issue of particular interest for us is the way that the Hilbert space for the canonical theory of a single free relativistic particle is reflected in the measure for the sum over histories. We shall find that knowledge of the local measure is not enough to define the sum correctly but that global issues are also involved. We will be able to cast the sum over histories into a variety of forms corresponding to different possibilities for the classical action but there will also be some classical actions for which no corresponding sum over histories seems available. In the extended Lagrangian form of the action with multiplier we shall find a measure which, like its analog in general relativity,[6–9] appears to be reparametrization noninvariant. It will prove to be invariant, however, on closer inspection.

II. CLASSICAL THEORY OF THE RELATIVISTIC PARTICLE

In this section we shall review the classical theory of the free relativistic particle setting out the various ways in which its dynamics can be summarized by a variational principle. The several possibilities correspond to whether redundant variables, such as a parametrized time, are used in addition to the physical ones to describe the motion and, if so, to whether the consequent constraints are enforced explicitly or with a Lagrange multiplier. For each case there are two forms of the action: One in which it is a functional of configuration space variables and velocities (the Lagrangian form); and a second in which it is a functional on phase space (the Hamiltonian form).

The physical degrees of freedom of a relativistic particle are its position $x^a(t)$ at a given time. The action which summarizes its dynamics is

$$S[x^a] = -m \int dt \left[1 - \left[\frac{dx^a}{dt}\right]^2\right]^{1/2}, \tag{2.1}$$

where m is the particle's rest mass and we have introduced the obvious notation that $(v^a)^2 = \delta_{ab} v^a v^b$. As throughout, we have used units where $\hbar = 1 = c$.

The action (2.1) is not manifestly Lorentz invariant. That can be achieved by parametrizing the time. We describe the motion by a parametrized path $x^\alpha = x^\alpha(\tau)$ in Minkowski space $x^\alpha = (t, x^a)$. The parametrized action is

$$S[x^\alpha] = -m \int d\tau (-(\dot{x}^\alpha)^2)^{1/2}, \tag{2.2}$$

where a dot denotes a derivative with respect to the parameter τ. As physically it must be, this action is invariant under reparametrizations of the label time,

$$\tau \to \tau' = f(\tau), \quad x^\alpha(\tau) \to x'^\alpha(\tau) = x^\alpha(f(\tau)), \tag{2.3}$$

for an arbitrary monotonically increasing function $f(\tau)$ which leaves the end points of integration unchanged.

The action (2.2) is not the only Lorentz-invariant action which summarizes the dynamics of the particle. A form more closely analogous (Table I) to the Hilbert action for general relativity is obtained by introducing a "lapse" multiplier $N(\tau)$:

$$S[x^\alpha, N] = \tfrac{1}{2} m \int d\tau (N^{-1}(\dot{x}^\alpha)^2 - N). \tag{2.4}$$

Variation of this action with respect to the x^α and N yields equations which are equivalent to the equations of motion plus the statement that N is the rate of change of the proper time with respect to the label time τ. The action (2.4) is invariant under reparametrizations of the label time which, besides Eq. (2.3), also imply the change of the lapse:

$$N(\tau) \to N'(\tau) = \dot{f}(\tau) N(f(\tau)). \tag{2.5}$$

Further insight into the relation between the three Lagrangian forms of the action (2.1), (2.2), and (2.4) can be obtained by studying their Hamiltonian counterparts. A physical phase-space path is described by the particle's position $x^a(t)$ and the conjugate momentum $p_a(t)$. The physical Hamiltonian is

$$h(x^a, p_a) = ((p_a)^2 + m^2)^{1/2}. \tag{2.6}$$

An action which summarizes the Hamiltonian dynamics of the particle is

$$S[x^a, p_a] = \int dt \left[p_a \frac{dx^a}{dt} - h(x^a, p_a)\right]. \tag{2.7}$$

Independent variations with respect to the x^a and p_a yield Hamilton's equations of motion. This action is the Hamiltonian form of (2.1).

TABLE I. General relativity and the relativistic particle.

		General relativity in 3+1 form	Relativistic particle
Variables		$g_{\alpha\beta}$	x^α, N
Physical variables	+ Time	g_{ab}	x^a
	+ Multipliers	N, N^a	N
Action		$S=\int dt \int d^3x\, N(g^{1/2}(K_{ab}K^{ab}-K^2)+R)$ $K_{ab}\equiv\frac{1}{2}N^{-1}(-\dot{g}_{ab}+N_{(a\mid b)})$	$S=\frac{m}{2}\int d\tau(N^{-1}(\dot{x}^\alpha)^2-N)$
Invariance		$x'^\alpha=f^\alpha(x^\beta)$	$\tau'=f(\tau)$
Constraints		$H\equiv g^{-1/2}(\pi_{ab}\pi^{ab}-\pi^2)-g^{1/2}R=0$ $H_a\equiv-2\pi_{a\mid b}^b=0$	$H\equiv\frac{1}{2m}((p_\alpha)^2+m^2)=0$
Physical degrees of freedom		"g_{ab}^{TT}"	x^a

The Hamiltonian form of (2.2) may be obtained by first constructing the momenta p_α conjugate to x^α in the usual way. As a consequence of the reparametrization invariance there is a constraint which fixes p_0 for a given p_a. We can write it in the quadratic form

$$H\equiv\frac{1}{2m}((p_\alpha)^2+m^2)=0\ , \tag{2.8}$$

if we agree that p_α is future pointing, $p_0\leq 0$. Equation (2.8) is the condition that the super-Hamiltonian H on the extended phase space x^α, p_α vanish. Consequently, the action

$$S[x^\alpha,p_\alpha]=\int d\tau\, p_\alpha\dot{x}^\alpha \tag{2.9}$$

summarizes the dynamics on the extended phase space provided that x^α and p_α are varied subject to the constraint (2.8).

With the action (2.9) the constraint (2.8) must be enforced explicitly. It can be enforced implicitly by using a Lagrange multiplier $N(\tau)$:

$$S[x^\alpha,p_\alpha,N]=\int d\tau(p_\alpha\dot{x}^\alpha-NH(x^\alpha,p_\alpha))\ . \tag{2.10}$$

Variation with respect to N yields the constraint (2.8) and independent variations with respect to p_α and x^α yield the equations of motion. The action (2.10) is the Hamiltonian version of (2.4) and N in that expression is seen to be a multiplier enforcing the constraint.

We have presented six forms of the action, (2.1), (2.2), (2.4), (2.7), (2.9), and (2.10), as though plucked from the air. In fact, it is possible to proceed systematically from one to the other and we shall briefly indicate how to do so. Start with (2.7); it is a general form for the action written on physical phase space. The expression (2.9) is the same action rewritten with $t=x^0(\tau)$ and $h=p_0(\tau)$ as follows from the constraint (2.8). The expression (2.10) is again the same action but with the constraint enforced by a multiplier. The corresponding Lagrangian forms (2.1), (2.2), (2.4) are obtained from these Hamiltonian actions by a Legendre transformation in the usual way.

III. HAMILTONIAN QUANTUM MECHANICS

Is it possible to write down directly the sum over histories quantum mechanics of the relativistic particle which is equivalent to its canonical (Hamiltonian) formulation? In this section we shall briefly review the canonical formulation. In the next section we shall pass from it to the path-integral formulation.

The Hilbert space of states of a single free relativistic particle is spanned by the states of definite momentum $|\mathbf{p}\rangle$. We choose the momentum eigenstates to have relativistically invariant normalization

$$\int\frac{d^3p}{2p^0}|\mathbf{p}\rangle\langle\mathbf{p}|=1\ , \tag{3.1}$$

where $p^0=(\mathbf{p}^2+m^2)^{1/2}$.

There are two interesting definitions of states with position labels. First there are the Newton-Wigner states[13]

$$|\mathbf{x},t\rangle=(2\pi)^{-3/2}\int d^3p\,(2p^0)^{-1/2}\exp(ip_\alpha x^\alpha)|\mathbf{p}\rangle\ . \tag{3.2}$$

For fixed t, these are complete, orthogonal, and normalized so that

$$\langle\mathbf{x}',t|\mathbf{x},t\rangle=\delta(\mathbf{x}-\mathbf{x}')\ . \tag{3.3}$$

The Newton-Wigner wave function ψ corresponding to a state $|\psi\rangle$ is

$$\psi(\mathbf{x},t)=\langle \mathbf{x},t \mid \psi\rangle \tag{3.4}$$

and it obeys

$$i\partial_t\psi=(m^2-\nabla^2)^{1/2}\psi \ . \tag{3.5}$$

The inner product between two states in terms of their Newton-Wigner wave functions is

$$\langle \psi_1 \mid \psi_2\rangle=\int d^3x\, \psi_1^*(\mathbf{x},t)\psi_2(\mathbf{x},t) \ . \tag{3.6}$$

The Newton-Wigner states are not relativistically invariant in the sense that the construction (3.2) carried out on a Lorentz-boosted slice will yield a different state. The closely related states

$$\mid x^\alpha\rangle=(2\pi)^{-3/2}\int d^3p(2p^0)^{-1}\exp(ip_\alpha x^\alpha)\mid \mathbf{p}\rangle \tag{3.7}$$

are relativistically invariant but they are not orthogonal. The corresponding wave functions

$$\phi(x^\alpha)=\langle x^\alpha \mid \phi\rangle \tag{3.8}$$

are easily seen to be positive-frequency solutions of the Klein-Gordon equation

$$(m^2-\nabla^2)\phi=0 \ . \tag{3.9}$$

The inner product is then

$$\langle \phi_1 \mid \phi_2\rangle=-i\int d^3x\, \phi_1^*(x^\alpha)\overleftrightarrow{\partial}_t\phi_2(x^\alpha) \ . \tag{3.10}$$

Quantum dynamics of the free relativistic particle may be thought of as defined by either (3.5) or (3.9). It is most conveniently specified by giving the propagator. There are two forms corresponding to the two definitions of position states: the Newton-Wigner propagator $\langle \mathbf{x}'',t'' \mid \mathbf{x}',t'\rangle$ and the invariant propagator $\langle x''^\alpha \mid x'^\alpha\rangle$. They are, of course, but different representations of the same object. The Newton-Wigner product enjoys a simple composition law

$$\langle \mathbf{x}'',t'' \mid \mathbf{x}',t'\rangle=\int d^3x\langle \mathbf{x}'',t'' \mid \mathbf{x},t\rangle\langle \mathbf{x},t \mid \mathbf{x}',t'\rangle \ , \tag{3.11}$$

but is not relativistically invariant. The invariant propagator has a more complicated composition law:

$$\langle x''^\alpha \mid x'^\alpha\rangle=-i\int d^3x\langle x''^\alpha \mid x^\alpha\rangle\overleftrightarrow{\partial}_t\langle x^\alpha \mid x'^\alpha\rangle \ . \tag{3.12}$$

IV. PATH-INTEGRAL QUANTUM MECHANICS

The starting point for a sum over histories formulation of the quantum mechanics of a free relativistic particle is a path integral for the propagator of the schematic form

$$(\text{propagator})=\sum_{\text{paths}}\exp(iS) \ , \tag{4.1}$$

where S is an action. One expects different forms of (4.1) corresponding to which of the several forms of the action is used and to whether the integration is over configuration space paths or phase-space paths. In this section, we shall derive those forms which reproduce the Hamiltonian quantum mechanics discussed in the previous section.

We begin with physical phase space. In the Newton-Wigner representation the relativistic particle may be thought of as a physical system with the Hamiltonian (2.6), orthonormal position states $\mid \mathbf{x},t\rangle$, and orthonormal momentum states $(2p^0)^{-1/2}\mid \mathbf{p}\rangle$. It then follows from quantum mechanics that the propagator can be represented as the phase-space path integral:

$$\langle \mathbf{x}'',t'' \mid \mathbf{x}',t'\rangle=\int \frac{\delta^3p\delta^3x}{[(2\pi)^3]}\exp(iS[\mathbf{x},\mathbf{p}]) \ . \tag{4.2}$$

Here, $S[\mathbf{x},\mathbf{p}]$ is the canonical action (2.7) for the physical degrees of freedom. The sum is over phase-space paths which move forward in the time t and can be specified concretely as follows: Divide the time interval $[t',t'']$ up into slices $t(K)$, $K=0,1,\ldots,N$, with $t(0)=t'$ and $t(N)=t''$. The path integral (4.2) is the limit as $N\to\infty$ of multiple integral

$$\langle \mathbf{x}'',t'' \mid \mathbf{x}',t'\rangle=\lim_{N\to\infty}\int \frac{d^3p(N)}{(2\pi)^3}\prod_{K=1}^{N} d^3p(K)d^3x(K)C(\mathbf{x}(K),t(K) \mid \mathbf{x}(K-1),\mathbf{p}(K-1),t(K-1)) \ , \tag{4.3}$$

where $C(\mathbf{x}'',t'' \mid \mathbf{x}',\mathbf{p}',t')$ is the "classical propagator" to go from one slice to the next. The classical propagator is defined in terms of the classical action by

$$C(\mathbf{x}'',t'' \mid \mathbf{x}',\mathbf{p}',t')=(2\pi)^{-3}\exp(iS(\mathbf{x}'',t'' \mid \mathbf{x}',\mathbf{p}',t')) \ , \tag{4.4}$$

where $S(\mathbf{x}'',t'' \mid \mathbf{x}',\mathbf{p}',t')$ is the phase-space principal function defined by integrating the action along the classical configuration space path with constant momentum. In the case of the relativistic particle,

$$S(\mathbf{x}'',t'' \mid \mathbf{x}',\mathbf{p}',t')=\mathbf{p}'\cdot(\mathbf{x}''-\mathbf{x}')-((\mathbf{p}')^2+m^2)^{1/2}(t''-t') \ . \tag{4.5}$$

The form (4.3), which is an interpretation of (4.2), can be derived as follows. The propagator from t' to t'' can be represented as a composition of propagators from t' to $t(1)$, from $t(1)$ to $t(2)$, and so on. For small time steps one discovers that the propagator can be represented as the integral $\int d^3p'\,C(\mathbf{x}'',t'' \mid \mathbf{x}',\mathbf{p}',t')$, whence the identity (4.3) in the limit. For the relativistic particle, whose Hamiltonian depends only on momentum, this approximation is in fact an identity for all times. That is

$$\langle \mathbf{x}'',t'' \mid \mathbf{x}',t'\rangle =\int d^3p\, C(\mathbf{x}'',t'' \mid \mathbf{x}',\mathbf{p}',t') \tag{4.6}$$

$$=\int \frac{d^3p'}{(2\pi)^3}\exp\{i(\mathbf{p}'\cdot(\mathbf{x}''-\mathbf{x}') -(\mathbf{p}'^2+m^2)^{1/2}(t''-t'))\} \ , \tag{4.7}$$

for all t' and t''. The path integral (4.2) or (4.3) is then just an expression for the composition of propagators.

Equations (4.2) and (2.7) yield an expression for the Newton-Wigner propagator. The left-hand side of (4.2) is not relativistically invariant because the Newton-Wigner states are not relativistically invariant. The right-hand side of (4.2) is not relativistically invariant because the sum is over paths which move forward in the time of a particular Lorentz frame—the frame in which the Newton-Wigner states are defined. This property of paths is characteristic of single-particle theories. Of course, the quantum theory *is* relativistically invariant because there is a unitary connection between the theories formulated in any two Lorentz frames.

Equations (4.2) and (2.7) give an expression for the propagator as a path integral over the physical phase space. We shall now show how this can be converted into equivalent path-integral expressions involving parametrized time and the Hamiltonian actions of Sec. II. The method in each case is the same.[16] We add to the path integral integrations over redundant variables in such a way that the action is modified to the desired form but the value of the integral is unchanged.

We first parametrize the time. To do this we pick a function $F(t,\tau)$ of the form $F(t,\tau)=\tau-f(t)$ which through

$$F(t,\tau)=\tau-f(t)=0 \tag{4.8}$$

assigns a unique value of τ to each t. Without loss of generality, $f(t)$ may be taken to be a monotonically increasing function. We then introduce a functional integration over $x^0=t$ by substituting into the path integral (4.2) the identity

$$1=\int dx^0\left|\frac{\partial F}{\partial x^0}\right|\delta(F(x^0,\tau)) \tag{4.9}$$

on each internal time slice $K=1,\ldots,N-1$. The result may be written

$$\langle x'',t''\,|\,x',t'\rangle=\int\frac{\delta^3p\,\delta^4x}{[(2\pi)^3]}\left[\left|\frac{\partial F}{\partial x^0}\right|\delta(F(x^0,\tau))\right]\times\exp\left[i\int_{\tau'}^{\tau''}d\tau\,p_\alpha\dot{x}^\alpha\right]. \tag{4.10}$$

The notation is as follows: $\dot{x}^\alpha=dx^\alpha/d\tau$, τ' and τ'' are the values of τ corresponding through (4.8) to t' and t'', and p_0 is defined to be $-(\mathbf{p}^2+m^2)^{1/2}$. The integral is understood to be defined by a skeletonization process similar to (4.3) with one of each factor in square brackets in the measure included on each internal slice.

The action in (4.10) is the action (2.9) on the extended phase space x^α,p_α but the integration is over the subspace restricted by the vanishing of the super-Hamiltonian (2.8). We can extend the integration by introducing a δ function to enforce the constraint. If the identity

$$m^{-1}\int_{-\infty}^{\infty}dp_0\,|\,p_0\,|\,\theta(-p_0)\delta(H)=1 \tag{4.11}$$

is inserted on each slice but the first, there results

$$\langle \mathbf{x}'',t''\,|\,\mathbf{x}',t'\rangle=\int\frac{\delta^4p\,\delta^4x}{[(2\pi)^3]}[\theta(-p_0)\,|\,\{F,H\}\,|\,\delta(F)\delta(H)]\times\exp\left[i\int_{\tau'}^{\tau''}d\tau\,p_\alpha\dot{x}^\alpha\right], \tag{4.12}$$

where $\{A,B\}$ is the Poisson brackets of A and B. The square-bracket notation in (4.12) is intended to indicate "one such factor for each appropriate time slice." To avoid introducing cumbersome notation we have not tried to indicate whether a factor would be included on the last slice or whether it should be omitted. For that the reader should refer to the derivation. In (4.12), the propagation is expressed as an integral over the extended phase space with the appropriate action (2.9) and the constraint enforced explicitly. The Poisson brackets

$$\{F,H\}=\frac{\partial F}{\partial x^0}\frac{p^0}{m} \tag{4.13}$$

are the analog of the "Faddeev-Popov determinant" in gauge theories and $\delta(F)$ is the analog of the "gauge-fixing δ function." The forms of F which will reproduce canonical quantum mechanics are, however, much more restricted than the possible forms which can fix a gauge in a gauge theory. F can depend on no variables other than t and τ and must single out a unique slicing in t for the given slicing in τ. Note also the $\theta(-p_0)$ function which has no counterpart in gauge theories, but which is needed to restrict the solutions of the constraint equation to the positive mass shell. Its inclusion, of course, amounts to limiting the range of the p_0 integration.

The constraint of the classical theory of a relativistic particle can either be enforced explicitly in variations of the action (2.9) or implicitly by introducing a Lagrange multiplier as in (2.10). Equation (4.12) is the form of the path integral in which the constraint is enforced explicitly. Forms may be found in which it is enforced implicitly by "exponentiating" the δ function. There are several ways to do this.

First, we can write

$$\delta(H)=\frac{1}{2\pi}\int_{-\infty}^{\infty}dN\,\epsilon\exp(-iN\epsilon H) \tag{4.14}$$

for any ϵ. If this identity is used to replace $\delta(H)$ in (4.12) on each slice where it occurs, and ϵ is chosen to be the spacing in τ between that slice and the previous one, then

$$\langle \mathbf{x}'',t''\,|\,\mathbf{x}',t'\rangle=\int\frac{\delta N\,\delta^4p\,\delta^4x}{[(2\pi)^4]}[\epsilon\theta(-p_0)\,|\,\{F,H\}\,|\,\delta(F(x^0,\tau))]\times\exp\left[i\int_{\tau'}^{\tau''}d\tau(p_\alpha\dot{x}^\alpha-NH)\right]. \tag{4.15}$$

By δ^4p we mean a factor of $dp_0dp_1dp_2dp_3$ on each appropriate slice. Equation (4.15) is a path integral for the propagator over all curves in the extended phase space x^α,p_α consistent with the propagator's arguments together with a functional integral over a multiplier $N(\tau)$ not fixed on one end of the range $[\tau',\tau'']$. The range of this integration is important: N ranges over the whole real line. The action in (4.15) is the action (2.10), i.e., the parametrized action (2.9) with a multiplier enforcing the constraint.

Equation (4.14) is not the only way of exponentiating a δ function and (4.15) not the only path integral for the propagator with lapse multiplier. As one alternative we can write

$$\int_{-\infty}^{\infty} dp_0 \, |p_0| \, \theta(-p_0)\delta(H)\exp(ip_\alpha \dot{x}^\alpha)$$
$$= \frac{i}{2\pi}\int_{-\infty}^{\infty} dp_0 \frac{p_0 \exp(ip_\alpha \dot{x}^\alpha)}{H - i\eta}$$
$$= \int_{-\infty}^{\infty} dp_0 p_0 \int_{-\infty}^{\infty} \frac{\epsilon \, dN}{2\pi}\theta(N) \times \exp\{i(p_\alpha \dot{x}^\alpha - \epsilon N(H - i\eta))\} \tag{4.16}$$

in the limit $\eta \to 0+$. Using this identity to replace the combination on the left-hand side on every slice it occurs in (4.12) we have

$$\langle \mathbf{x}'',t'' \,|\, \mathbf{x}',t' \rangle = \int \frac{\delta N \, \delta^4 p \, \delta^4 x}{[(2\pi)^4]}[\epsilon\theta(N) \,|\, \{F,H\} \,|\, \delta(F(x^0,\tau))] \times \exp\left[i \int_{\tau'}^{\tau''} d\tau (p_\alpha \dot{x}^\alpha - NH) \right]. \tag{4.17}$$

Equation (4.17) is essentially the same as (4.16) except that the ranges of integration are different. The p_0 integration is unrestricted, while the N integration is restricted by the $\theta(N)$ function to be positive; a restriction ultimately arising from the restriction of the configuration paths to move forward in time. In (4.15) and (4.17) we see that even with the same action there may be several different path integrals expressing quantum dynamics.

The expressions (4.2), (4.12) and (4.15), (4.17) are phase-space path integrals involving the Hamiltonian actions (2.7), (2.9), and (2.10) introduced in Sec. II. One can attempt to find the corresponding Lagrangian forms by integrating out the momenta in the phase-space path integrals. This is not always possible. There is a momentum integration to be carried out on each internal slice of the iterated integral (4.3). To reproduce a Lagrangian path integral the result of this integration must be of the form $\exp(iS)$ for some Lagrangian action at least for small values of the spacing ϵ between the slices. If the action becomes large as ϵ becomes small this form naturally emerges as a steepest-descent approximation to the integral over momenta. If, however, the action does not become large as ϵ becomes small one would expect the necessary form to emerge only "accidentally."

The integrating out the momenta in the path integral on the physical phase space built on D spatial dimensions provides a good example. The relevant integral is

$$\int \frac{d^D p}{(2\pi)^D} e^{i(\mathbf{p}\cdot\mathbf{x} - (\mathbf{p}^2+m^2)^{1/2}t)}, \tag{4.18}$$

where the exponent is the phase-space principal function $S(\mathbf{x},t \,|\, 0,\mathbf{p},0)$ given by (4.6). We first note that the principal function evaluated at the extremizing $\mathbf{p}$ does not become large for small t. The integral cannot be legitimately approximated by steepest descents. In fact, its explicit evaluation reveals that it is proportional to $e^{iS(\mathbf{x},t\,|\,0,0)}$, where S is the principal function of the action (2.1) for even D, but not for an odd D. For an odd D, the proportionality does not hold even in a small t limit and a standard Lagrangian path integral cannot be constructed.

Integrating out the momenta in (4.10) or (4.12) is no easier because they are of the same form as those in (4.2). There remain the forms (4.15) and (4.16) in which the constraints are enforced by a multiplier. The crucial integral in (4.15) is that over p_0:

$$I = \int_0^\infty dp_0 p_0 \exp\left[i \left[p_0 \dot{x}^0 + \frac{N}{2m} p_0^2 \right] \epsilon \right]. \tag{4.19}$$

For small ϵ this has an expansion of the form

$$I = \frac{2m}{\epsilon N}\left[C_0 + C_1 \left[\frac{2m\epsilon}{N}\right]^{1/2} \dot{x}^0 + C_2 \left[\frac{2m\epsilon}{N}\right](\dot{x}^0)^2 + \cdots \right], \tag{4.20}$$

where C_0, C_1, and C_2 are nonvanishing numerical constants. Here we have used in an important way the fact that in (4.15) any smooth function F will restrict $\dot{x}^0$ to be of order unity, not of order ϵ. If C_1 vanished, (4.20) would be an expansion of $\exp(iS)$ for S an integral over time. But C_1 does not vanish and we do not recover naturally a Lagrangian path integral from the momentum integrals in (4.15).

Equation (4.17) is the one form of the phase-space path integral for which the integrals over momentum can be carried out. They are Gaussian. The result is

$$\langle \mathbf{x}'',t'' \,|\, \mathbf{x}',t' \rangle = \int \frac{\delta N \, \delta^4 x}{[-(2\pi)^2 i]}\left[\frac{1}{N}\left[\frac{m}{\epsilon N}\right]^2 \theta(N)(\epsilon \dot{x}^0) \left| \frac{\partial F}{\partial x^0} \right| \delta(F(x^0,\tau)) \right] \exp\left[i\tfrac{1}{2}m \int d\tau (N^{-1}(\dot{x}^\alpha)^2 - N) \right]. \tag{4.21}$$

This is the Lagrangian form of the path integral corresponding to the action (2.4). It has a nontrivial measure.

V. REMARKS AND CONCLUSIONS

The sum over histories quantum mechanics of the single free relativistic particle can be cast into many forms. There are phase-space forms in the physical phase space, in the physical phase space augmented by a parametrized time and its conjugate momentum, and on that extended phase space augmented further by a multiplier to enforce the constraint. There can even be two forms on the same space as (4.15) and (4.17) show. The Lagrangian forms are more limited. We were able to construct a Lagrangian path integral only for that form of the action which was quadratic in the velocities.

With which path integral should one begin a study of the sum over histories formulation of the relativistic particle? It does not matter. As we have seen, one can pass from one form to another. The different forms, however, are useful because they display different aspects of the theory in a manifest way. The path integral over the physical phase space expresses the quantum mechanics of the relativistic particle in the most direct way. In a theory like general relativity, however, such a form is not available. There one must start from a form such as (4.17) or (4.21) in which time is parametrized. None of the integrals is manifestly relativistically invariant.

We have derived the various forms of the sum over histories quantum mechanics starting from the standard canonical theory. To what extent is there a prescription for arriving at the sum over histories directly? How elegantly simple the transition from classical to quantum mechanics would be if there were a unique way of constructing quantum amplitudes, given a classical action. How useful such a prescription would be for the construction of a quantum theory of gravity whose canonical quantum mechanics presents so many problematical aspects. Such a prescription may be said to have been given for gauge theories by Faddeev.[16] There, given an action, one finds its invariances, chooses an invariant measure, and uses an appropriate gauge fixing so that physically distinct configurations are counted only once in the sum over histories. In our concluding remarks we shall discuss the question of whether an analogous prescription exists for the quantum mechanics of the single relativistic particle, and, therefore, by extension whether it is likely to exist for other parametrized theories.

The choice of variables, action, and class of paths may be taken as the given starting points of a sum over histories prescription. The crucial question is then whether one can pick a measure such that the sum over histories reproduces the canonical quantum mechanics of the free relativistic particle. As we have seen, the possibility of doing so depends on the choice of variables, for example, whether we use physical phase space, physical configuration space, extended phase space, or whatever. Indeed, unless the action is quadratic in the velocities we did not (in general dimension) find a path integral for the propagator at all. Suppose, however, we choose a representation in which the action is quadratic. How then do we choose the appropriate measure? There are three parts to this issue: the local form of the measure, fixing the gauge, and the domain of integration.

A. Local form of the measure

The Lagrangian sum over histories on the extended configuration space with multiplier provides an easily analyzable example of a path integral with a nontrivial invariant measure. The invariance at issue is the reparametrization invariance (2.5). The action is invariant under such a transformation and so is clearly the result of the integration. The measure in (4.21) would seem to be not invariant. If we wrote it schematically as

$$\text{const}\times\prod_{\tau} d^4x\frac{dN}{N^3}\,|\{F,H\}|\,\delta(F)\ , \tag{5.1}$$

where the product is over time slices, the measure would appear to acquire several factors of $(\dot{f})^2$ on the execution of a transformation (2.5).

There is an analogous problem in quantum gravity. There the canonical measure for a coordinate grid whose constant t surfaces define the foliation needed in Hamiltonian quantum mechanics is[5,7,8]

$$\text{const}\times\prod_{x} N^{-2}(-g)^{3/2}\prod_{\alpha\beta} dg_{\alpha\beta}(x)\times(\text{gauge-fixing terms})\ . \tag{5.2}$$

Here, g is the determinant of the space-time metric and N is the lapse function $N=(g^{00})^{-2}$ which in the action for general relativity is the multiplier enforcing the Hamiltonian constraint (see Table I). Our choice of notation to stress the analogy between (5.1) and (5.2) is thus not artificial. Equation (5.2) appears to be not invariant under general coordinate transformations because it prefers a particular set of spacelike surfaces associated with the lapse N.

As stressed by Misner,[4] Leutwyler,[5] and, in greatest detail, by Fradkin and Vilkovisky,[8] however, appearances can be deceiving. The measures (5.1) and (5.2) do not lack invariance any more than the standard volume element on the sphere—$\sin\theta\, d\theta\, d\phi$—lacks invariance because it involves a factor $\sin\theta$ which changes under rotations. To understand this in the present simple situation, it is useful to think formally in terms of the space $\mathscr{P}(I)$ of histories $x^\alpha(\tau),N(\tau)$ on a fixed τ interval I. A single point P in $\mathscr{P}(I)$ thus represents a set of five functions. The action is a function on this space. The action has the same value at any two points corresponding to histories connected by a reparametrization of τ as in (2.5). One sees this by changing coordinates from τ to $f(\tau)$ in the action integral involving the transformed histories. It is then manifestly equal to its untransformed value.

To evaluate an integral over $\mathscr{P}(I)$ we can proceed in a fashion analogous to evaluating a Riemann integral of a function on a line. We first establish a "coordinate grid" on $\mathscr{P}(I)$. A particular time slicing specified by N and $\epsilon(K)$, $K=1,\dots,N$ gives such a grid, the "coordinates" being the values $x^\alpha(K),N(K)$ specifying skeletonized histories. One then defines a sum and takes an appropriate limit. As with the action, one demonstrates the invariance of the measure by changing τ to $f(\tau)$ and calculating how the measure changes. Under such a change not only do $x^\alpha(K)$ and $N(K)$ change as in (2.5) but also

$$\epsilon'(K)=\dot{f}(K)\epsilon(K)\ . \tag{5.3}$$

The issue of the invariance of the measure is the issue of its invariance under *both* (2.5) and (5.3). An inspection of (4.17) shows that the factors of ϵ enter in such a way as to make the measure reparametrization invariant. A similar in spirit, though much more involved analysis has been given by Fradkin and Vilkovisky[8] for general relativity. The existence of combinations, such as ϵN, which are invariant under ıeparametrizations means that, unlike in gauge theories, the Lagrangian form of the measure cannot be guessed from arguments of invariance and locality alone.

The situation with regard to the local form of the measure is considerably simpler in the extended phase space with multiplier. The candidate for a prescription is that of Faddeev for gauge theories. There one would write for the local measure on the extended phase space

$$\text{const}\times\delta N\,\delta^4 p\,\delta^4 x\,|\{F,H\}|\,\delta(F)\,, \tag{5.4}$$

where $F=0$ fixes the parametrization of time and $\{F,H\}$ is the Poisson brackets of F with the Hamiltonian constraint. Indeed the measure in (4.15) has this local form.

B. Gauge fixing

Even with a choice for the local measure considerable care must be taken with the choice of gauge-fixing conditions F in order to arrive at correct results. This point has been stressed by Teitelboim.[15] Consider, for example, the path integral for the field-theoretic (Feynman) propagator considered by him:

$$\Delta_F(x'',x')=\int \delta^4x\,\delta^4p\,\delta N[\theta(N)\,|\{F,H\}|\,\delta(F)] \times\exp(iS[x^\alpha,p_\alpha,N])\,, \tag{5.5}$$

where S is the action (2.10). The difference between (5.5) for the Feynman propagator and (4.15) is chiefly in the class of paths. In (5.5) the integration is over paths which move forward and backward in time while for the Newton-Wigner propagator it is over paths which move forward in time.

The gauge-fixing condition should assign a unique parametrization to every path entering the sum over histories. As emphasized by Teitelboim, not every condition which fixes a parametrization locally does the job globally. For example, linear gauge-fixing conditions of the form

$$F(x^\alpha,\tau)=\tau-Ax^0-B \tag{5.6}$$

with A,B constant will not work in (5.5). Condition (5.6) will not fix a parametrization for a path which moves both forward and backward in time.

On the other hand, a condition such as (5.6) can be used in the sum over histories giving the Newton-Wigner propagator because there the paths are restricted to move forward in time. If one calculates the propagator $\langle \mathbf{x}'',t''|\mathbf{x}',t'\rangle$ for fixed t'' and t' then the permissible values of A and B depend on whether one thinks of the limits τ' and τ'' as fixed *a priori* or as determined from t' and t'' by the condition $F=0$. In the latter case any values of A and B are permissible. In the former A and B must be chosen to reproduce t' and t'' at the given values of τ' and τ''. A and B will thus depend[17] on t' and t''.

The restriction that paths move forward in the time of a particular Lorentz frame leads to a very restricted class of functions F which yield a unique parametrization. They are of the form (4.8) with monotonically increasing f. If one calculated a path integral such as (4.17) with an f which was not monotonically increasing or dependent on x one would not be summing over paths which move forward in time. This choice of paths or equivalently the restricted form of the parametrization functions F is one important way in which the canonical Hilbert space is reflected in the path-integral measure.

C. Domain of integration

Even if the local form of the measure, including a correct gauge fixing, is known, there is still the domain of integration to be specified. The restricted ranges of integration in (4.15) and (4.17), and indeed in (5.5), are essential to obtaining correct results. If in any of these sums the ranges are extended to untrue values of p_0 and N the correct propagators are not obtained. Thus an additional element enters into the specification of the sum over histories.

D. Conclusion

Can the canonical quantum mechanics of parametrized theories such as the relativistic particle and general relativity be formulated directly in terms of path integrals? It may be that a set of principles can be developed for doing this. Such principles must specify both the allowed forms of the action and the allowed measures for the sum over histories. The example of the relativistic particle shows that if these principles are to reproduce the Hamiltonian quantum mechanics of simple examples, it will not be enough to specify the measure locally to ensure invariance. One will also have to know it in the large.

ACKNOWLEDGMENTS

One of the authors (J.H.) would like to thank the Department of Applied Mathematics and Theoretical Physics, University of Cambridge, for hospitality and the Science and Engineering Research Council, United Kingdom (SERC) for support while some of this work was carried out. We both would like to thank the Institute of Theoretical Physics, University of California Santa Barbara, for hospitality and support while another portion was carried out. Additional support was provided by National Science Foundation Grants Nos. PHY 85-06686 and PHY 85-03653.

[1] J. B. Hartle and K. V. Kuchař, J. Math Phys. **25**, 57 (1984); in *Quantum Theory of Gravity,* edited by S. Christensen (Hilger, Bristol, England, 1984).

[2] R. F. Baierlein, D. H. Sharp, and J. A. Wheeler, Phys. Rev. **126**, 1864 (1962).

[3] The reader not familiar with the standard notation for 3 + 1 relativity can consult, for example, C. Misner, K. Thorne, and J. A. Wheeler, *Gravitation* (Freeman, San Francisco, California, 1970), Chap. 21 for more details.

[4] C. Misner, Rev. Mod. Phys. **29**, 497 (1957).

[5]H. Leutwyler, Phys. Rev. **134**, B1155 (1964).
[6]L. Faddeev and V. Popov, Usp. Fiz. Nauk. **111**, 427 (1973) [Sov. Phys. Usp. **16**, 777 (1974)].
[7]E. Fradkin and G. Vilkovisky, Phys. Rev. D **8**, 4241 (1973).
[8]E. Fradkin and G. Vilkovisky, CERN Report No. TH-2332, 1977 (unpublished).
[9]M. Kaku, Phys. Rev. D **15**, 1019 (1977).
[10]M. Henneaux, Phys. Rep. **126**, 1 (1985).
[11]J. B. Hartle and K. Schleich (unpublished).
[12]A. Barvinsky and V. N. Ponomariov, Phys. Lett. **167B**, 289 (1986).
[13]See, e.g., T. Newton and E. Wigner, Rev. Mod. Phys. **21**, 400 (1949).
[14]See L. Schulman, *Techniques and Applications of Path Integration* (Wiley, New York, 1982), for a representative discussion and further references.
[15]C. Teitelboim, Phys. Rev. D **25**, 3159 (1982); Ann. Phys. (N.Y.) **143**, 127 (1983).
[16]L. Faddeev, Teor. Mat. Fiz. **1**, 3 (1969) [Theor. Math. Phys. **1**, 1 (1969)].
[17]In Ref. 15 the point of view seems to be taken that τ' and τ'' should be fixed *a priori and* the gauge condition F should be independent of t' and t''. We find no such restriction is needed for the problem considered here. Indeed, even for the Feynman propagator if one considers the action as a function of the end points τ' and τ'' and sums over these in the sum over parametrized paths, more general gauge conditions than those of Ref. 15 can be used and still lead to correct results.