Hadamard function, stress tensor, and de Sitter space

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Using the "Hadamard" formalism, we derive the general expression of the renormalized vacuum expectation value of the stress tensor of a massive scalar field. We apply this formalism to the O(1,4)- and E(3)-invariant vacua of de Sitter space.

I. INTRODUCTION

The vacuum expectation value of the stress tensor $\langle T_{ab} \rangle$ is the central object of the quantum field theory in curved spacetime because (i) the particle concept is very nebulous in curved spacetime and a quantity such as $\langle T_{ab} \rangle$ is more objective,[1] and (ii) the $\langle T_{ab} \rangle$ expectation value acts as the source in Einstein's field equations, and it governs the back reaction of the quantum field on the spacetime geometry. Therefore, in order to evaluate this effect, a compact expression of $\langle T_{ab} \rangle$, showing in detail the influence of the background geometry but also of the quantum state, would be very helpful.

Because the quantum stress tensor, which is a product of distributions, is ill defined, renormalization is required. Much work has been done since the mid-1970s on computing $\langle T_{ab} \rangle$ (for a review see Ref. 2). Among all the employed methods, the axiomatic approach introduced by Wald[3] and developed in connection with the Hadamard formalism by Adler *et al.*,[4] Wald,[3] and also Brown and Ottewill[5] seems to be the most powerful.

Contrary to the commutator function which is state independent in a globally hyperbolic spacetime,[6] the vacuum expectation value of the anticommutator,

$$G^{(1)} | (x,x') = \langle \ | \{\phi(x),\phi(x')\} | \ \rangle ,$$

depends upon the state $| \ \rangle$. The Hadamard formalism assumes that the singular part of $G^{(1)}(x,x')$ is given by the geometrical Hadamard ansatz.[7] Despite the constraints imposed by this hypothesis on the Fock-space structure, there is always a large class of Hadamard vacuum states.[8] Fulling, Sweeny, and Wald have shown that the Hadamard singularity is preserved by time evolution.[9] This ensures that the Hadamard vacua are unitarily equivalent[10] and that the renormalization of T_{ab} is well defined.

By extending the Brown and Ottewill approach,[5] we present the Hadamard definition of the renormalized stress tensor for the nonconformally invariant case and we discuss its ambiguities (Secs. II and III). In Sec. IV we illustrate the power of this method by computing several stress tensors in de Sitter space. The renormalized stress tensor for the full family of de Sitter-invariant vacua,[11] and also for the E(3)-invariant vacua,[12] are evaluated. In particular, the Bunch and Davies result[13] is readily obtained. In the following, we shall use the conventions of Hawking and Ellis.[14]

II. HADAMARD DEVELOPMENT OF $G^{(1)}(x,x')$

Let us consider a free massive scalar field $\phi(x)$ propagating in a curved spacetime, whose action functional is

$$S[\phi] = -\tfrac{1}{2} \int d^4x\, g^{1/2}(g_{ab}\phi^{;a}\phi^{;b} + \xi R\phi^2 + m^2\phi^2) . \tag{2.1}$$

This gives rise to the field equation

$$(\Box - m^2 - \xi R)\phi(x) = 0 . \tag{2.2}$$

The anticommutator function $G^{(1)}(x,x')$, which contains all the information about the Fock-space structure, also satisfies the wave equation (2.2). We shall suppose that $G^{(1)}(x,x')$ has a singular structure represented by the Hadamard development:[7] i.e., $G^{(1)}(x,x')$ can be written, at least for x', in a small normal neighborhood of x, as

$$G^{(1)}(x,x') = (2\pi)^{-2} \left[\frac{\Delta^{1/2}(x,x')}{\sigma(x,x')} + V(x,x')\ln\sigma(x,x') + W(x,x') \right] , \tag{2.3}$$

where $\sigma(x,x')$ is one-half the square of the geodesic distance between x and x' and where $\Delta(x,x')$ is the Van-Vleck determinant given by

$$\Delta(x,x') = -[g(x)]^{-1/2}\mathrm{Det}\left[-\frac{\partial^2\sigma(x,x')}{\partial x^a \partial x'^b} \right] [g(x')]^{-1/2} .$$

$V(x,x')$ and $W(x,x')$ are both smooth symmetric functions which can be expanded in powers of σ as

$$V(x,x') = \sum_{n=0}^{+\infty} V_n(x,x')\sigma^n , \tag{2.4}$$

$$W(x,x') = \sum_{n=0}^{+\infty} W_n(x,x')\sigma^n . \tag{2.5}$$

Applying the wave equation to $G^{(1)}(x,x')$ yields the Hadamard recursion relations[7] for the coefficients $V_n(x,x')$ and $W_n(x,x')$:

$$V_0+V_{0;a}\sigma^{;a}-V_0\Delta^{-1/2}\Delta^{1/2}{}_{;a}\sigma^{;a}+\tfrac{1}{2}(\Box-m^2-\xi R)\Delta^{1/2}=0\ , \tag{2.6}$$

$$(n+2)(n+1)V_{n+1}+(n+1)V_{n+1;a}\sigma^{;a}-(n+1)V_{n+1}\Delta^{-1/2}\Delta^{1/2}{}_{;a}\sigma^{;a}+\tfrac{1}{2}(\Box-m^2-\xi R)V_n=0\ , \tag{2.7}$$

$$(n+2)(n+1)W_{n+1}+(n+1)W_{n+1;a}\sigma^{;a}-(n+1)W_{n+1}\Delta^{-1/2}\Delta^{1/2}{}_{;a}\sigma^{;a}+\tfrac{1}{2}(\Box-m^2-\xi R)W_n$$
$$+(2n+3)V_{n+1}+V_{n+1;a}\sigma^{;a}-V_{n+1}\Delta^{-1/2}\Delta^{1/2}{}_{;a}\sigma^{;a}=0\ . \tag{2.8}$$

The differential equations for $V_n(x,x')$ are a set of transport equations which can be integrated along the geodesic from x to x'. These equations determine $V_n(x,x')$ uniquely. By giving $W_0(x,x')$ all of the remaining $W_n(x,x')$ $(n>1)$ are uniquely determined by the recursion relations above. $W_0(x,x')$ is not completely arbitrary: It must give rise to a symmetric function $W(x,x')$ satisfying (2.8). $\Delta(x,x')$ and the coefficients $V_n(x,x')$ are determined in terms of the local geometry, and therefore the singular part of $G^{(1)}(x,x')$ is also determined by the local geometry. The vacuum state dependence is contained in the function $W(x,x')$ through the choice of $W_0(x,x')$.

Let us now recall some property of the covariant Taylor-series expansion. If $F(x,x')$ is a symmetric biscalar possessing a covariant Taylor-series expansion in a neighborhood of the point x, we can write[15]

$$F(x,x')=f(x)-\tfrac{1}{2}f(x)_{;a}\sigma^{;a}+\tfrac{1}{2}f_{ab}(x)\sigma^{;a}\sigma^{;b}$$
$$-\tfrac{1}{4}[f_{ab}(x)_{;c}-\tfrac{1}{6}f(x)_{;abc}]\sigma^{;a}\sigma^{;b}\sigma^{;c}+O(\sigma^2)\ , \tag{2.9}$$

where

$$f(x)=\lim_{x'\to x}[F(x,x')]\ ,$$

$$f_{ab}(x)=\lim_{x'\to x}[F(x,x')_{;ab}]\ .$$

We shall write the covariant Taylor-series expansions for the symmetric biscalars $V_0(x,x'), V_1(x,x'), V(x,x'')$ and $W_0(x,x'), W_1(x,x'), W(x,x')$ and introduce their Taylor coefficients $v_0(x), v_{0ab}(x), \dots$, with the notations of (2.9). From $2\sigma=g_{ab}\sigma^{;a}\sigma^{;b}$, we get the identities

$$v(x)=v_0(x)\ ,$$

$$v_{ab}(x)=v_0(x)_{ab}+v_1(x)g_{ab}\ ,$$

and similar relations between $w(x)$, $w_0(x)$, $w_{0ab}(x)$, and $w_{ab}(x)$.

Use of the Taylor series for the Van-Vleck determinant (see the Appendix) in the recursion equations (2.6) and (2.7) yields the Taylor expansions of the $V_n(x,x')$ coefficients. The boundary relation (2.6), which provides the "starting term" V_0 for the recursion relations, gives

$$v_0(x)=\tfrac{1}{2}[m^2+(\xi-\tfrac{1}{6})R]\ , \tag{2.10}$$

$$v_{0ab}(x)=\tfrac{1}{3}v_{o;ab}+\tfrac{1}{6}v_0R_{ab}-C_{ab}\ , \tag{2.11}$$

where C_{ab} is defined in the Appendix. The relation (2.7) with $n=0$ gives

$$v_1(x)=-\tfrac{1}{12}\Box v_0+\tfrac{1}{2}v_0{}^2+\tfrac{1}{4}C_a{}^a\ . \tag{2.12}$$

The expressions for $v_{0ab}(x)$ and $v_1(x)$ could be expressed in terms of the Riemann tensor and its derivatives but we prefer to keep them in these more compact forms. Later we shall also need the expression for v_{ab}:

$$v_{ab}(x)=-(C_{ab}-\tfrac{1}{4}g_{ab}C_d{}^d)+\tfrac{1}{3}(v_{0;ab}-\tfrac{1}{4}g_{ab}\Box v_0)$$
$$+\tfrac{1}{6}v_0R_{ab}+\tfrac{1}{2}v_0{}^2g_{ab}\ . \tag{2.13}$$

It should be noted that in the conformally invariant case $(m^2=0,\xi=\tfrac{1}{6})$ in v_0 function and the trace of the v_{ab} tensor vanish.

Similarly, using (2.8) with $n=0$ one can obtain the expressions of the w_{ab} tensor in terms of the Taylor coefficients w_0 and w_{0ab} of the biscalar $W_0(x,x')$:

$$w_{ab}(x)=(w_{0ab}-\tfrac{1}{4}g_{ab}w_{0d}{}^d)$$
$$+\left[\frac{R}{24}w_0+\tfrac{1}{2}v_0w_0-\tfrac{3}{2}v_1\right]g_{ab}\ . \tag{2.14}$$

It should be noted that w_{0ab} does not contribute to the trace of w_{ab}.

Now we have the necessary quantities for the computation of the stress tensor at our disposal. However, in order to demonstrate the conservation of the stress tensor in the next section, we first need to derive an important relation about the divergence of w_{ab}. This relation follows from the symmetry property of $W(x,x')$ which allows us to express the third Taylor coefficient in terms of w and w_{ab}, (2.9). $W(x,x')$, unlike $V(x,x')$, is not a solution of the wave equation but it has to satisfy

$$(\Box-m^2-\xi R)W(x,x')=-6v_1(x)+2v_{1;a}\sigma^{;a}+O(\sigma)\ .$$

Inserting the Taylor expansion of $W(x,x')$ up to the third order in $\sigma^{;a}$ into the previous equation yields

$$[w_{ab}-\tfrac{1}{2}g_{ab}(w^d{}_d-m^2w+\tfrac{1}{2}\Box w)]^{;b}$$
$$=2v_{1;a}+\tfrac{1}{2}R_{ab}w^{;b}-\tfrac{1}{2}\xi Rw_{;a}\ . \tag{2.15}$$

Equation (2.15) can be expressed in terms of $w_0(x)$ and $w_{0ab}(x)$ and then appears as a constraint on $W_0(x,x')$.[15] Additional constraints could be obtained from the higher order of the Taylor expansions. They follow by demanding that $W_0(x,x')$ be chosen in such a way that the resulting two-point function is symmetric.

III. "HADAMARD" DEFINITION OF THE STRESS TENSOR

Armed with the properties of the Hadamard development of $G^{(1)}(x,x')$, we can give a definition of the renormalized expectation values $\langle\phi^2\rangle_{\rm ren}$ and $\langle T_{ab}\rangle_{\rm ren}$. The principle behind the computation of the renormalized stress tensor from the Hadamard function was already described by Brown and Ottewill in Refs. 5 and 15 in the conformally invariant case. We shall present this method in the massive case and describe its ambiguities.

We first define the renormalized expectation value $\langle\phi^2\rangle_{\rm ren}$. For the Hadamard states, the divergence of ϕ^2 is always of the same type. It is therefore quite natural to define $\langle\phi^2\rangle_{\rm ren}$ simply by substracting the geometrical divergences in the $G^{(1)}(x,x')$ function. Thus,

$$8\pi^2\langle\phi^2(x)\rangle_{\rm ren}=w_0(x)-v_0(x)\ln\mu^2 \;. \qquad (3.1)$$

The second term above arises because there must exist in $W(x,x')$ a term of the form $[V(x,x')\ln\mu^2]$ in order to make dimensionless the argument of the logarithm term $V(x,x')\ln\sigma$ in $G^{(1)}(x,x')$. The "renormalization" mass μ may be partially fixed by demanding that in the flat-space limit, $R_{abcd}\to 0$, we find the Minkowski result $\langle\phi^2\rangle_{\rm ren}=0$.

The definition of the renormalized stress tensor T_{ab} is in the same spirit. It is formally defined by

$$\begin{aligned}T_{ab}&=2g^{-1/2}\frac{\delta}{\delta gab}S[\phi]\\ &=(1-2\xi)\phi_{;a}\phi_{;b}+(2\xi-\tfrac{1}{2})g_{ab}\phi_{;d}\phi^{;d}-2\phi\phi_{;ab}\\ &\quad+2\xi g_{ab}\phi\Box\phi+\xi\left[R_{ab}-\frac{R}{2}g_{ab}\right]\phi^2-\frac{m^2}{2}g_{ab}\phi^2 \;.\end{aligned}$$

The standard point-splitting renormalization method[3] defines $\langle T_{ab}\rangle_{\rm ren}$ as the limit

$$\langle T_{ab}(x)\rangle_{\rm ren}=\lim_{x'\to x}\tfrac{1}{2}\mathscr{D}_{ab'}(x,x')\times[G^{(1)}(x,x')-G^{(1)}_{\rm ref}(x,x')] \;, \qquad (3.2)$$

where the differential operator $\mathscr{D}_{ab'}(x,x')$ is given by

$$\begin{aligned}\mathscr{D}_{ab'}&=(1-2\xi)\nabla_a\nabla_{b'}+(2\xi-\tfrac{1}{2})g_{ab'}g_{dd'}\nabla^d\nabla^{d'}\\ &\quad-2\xi\nabla_{a'}\nabla_{b'}+2\xi g_{ab'}\nabla_{d'}\nabla^{d'}\\ &\quad+\xi(R_{ab}-\tfrac{1}{2}g_{ab}R)-\frac{m^2}{2}g_{ab}\end{aligned} \qquad (3.3)$$

and where $g_a{}^{b'}$ is the geodetic parallel displacement bivector.[16] The $G^{(1)}_{\rm ref}$ function is a reference two-point function introduced in order to remove the singularities from $G^{(1)}$ and its derivatives. In particular one would expect that the difference between two renormalized stress tensors is given by

$$8\pi^2\Delta\langle T_{ab}(x)\rangle_{\rm ren}=\lim_{x'\to x}\mathscr{D}_{ab'}(x,x')[\Delta W(x,x')] \;.$$

Therefore, the renormalized stress tensor must be an affine functional of $W(x,x')$. That is to say, the definition (2.2) of $\langle T_{ab}\rangle$ reduces to

$$8\pi^2\langle T_{ab}(x)\rangle_{\rm ren}=\lim_{x'\to x}\mathscr{D}_{ab'}(x,x')[W(x,x')]+\hat{O}_{ab}(x) \;, \qquad (3.4)$$

where $\hat{O}_{ab}$ is state independent. It can only depend on the parameters of the theory, m^2 and ξ, and on the local geometry. The condition that $\langle T_{ab}\rangle_{\rm ren}$ be divergenceless determines the $\hat{O}_{ab}$ tensor up to a conserved tensor.

Using the expansion (2.9) for $W(x,x')$ one can show that

$$\begin{aligned}\mathscr{T}_{ab}[W]&\equiv\lim_{x'\to x}\mathscr{D}_{ab'}(x,x')[W(x,x')]\\ &=-(w_{ab}-\tfrac{1}{2}g_{ab}w_d{}^d)+\tfrac{1}{2}(1-2\xi)w_{0;ab}\\ &\quad+\tfrac{1}{2}(2\xi-\tfrac{1}{2})\Box w_0g_{ab}\\ &\quad+\xi(R_{ab}-\tfrac{1}{2}g_{ab}R)w_0-\frac{m^2}{2}w_0g_{ab} \;.\end{aligned} \qquad (3.5)$$

Using Eq. (2.15) one obtains the divergence of $\mathscr{T}_{ab}[W]$:

$$\mathscr{T}_{ab}[w]^{;b}=-2v_{1;a}=(-2v_1g_{ab})^{;b} \;. \qquad (3.6)$$

The state independence of this divergence comes from the properties of the Hadamard development. (Here divergence does not refer to any potential infinities, but only to a traced covariant derivative. This divergence can be removed by incorporating the term $2v_1g_{ab}$ into the $\hat{O}_{ab}$ tensor. The $\hat{O}_{ab}$ tensor is written as $\hat{O}_{ab}=2v_1g_{ab}+O_{ab}$ where O_{ab} is a divergenceless geometrical tensor. The $\langle T_{ab}\rangle_{\rm ren}$ so constructed is divergenceless.

The trace $\langle T_a{}^a\rangle_{\rm ren}$ of the renormalized stress tensor is given by [using (2.14)]

$$8\pi^2\langle T_a{}^a(x)\rangle_{\rm ren}=2v_1+3(\xi-\tfrac{1}{6})\Box w_0-m^2w_0+O_a{}^a \;, \qquad (3.7)$$

which reduces, in the conformally invariant case, simply to $2v_1$ up to the trace of the auxiliary tensor O_{ab}.

The auxiliary O_{ab} tensor is composed of two terms. The first one comes from the presence of the renormalization mass term, i.e., from the indeterminacy in the $W(x,x')$ function which is of the form $[-V(x,x')\ln\mu^2]$. This indeterminacy gives rise to a term $(-\mathscr{T}_{ab}[v]\ln\mu^2)$ in the expression of O_{ab}, where

$$\begin{aligned}\mathscr{T}_{ab}[v]&=(C_{ab}-\tfrac{1}{4}g_{ab}C_d{}^d)-\tfrac{1}{4}(\xi-\tfrac{1}{6})^{2\,(1)}H_{ab}\\ &\quad+\frac{m^2}{2}(\xi-\tfrac{1}{6})\left[R_{ab}-\frac{R}{2}g_{ab}\right]-\frac{m^4}{8}g_{ab} \;,\end{aligned} \qquad (3.8)$$

where ${}^{(1)}H_{ab}$ is defined in (3.9). It should be noted that $\mathscr{T}_{ab}[V]$ is a conserved tensor, and its trace vanishes in the conformally invariant case. The second term directly comes from the definition of O_{ab} as a geometrical conserved tensor. Following Wald's arguments,[3(b)] one can show that the only geometrical conserved tensors are those obtained from a Lagrangian of dimension $(\text{length})^{-4}$. In four dimensions, there are only four independent geometrical Lagrangians of dimension $(\text{length})^{-4}$ which remain finite in the massless limit. These are $\mathscr{L}=m^4,m^2R,R^2$, and R_2R^{ab}, and they define the conserved tensors $m^4g_{ab},m^2(R_{ab}-\tfrac{1}{2}g_{ab}R)$ and the following well-known ones:[2]

$$^{(1)}H_{ab}=g^{-1/2}\frac{\delta}{\delta g^{ab}}\int d^4x\, g^{1/2}R^2$$
$$=2R_{;ab}-2g_{ab}\Box R-2R(R_{ab}-\tfrac{1}{4}Rg_{ab})\ , \qquad (3.9)$$

$$^{(2)}H_{ab}=g^{-1/2}\frac{\delta}{\delta g^{ab}}\int d^4x\, g^{1/2}R_{ab}R^{ab}$$
$$=R_{;ab}-\Box R_{ab}+\tfrac{1}{2}g_{ab}(R_{cd}R^{cd}-\Box R)$$
$$-2R^{cd}R_{cadb}\ . \qquad (3.10)$$

Thus, the $\langle T_{ab}\rangle_{\rm ren}$ can be written as

$$8\pi^2\langle T_{ab}(x)\rangle_{\rm ren}=\mathcal{T}_{ab}[w]-\mathcal{T}_{ab}[v]\ln\mu^2+2v_1g_{ab}$$
$$+\alpha m^4g_{ab}+\beta m^2(R_{ab}-\tfrac{1}{2}Rg_{ab})$$
$$+\gamma\,^{(1)}H_{ab}+\delta\,^{(2)}H_{ab}\ . \qquad (3.11)$$

We shall now study the flat-space limits of the renormalized quantities $\langle\phi^2\rangle_{\rm ren}$ and $\langle T_{ab}\rangle_{\rm ren}$. This allows us to specify the value of certain coefficients. We assume that the vacuum state tends to the Minkowskian vacuum and thus that the limit of w_0 is a constant.

In the flat-space limit, we have the relation

$$8\pi^2\langle T_a{}^a(x)\rangle_{\rm ren}\simeq\frac{m^4}{4}(1+16\alpha)-m^2\langle\phi^2\rangle\ . \qquad (3.12)$$

Requiring that both $\langle\phi^2\rangle_{\rm ren}$ and $\langle T_{ab}\rangle_{\rm ren}$ vanish in the flat-space limit gives the value of μ^2 and the value of α, which is $\alpha=-\frac{1}{16}$. In the massless case this procedure does not determine the value of μ^2.

As the three tensors $(R_{ab}-\frac{1}{2}g_{ab}R)$, $^{(1)}H_{ab}$, and $^{(2)}H_{ab}$ occur in the infinite renormalization of $\langle T_{ab}\rangle$, we may choose to cancel their coefficients in (3.11) by finite renormalization. This argument is supported by the analysis of the conformally invariant case. In that case, only $^{(1)}H_{ab}$ and $^{(2)}H_{ab}$ occur in O_{ab}. As their traces are both proportional to $\Box R$, the $\Box R$ coefficient in the trace anomaly is arbitrary. This agrees with the cohomological study of the conformal anomaly which shows that $c\Box R$ is a Becchi-Rouet-Stora (BRS) exact form[17] and hence $\Box R$ is not a true anomaly. The traceless combination, $^{(1)}H_{ab}-3\,^{(2)}H_{ab}$, is proportional to $\mathcal{T}_{ab}[v]$ and its coefficient can be absorbed in $\ln\mu^2$. The crucial assumption is that we might identify the two renormalization masses appearing in (3.1) and (3.11).

The final expression for the renormalized stress tensor is

$$8\pi^2\langle T_{ab}(x)\rangle_{\rm ren}=\mathcal{T}_{ab}[w]-\mathcal{T}_{ab}[v]\ln\mu^2$$
$$+2v_1g_{ab}-\frac{m^4}{16}g_{ab}\ , \qquad (3.13)$$

where $v_1(x)$ and $\mathcal{T}_{ab}[\text{ab}]$ are given, respectively, in (2.12) and (3.8). Thus, the evaluation of $\langle T_{ab}\rangle_{\rm ren}$ only involves the computation of $\mathcal{T}_{ab}[\text{w}]$.

IV. STRESS TENSORS IN DE SITTER SPACE

In this section we shall apply the previous result to de Sitter space and calculate the expectation values of the stress tensor in de Sitter-invariant states and in E(3)-invariant states.[12] Let us first recall a few results about quantum field theory in de Sitter space.[2,11,12] In de Sitter space, there is generally a one-real-parameter family of vacua invariant under the full disconnected de Sitter group O(1,4). The anticommutator functions in these states can be written as

$$G_\alpha^{(1)}(x,x')=\cosh 2\alpha\, G_0^{(1)}(z(x,x'))$$
$$+\sinh 2\alpha\, G_0^{(1)}(-z(x,x'))\ , \qquad (4.1)$$

where α is a real parameter, $G_0^{(1)}(z)$ the anticommutator function in the "Euclidean" vacuum or Bunch-Davies vacuum, and where $z(x,x')$ is a de Sitter-invariant quantity which can be expressed in terms of $\sigma(x,x')$, for x and x' spacelike separated:

$$z(x,x')=\cos\left[\frac{R\sigma}{6}(x,x')\right]^{1/2}\ , \qquad (4.2)$$

where R is the constant scalar curvature of de Sitter space: The Euclidean anticommutator function is

$$G_0^{(1)}(z)=\frac{R}{96\pi}\frac{(\frac{1}{4}-\nu^2)}{\cos(\pi\nu)}F\left[\tfrac{3}{2}+\nu,\tfrac{3}{2}-\nu;2;\frac{1+z}{2}\right]\ , \qquad (4.3)$$

where F is the hypergeometric function and where $\nu^2=\frac{9}{4}-12(\xi+m^2/R)$. In the massless minimally coupled case, $m^2=0$ and $\xi=0$, $G_0^{(1)}(z)$, and therefore $G_\alpha^{(1)}(z)$, has an infrared divergence. Allen[12] has proven that in this case there is no de Sitter-invariant vacuum state although a de Sitter-invariant $G^{(1)}$ function exists. However, he has found a one-real-parameter family of vacuum states invariant under E(3), a maximal subgroup of O(1,4) which leaves invariant the spatially flat hypersurfaces $t=$const of de Sitter space. Here t is the time coordinate in a coordinate system that only covers half of the de Sitter manifold, such that the metric on de Sitter space is

$$ds^2=\frac{12}{Rt^2}[-dt^2+(d\mathbf{x})^2]\ . \qquad (4.4)$$

The symmetric two-point functions in the E(3)-invariant vacua can be written[12] as

$$G_\alpha^{(1)}(t,\mathbf{x};t',\mathbf{x}')=\frac{R}{48\pi^2}\left[\cosh 2\alpha\left[\frac{1}{1-z}-\ln(1-z)-\ln 2tt'\right]+\sinh 2\alpha\left[\frac{1}{1+z}-\ln(-1-z)-\ln 2tt'\right]\right]\ . \qquad (4.5)$$

The expectation value of the stress tensor in the Euclidean vacuum can easily be calculated. The anticommutator function $G_0^{(1)}(z)$ given in (3.3) has the Hadamard development and therefore the formalism displayed in the previous sections can be applied. $G_0^{(1)}(z)$ is invariant under the de Sitter group; hence we have $\langle T_{ab}\rangle_{\rm ren}=\frac{1}{4}g_{ab}\langle T_c{}^c\rangle_{\rm ren}$. But in de Sitter space we can write, using (3.8) and (3.13),

$$8\pi^2\langle T_a{}^a\rangle_{\rm ren}=2v_1-\frac{m^4}{4}+3(\xi-\tfrac{1}{6})\Box w-m^2\{w+\tfrac{1}{2}[m^2+(\xi-\tfrac{1}{6})R]\}\ln\mu^2\ , \tag{4.6}$$

where

$$2v_1=\frac{1}{2}\left[\frac{m^4}{2}+m^2(\xi-\tfrac{1}{6})R+\tfrac{1}{2}(\xi-\tfrac{1}{6})^2R^2-\frac{R^2}{2160}\right]. \tag{4.7}$$

Knowing the expression of the Van-Vleck determinant in de Sitter space,[7(b)]

$$\Delta(\sigma)=\left[\frac{R\sigma}{6}\right]^{3/2}\left[\sin\left[\frac{R\sigma}{6}\right]^{1/2}\right]^{-3},$$

and using (4.3) and the properties of hypergeometric functions[18] it is straightforward to obtain

$$w_0=-\frac{R}{36}+\frac{m^2+(\xi-\frac{1}{6})R}{2}\left[\psi(\tfrac{3}{2}+\nu)+\psi(\tfrac{3}{2}-\nu)+\ln\frac{R}{24}-\psi(1)-\psi(2)\right]. \tag{4.8}$$

Thus, absorbing $\psi(1)$, $\psi(2)$, and 24 into the renormalization mass μ^2, we obtain

$$\langle\phi^2(x)\rangle_{\rm ren}=\frac{1}{16\pi^2}\left[\frac{-R}{18}+[m^2+(\xi-\tfrac{1}{6})R]\left[\psi(\tfrac{3}{2}+\nu)+\psi(\tfrac{3}{2}-\nu)+\ln\frac{R}{\mu^2}\right]\right], \tag{4.9}$$

$$\langle T_{ab}(x)\rangle_{\rm ren}=-\frac{g_{ab}}{64\pi^2}\Big\{m^2[m^2+(\xi-\tfrac{1}{6})R][\psi(\tfrac{3}{2}+\nu)+\psi(\tfrac{3}{2}-\nu)+\ln(R/\mu^2)]$$
$$-(\xi-\tfrac{1}{6})m^2R-\tfrac{1}{18}m^2R-\tfrac{1}{6}(\xi-\tfrac{1}{6})^2R^2+\frac{R^2}{2160}\Big\}. \tag{4.10}$$

The arbitrary value of μ^2 can be removed by requiring that $\langle\phi^2\rangle_{\rm ren}$ and $\langle T_{ab}\rangle_{\rm ren}$ vanish in the flat-space limit. Use of the limit

$$\psi(\tfrac{3}{2}+\nu)+\psi(\tfrac{3}{2}-\nu)+\ln(R/\mu^2)=\ln(12m^2/\mu^2)+O(R)$$

gives the value $\mu^2=12m^2$. The expression (4.10) becomes exactly the Bunch-Davies one.[13]

For $\alpha\neq0$, $G_\alpha^{(1)}(x,x')$, given in (4.1), does not possess the Hadamard development. But, the two-point symmetric function $G_0^{(1)}(z)+\tanh2\alpha G_0^{(1)}(-z)\equiv F(z)$ does. It is therefore possible to apply the previous method to this function. But, before computing the value of the stress tensor, it should be noted that the singular part of $\langle T_{ab}\rangle$ is proportional to $\cosh2\alpha$. In other words, the reference two-point function which occurs in the definition (3.2) must be α dependent. That is to say, the renormalization becomes state dependent. After all, one can use the fact that $F(z)$ has the Hadamard development to formally define $\langle T_{ab}\rangle$ in the $|\alpha\rangle$ vacuum as $\cosh2\alpha$ times the renormalized stress tensor associated to $F(z)$ [defined by Eq. (3.13)]. Thus, taking into account this prefactor, $\cosh2\alpha$, we obtain the expectation value $\langle T_{ab}\rangle(\alpha)$ using (3.13) with $\mu^2=12m^2$:

$$\langle T_{ab}\rangle_{\rm ren}(\alpha)=-g_{ab}\frac{\cosh2\alpha}{64\pi^2}\Big\{m^2[m^2+(\xi-\tfrac{1}{6})R]\left[\psi(\tfrac{3}{2}+\nu)+\psi(\tfrac{3}{2}-\nu)+\ln(R/12m^2)+\frac{\pi}{\cos\pi\nu}\tanh2\alpha\right]$$
$$-m^2(\xi-\tfrac{1}{6})R-\frac{m^2R}{18}-\tfrac{1}{2}(\xi-\tfrac{1}{6})^2R^2+\frac{R^2}{2160}\Big\}. \tag{4.11}$$

Now let us consider the $\alpha=0$, E(3)-invariant vacuum. The $G^{(1)}$ function

$$G^{(1)}(t,\mathbf{x};t',\mathbf{x}')=\frac{R}{48\pi^2}\left[\frac{1}{1-z}-\ln(1-z)-\ln(2tt')\right]$$

has the Hadamard development. We can show, using (3.3) with $m^2=0$ and $\xi=0$ that the $\ln(2tt')$ term does not contribute towards the vacuum expectation value of the stress tensor. Therefore $\langle T_{ab}\rangle_{\rm ren}$ is invariant under the de Sitter group, we can write $\langle T_{ab}\rangle_{\rm ren}=\frac{1}{4}\langle T_c{}^c\rangle g_{ab}$ and compute it by using (4.6) with $m^2=0$ and $\xi=0$. (Note that the arbitrariness in $\ln\mu^2$ vanishes.) Here we have $w=-(R/6)\ln(t)+\text{const}$ which satisfies $\Box w=-R^2/24$ and therefore

$$\langle T_{ab}\rangle_{\rm ren}(\mathrm{E}(3);\alpha=0)=\frac{119}{138\,240\pi^2}R^2g_{ab}\ ; \tag{4.12}$$

it should be noted that it is not the finite limit ($m^2\to0$, $\xi\to0$) of the value of the Bunch-Davies stress tensor (4.10). This is not too surprising because the Bunch-Davies two-point function (4.3) is ill defined in the massless minimally coupled case.

We can also calculate the vacuum expectation value of the stress tensor for $\alpha\neq 0$, E(3)-invariant vacua. One easily finds

$$\langle T_{ab}\rangle_{\rm ren}(E(3);\alpha\neq 0)=\frac{119\cosh 2\alpha+90\sinh 2\alpha}{138\,240\pi^2}R^2 g_{ab}\ . \tag{4.13}$$

V. CONCLUSION

In conclusion, following the method developed in the conformally invariant case by Wald, Brown, and Ottewill, we have presented the Hadamard definition of the stress tensor for a massive scalar field. The properties of the Hadamard development allowed us to systematically take into account the infinite counterterm necessary in the definition of $\langle T_{ab}\rangle_{\rm ren}$. By looking at the flat-space limit of both $\langle\phi^2\rangle_{\rm ren}$ and $\langle T_{ab}\rangle_{\rm ren}$, we have shown that some of the ambiguities in the definition of $\langle T_{ab}\rangle_{\rm ren}$ can be removed. Furthermore, the state dependence of $\langle T_{ab}\rangle_{\rm ren}$ is explicitly detailed using this definition. This property could be helpful, for instance, to study the influence of the vacuum state in the back-reaction problem. We have illustrated the power of this method by evaluating the renormalized stress tensors in de Sitter space for various vacuum states. It would now be interesting to apply this approach to black holes with the possibility to use an asymptotic development near the horizon. The results thus obtained could be compared to those found by Candelas and Howard[19] and Frolov and Sanchez,[20] using Page's approximation.[21]

Note added. After this work was completed, we became aware of a related paper by Brown and Ottewill.[23]

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APPENDIX

In this appendix we give a few relations and notations used in the derivation of the equations in the text:[16,22]

$$\sigma_{;ab}=g_{ab}-\tfrac{1}{3}R_{acbd}\sigma^{;c}\sigma^{;d}+O(\sigma^{3/2})\ , \tag{A1}$$

$$\Delta^{1/2}=1+\tfrac{1}{12}R_{ab}\sigma^{;a}\sigma^{;b}+O(\sigma^{3/2})\ , \tag{A2}$$

$$\Delta^{1/2}{}_{;a}=\tfrac{1}{6}R_{ac}\sigma^{;c}+O(\sigma)\ , \tag{A3}$$

$$\Box\Delta^{1/2}=\frac{R}{6}+\left[3C_{ab}+\frac{RR_{ab}}{72}\right]\sigma^{;a}\sigma^{;b}+O(\sigma^{3/2})\ , \tag{A4}$$

where

$$C_{ab}-\tfrac{1}{4}g_{ab}C_d{}^d=-\tfrac{1}{120}(C_{c(ab)d}R^{cd}+2C_{c(ab)d}{}^{;cd}) \tag{A5}$$

and

$$C_d{}^d=\tfrac{1}{180}(\Box R-R_{ad}R^{ad}+R_{abcd}R^{abcd})\ . \tag{A6}$$

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