Consistency of canonical quantization of gravity and boundary conditions for the wave function of the Universe

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By using the expression given by Halliwell and Hawking, we discuss consistency of the constraint equations of the superspace-quantized gravity coupled to the minimal massive scalar field, in the lowest nontrivial order in the cosmological perturbation. We show that the consistency imposes highly nontrivial conditions on the behavior of the physical state (the wave function of the Universe) at a (the cosmic scale factor) $=0$.

I. INTRODUCTION

Quantum gravity in the path-integral form is well known but its rigorous canonical form at the operator level has not been established yet.[1] A main difficulty lies in the fact that one class of constraints which come from general covariance is quadratic in canonical momenta and that consistency of the constraints is intricately related to the operator-ordering problem and to possible existence of counterterms.

Recently Halliwell and Hawking presented explicit expressions of the constraints to the lowest nontrivial order of the perturbations in the inhomogeneous parts of the metric and a matter field (the minimal massive scalar), while treating the homogeneous and isotropic parts exactly.[2] In this order, at least, we do not need to introduce gauge-fixing and ghost fields. We can apply the superspace quantization formalism of Wheeler and DeWitt in its original form.[3] However, various problems due to the quadraticity of the constraints in canonical momenta already appear in this order and the consistency requirement imposes highly nontrivial constraints on the theory. The purpose of this paper is to discuss how the boundary conditions for the physical state and the operator ordering of the constraint equations are restricted from the consistency of the theory.

In the next section we recapitulate the result of Ref. 2 and present the momentum constraints and the energy constraints (the Wheeler-DeWitt equations) of the superspace-quantized canonical gravity. In Sec. III we explicitly solve the momentum constraints and define a quotient space, i.e., original superspace/spatial diffeomorphism. The Wheeler-DeWitt equations can be rewritten in this quotient space. In Sec. IV we show that the Wheeler-DeWitt equations, which consist of an infinite number of second-order differential equations, can be imposed on the physical state Ψ consistently only when the latter satisfies certain boundary conditions. There are two classes of the solutions. One is the "singularity-free" solution in which Ψ vanishes when the cosmic scale factor does. The other includes the one proposed by Hartle and Hawking.[4]

II. COSMOLOGICAL PERTURBATIONS AND CONSTRAINTS

First we decompose the metric and the minimal massive scalar field Φ into the homogeneous isotropic parts and the inhomogeneous and anisotropic perturbations. We follow the notation used in Ref. 2. The three-metric is written as

$$h_{ij}=e^{2\alpha(t)}(\Omega_{ij}+\epsilon_{ij}) ,$$

where Ω_{ij} is the metric on the unit S^3, $e^{2\alpha}\Omega_{ij}$ is the homogeneous isotropic part which we treat exactly, ϵ_{ij} is a perturbation which we expand in harmonics

$$\epsilon_{ij}=\sum_n(\sqrt{6}a_n \tfrac{1}{3}\Omega_{ij}Q^n+\sqrt{6}b_n P_{ij}^n+\sqrt{2}c_n S_{ij}^n+2d_n G_{ij}^n) .$$

Q^n are the scalar harmonics, P_{ij}^n, S_{ij}^n, and G_{ij}^n are the tensor harmonics of the scalar type, the vector type, and the tensor type, respectively. The harmonics are specified by three indices but we suppress two of them for simplicity. The coefficients $a_n - d_n$ depend on the time t. The scalar field is expanded as

$$\Phi \propto \frac{1}{\sqrt{2}\pi}\phi(t)+\sum_n f_n(t)Q^n .$$

ϕ is a homogeneous part and f_n describes perturbations. The canonical variables are α, ϕ, $a_n - d_n$, and f_n. Their conjugate momenta are denoted as π_α, π_ϕ, etc.

The canonical theory of classical gravity is described by four sets of constraints which come from the four degrees of general coordinate transformation. We symbolically write them as

$$H_\perp(x_j)=0 , \tag{2.1a}$$

$$H^i(x_j)=0 \quad (i=1-3) , \tag{2.1b}$$

each of which is defined at every spatial point x_j. H^i generates the general coordinate transformation in the spatial directions and is linear in canonical momenta. But $H_\perp$, which generates the one in the time direction, is quadratic in them. The Hamiltonian vanishes under Eqs. (2.1) and therefore the constraints are everything in the canonical theory (besides classical equations of motion of matter fields if any). Throughout this paper we assume the closedness of the spatial section and ignore possible surface effects.

In the superspace formulation of quantum gravity, we interpret Eqs. (2.1) as the constraints on the physical state Ψ ($H_\perp\Psi=0$, etc.). It is convenient to decompose the constraints by the harmonics. In the order given in Ref. 2 we can write them as

$$\left[H_{\perp 0}+\sum_n ({}^S H^n_{\perp 2}+{}^V H^n_{\perp 2}+{}^T H^n_{\perp 2})\right]\Psi=0 \,, \tag{2.2a}$$

$$H^n_{\perp 1}\Psi=0 \,, \tag{2.2b}$$

$${}^S H^n_{-1}\Psi=0 \,, \tag{2.2c}$$

$${}^V H^n_{-1}\Psi=0 \,. \tag{2.2d}$$

The suffixes 0–2 denote the order of the quantities in the perturbations. Equation (2.2a) is the homogeneous part of $H_\perp\Psi=0$. $H_{\perp 0}$ is its leading part

$$H_{\perp 0}=-\pi_\alpha{}^2+\pi_\phi{}^2-e^{4\alpha}+e^{6\alpha}m^2\phi^2 \,. \tag{2.3}$$

(Throughout this paper we drop a factor $e^{-3\alpha}$ in front of $H_\perp$ and H^i.) ${}^S H^n_{\perp 2}$, ${}^V H^n_{\perp 2}$, and ${}^T H^n_{\perp 2}$ are terms of second order in the scalar perturbations (a_n, b_n, f_n, π_{a_n}, π_{b_n}, and π_{f_n}), the vector perturbations (c_n,π_{c_n}), and the tensor perturbations (d_n,π_{d_n}), respectively. Equation (2.2b) is the inhomogeneous part of $H_\perp$ and, in lowest order, depends only on the scalar perturbations besides α and ϕ. ${}^S H^n_{-1}$ and ${}^V H^n_{-1}$ are the scalar and vector parts of H^i and, in lowest order, depend only on the scalar perturbations and the vector perturbations, respectively, besides α and ϕ.

III. SUPERSPACES M AND M/Diff_3

In the superspace representation Ψ is a function on the space, say M, which is an infinite-dimensional space of the three-metric h_{ij} and the scalar field Φ. M is redundant because four of the six components of h_{ij} are unphysical. Among the four unphysical components, three can be eliminated by the constraints of Eqs. (2.2c) and (2.2d) which express the invariance of Ψ under spatial diffeomorphism Diff_3. Explicitly we have

$$\left[\frac{\partial}{\partial a_n}-\frac{\partial}{\partial b_n}-\left[a_n+4\frac{n^2-4}{n^2-1}b_n\right]\frac{\partial}{\partial\alpha}-3f_n\frac{\partial}{\partial\phi}\right]\Psi=0 \,, \tag{2.2c$'$}$$

$$\left[\frac{\partial}{\partial c_n}+4(n^2-4)c_n\frac{\partial}{\partial\alpha}\right]\Psi=0 \,. \tag{2.2d$'$}$$

The solution of the above is

$$\Psi(\alpha,\phi,a_n,b_n,c_n,d_n,f_n)=\Psi(\tilde\alpha,\tilde\phi,d_n,f_n,s_n) \,,$$

where

$$\tilde\alpha\equiv\alpha+\tfrac{1}{2}\sum_n a_n{}^2-2\sum_n\frac{n^2-4}{n^2-1}b_n{}^2-2\sum_n(n^2-4)c_n{}^2 \,,$$

$$\tilde\phi\equiv\phi-3\sum_n b_nf_n \,,$$

$$s_n\equiv a_n+b_n \,.$$

This means that the theory should be redefined in the quotient space M/Diff_3 which is parametrized by $\tilde\alpha$, $\tilde\phi$, d_n, f_n, and s_n.

The first test of the consistency of the theory is whether we can rewrite the rest of the constraints Eqs. (2.2a) and (2.2b) in this quotient space. After some algebra we find that this is in fact possible and the result is

$$\begin{aligned}H_{\perp 0}(\alpha,\phi)+\sum_n H^n_{\perp 2}+\sum_n\Bigg[&\frac{3}{n^2-4}\pi_{s_n}{}^2-6\frac{n^2-1}{n^2-4}f_n\pi_{s_n}\pi_\phi+\pi_{f_n}{}^2-(n^2-1)f_n{}^2\pi_\alpha{}^2+\frac{(n^2-1)(n^2+5)}{n^2-4}f_n{}^2\pi_\phi{}^2\\&-6s_n\pi_{f_n}\pi_\phi+\tfrac{1}{3}(n^2-4)s_n{}^2\pi_\alpha{}^2-\tfrac{1}{3}(n^2-31)s_n{}^2\pi_\phi{}^2\\&+m^2e^{6\alpha}\left[f_n{}^2+(n^2-1)f_n{}^2\phi^2+6s_nf_n\phi-\frac{n^2-4}{3}s_n{}^2\phi^2\right]\Bigg]\Psi=0 \,,\end{aligned} \tag{3.1}$$

$$\left[\pi_{f_n}\pi_\phi-\pi_{s_n}\pi_\alpha+m^2e^{6\alpha}f_n\phi+\frac{s_n}{3}[(n^2-4)\pi_\alpha{}^2-(n^2+5)\pi_\phi{}^2-(n^2-4)m^2e^{6\alpha}\phi^2]\right]\Psi=0 \,. \tag{3.2}$$

Here we use the original symbols α and ϕ to express $\tilde\alpha$ and $\tilde\phi$ for simplicity.

The vector perturbations totally disappeared from Eq. (3.1). This is due to the relation

$$\begin{aligned}&{}^V H^n_{\perp 2}+[c_n{}^2 \text{ parts of } H_{\perp 0}(\alpha,\phi)-H_{\perp 0}(\tilde\alpha,\tilde\phi)]\\&\quad=8(n^2-4)c_n{}^2H_{\perp 0}(\tilde\alpha,\tilde\phi)+{}^V H^n_{-1}{}^V H^n_{-1} \,.\end{aligned} \tag{3.3}$$

The right-hand side vanishes in the present order of perturbations when it is multiplied from the right by Ψ in Eq. (2.2d). However note that we had to use a specific operator ordering in ${}^V H^n_{\perp 2}$ to get Eq. (3.3); i.e., $c_n\pi_{c_n}\pi_\alpha$ in ${}^V H^n_{\perp 2}$ should be $(c_n\pi_{c_n}+\pi_{c_n}c_n)\pi_\alpha/2$. Otherwise we would get a term proportional to $\hbar\pi_\alpha$ with a divergent coefficient (because of $\sum_n$), which should be subtracted by a counterterm. The counterterm is of higher order in $\hbar$ and therefore can be added to $H_{\perp 0}$ as a quantum correction. Alternatively we can regard it as a correction due to ambiguity in the operator ordering:

$$\pi_\alpha{}^2=e^{k\alpha}\pi_\alpha e^{-k\alpha}\pi_\alpha-k\hbar\pi_\alpha \,, \tag{3.4}$$

where k is an arbitrary constant. Elimination of one of the three scalar degrees of freedom is done through a similar procedure.

IV. BOUNDARY CONDITIONS

In this section we study the consistency of Eqs. (3.1) and (3.2). Both are quadratic in momenta and a main difficulty for constructing a consistent canonical quantum gravity lies in this point. First we rewrite Eq. (3.2) as

$$\pi_{s_n}\Psi=\pi_\alpha{}^{-1}(\mathcal{S}_n^{(0)}+s_n\mathcal{S}_n^{(1)})\Psi\ , \tag{4.1}$$

where

$$\mathcal{S}_n^{(0)}=\pi_\phi\pi_{f_n}+m^2e^{6\alpha}f_n\phi\ ,$$

$$\mathcal{S}_n^{(1)}=\tfrac{1}{3}[(n^2-4)\pi_\alpha{}^2(n^2+5)\pi_\phi{}^2-(n^2-4)m^2e^{6\alpha}\phi^2]\ .$$

In the superspace representation π_α is $-i\hbar\partial/\partial\alpha$ and so we identify $\pi_\alpha{}^{-1}$ with the integration

$$\pi_\alpha{}^{-1}f(\alpha)\equiv\frac{i}{\hbar}\int_{\alpha_0}^{\alpha}d\alpha' f(\alpha')\ , \tag{4.2}$$

where α_0 is a constant which we specify later from the consistency requirement. Note that this definition implies that $\pi_\alpha{}^{-1}$ is only a right inverse. We have $\pi_\alpha\pi_\alpha{}^{-1}=1$ but

$$\pi_\alpha{}^{-1}\pi_\alpha=1-P\quad(\neq 1)\ . \tag{4.3}$$

Here we introduced a new operator P which is defined by the relation

$$Pf(\alpha)\equiv f(\alpha_0)\ ,$$

for an arbitrary function f on the superspace.

We discuss the consistency through the following steps.

Step I. Eliminate π_{s_n} from Eq. (3.1) by using (4.1) and expand the result in power of s_n as

$$\left[H^{(0)}+\sum_n s_nH_n^{(1)}+\sum_n s_n{}^2H_n^{(2)}\right]\Psi=0\ . \tag{4.4}$$

$H_{(n)}^{(0)-(2)}$ depend only on α, ϕ, f_n, and their conjugate momenta. See the Appendix for their explicit forms.

Step II. Suppose that Ψ at $s_n=0$ is given by

$$H^{(0)}\Psi(s_n=0)=0\ . \tag{4.5}$$

Then $\Psi(s_n\neq 0)$ is calculated from Eq. (4.1) as

$$\Psi(s_n)=\prod_n T\exp\left[\frac{i}{\hbar}\int_0^{s_n}ds_n'\times\pi_\alpha{}^{-1}(\mathcal{S}_n^{(0)}+s_n'\mathcal{S}_n^{(1)})\right]\Psi(0)\ .$$

Step III. This is consistent with Eq. (4.4) if and only if

$$\left[\frac{i}{\hbar}\pi_\alpha{}^{-1}\mathcal{S}_n^{(0)},H^{(0)}\right]\Psi(0)=(H_n^{(1)}+\xi_nH^{(0)})\Psi(0)\ , \tag{4.6}$$

$$\left[\left[\frac{i}{\hbar}\pi_\alpha{}^{-1}\mathcal{S}_n^{(1)},H^{(0)}\right]+\left[\frac{i}{\hbar}\pi_\alpha{}^{-1}\mathcal{S}_n^{(0)},H_n^{(1)}\right]\right]\Psi(0)=(2H_n^{(2)}+\xi_n'H^{(0)})\Psi(0)\ , \tag{4.7}$$

where ξ_n and ξ_n' are arbitrary constants. Equations (4.6) and (4.7) come from the consistency of the terms linear in s_n and the ones quadratic in s_n, respectively.

A main issue is step III, but we need some caution in step I. The term $\pi_{s_n}{}^2$ in Eq. (3.1) cannot be replaced simply with the square of the right-hand side of Eq. (4.1) because $[\pi_{s_n},s_n]\neq 0$. Instead we have

$$\pi_{s_n}{}^2\Psi=[\pi_\alpha{}^{-1}(\mathcal{S}_n^{(0)}+s_n\mathcal{S}_n^{(1)})]^2\Psi-i\hbar\pi_\alpha{}^{-1}\mathcal{S}_n^{(1)}\Psi\ . \tag{4.8}$$

The summation of the last term in n gives rise to divergence in $H^{(0)}$ because $\mathcal{S}_n^{(1)}$ is independent of n except for numerical coefficients. Subtraction of this term is analogous to the one in the previous section. Here we use the formula

$$1=e^{-k\alpha}\pi_\alpha e^{k\alpha}\pi_\alpha{}^{-1}+i\hbar k\pi_\alpha{}^{-1}\ ,$$

where k is an arbitrary constant. This implies that we can get counterterms by adjusting the operator ordering of $H_{\perp 0}$. [For example, replace $\pi_\alpha{}^2$ in $H_{\perp 0}$ with

$$(1-Z)\pi_\alpha{}^2+Ze^{-k\alpha}\pi_\alpha e^{k\alpha}\pi_\alpha{}^{-1}\pi_\alpha{}^2=\pi_\alpha{}^2-Zi\hbar k\pi_\alpha{}^{-1}\pi_\alpha{}^2\ ,$$

where Z is a divergent constant.]

Now the remaining problem is the two relations Eqs. (4.6) and (4.7). After some tedious algebra we can show that the two relations would be correct to the order $\hbar^0$, if we ignore P in Eq. (4.3). This is nothing but the consistency of the classical theory. Higher-order corrections [$O(\hbar^1)$ terms] cause no trouble to Eqs. (4.6) and (4.7) because we are allowed to add such terms to $H_n^{(1)}$ and $H_n^{(2)}$ as quantum corrections. Therefore the only problem is the effects of P, whose contributions can be even $O(\hbar^{-1})$. [Note that P is an $O(\hbar^0)$ operator.] We should eliminate them by imposing boundary conditions on $\Psi(s_n=0)$ at $\alpha=\alpha_0$.

Below we show what kind of contributions we get in some detail. First consider the commutator

$$\left[\frac{1}{\hbar}\pi_\alpha{}^{-1}\pi_\phi\pi_{f_n},\pi_\alpha{}^2\right]=-\frac{1}{\hbar}\pi_\phi\pi_{f_n}P\pi_\alpha\ ,$$

which is in $[\pi_\alpha{}^{-1}\mathcal{S}_n^{(0)},H^{(0)}]$ of Eq. (4.6). The right-hand side is $O(\hbar^{-1})$ and therefore should vanish. This implies that

$$\pi_\phi\pi_{f_n}P\pi_\alpha\Psi=0\ . \tag{4.9}$$

[Here and below we understand that Ψ stands for $\Psi(s_n=0)$.] $[\pi_\alpha^{-1}\mathscr{S}_n^{(0)},H^{(0)}]$ also gives rise to

$$\left[\frac{1}{\hbar}\pi_\alpha^{-1}\pi_\phi\pi_{f_n},e^{4\alpha}\right]\propto\pi_\phi\pi_{f_n}\pi_\alpha^{-2}e^{4\alpha}+O(\hbar^1)\ .$$

We need $\pi_\alpha^{-2}\pi_\alpha^{2}=1$ to get Eq. (4.7) but in reality

$$\pi_\alpha^{-2}\pi_\alpha^{2}=1-P-\pi_\alpha^{-1}P\pi_\alpha\ .$$

Therefore we get

$$\pi_\phi\pi_{f_n}P\Psi=0$$

in addition to Eq. (4.9). Other important constraints come from

$$\begin{aligned}\left[\frac{1}{\hbar}\pi_\alpha^{-1}\pi_\alpha^{2},e^{4\alpha}\right]&=\frac{1}{\hbar}[\pi_\alpha,e^{4\alpha}]-\frac{1}{\hbar}P\pi_\alpha e^{4\alpha}+\frac{1}{\hbar}e^{4\alpha}P\pi_\alpha\\&=\frac{1}{\hbar}[\pi_\alpha,e^{4\alpha}]-\frac{1}{\hbar}e^{4\alpha_0}P\pi_\alpha+\frac{1}{\hbar}e^{4\alpha}P\pi_\alpha\\&\quad+4ie^{4\alpha_0}P\ ,\end{aligned}$$

which is in $[\pi_\alpha^{-1}\mathscr{S}_n^{(1)},H^{(0)}]$. This gives

$$P\pi_\alpha\Psi=e^{4\alpha_0}P\Psi=0\ .$$

Now we list all the constraints which come from a sequence of such calculations. From $[\pi_\alpha^{-1}\mathscr{S}_n^{(0)},H^{(0)}]$ we get

$$\begin{aligned}\pi_\phi\pi_{f_n}\pi_\alpha\Psi&=\pi_\phi\pi_{f_n}\Psi=\pi_\phi f_n\pi_\alpha\Psi\\&=\phi f_n\pi_\alpha\Psi=\phi f_n\Psi=0\end{aligned}\tag{4.10}$$

at $\alpha=\alpha_0$. Additional constraints from $[\pi_\alpha^{-1}\mathscr{S}_n^{(0)},H_n^{(1)}]$ are

$$\pi_\phi^{2}\Psi=\pi_\phi^{2}\pi_\alpha\Psi=0\tag{4.11}$$

at $\alpha=\alpha_0$. Additional ones from $[\pi_\alpha^{-1}\mathscr{S}_n^{(1)},H_n^{(0)}]$ are

$$e^{4\alpha_0}\Psi=\pi_\alpha\Psi=\pi_\alpha^{3}\Psi=0\tag{4.12}$$

at $\alpha=\alpha_0$.

If $\alpha_0\neq-\infty$, Eq. (4.12) means that $\Psi=\partial_\alpha\Psi=0$ at $\alpha=\alpha_0$. However, $\alpha=\alpha_0$ ($\neq\infty$) is a Cauchy surface for the hyperbolic differential equation $H_0\Psi=0$, and the above relation implies that Ψ vanishes identically. Therefore α_0 should be $-\infty$.

Next we divide the solutions of Eqs. (4.10)—(4.12) into two classes according to whether $\Psi(\alpha=-\infty)$ vanishes identically or not. When it does, the only additional requirement is $\pi_\alpha\Psi=0$ at $\alpha=-\infty$. But this is almost trivial when $\Psi(\alpha=-\infty)$ does not diverge. In this class of the solutions the wave function vanishes at the classically singular point $e^\alpha=0$. One may say that the singularity is "avoided" quantum mechanically here.

Next we consider the second class in which $\Psi(\alpha=-\infty)$ does not vanish identically. $\pi_\phi^2\Psi=0$ implies $\phi\Psi\neq0$ due to the uncertainty principle. Therefore we get $f_n\Psi=0$ from the last of Eq. (4.10) and also $\pi_\phi\Psi=0$ from the second of Eq. (4.10). Combining the above we get the following constraints.

(A) $\alpha_0=-\infty$. $\Psi(\alpha=-\infty)$ does not vanish identically.

(B) Boundary conditions at $\alpha=-\infty$:

$$f_n\Psi=\pi_\phi\Psi=0\ .$$

(C) Additional conditions at $\alpha=-\infty$:

$$\pi_\alpha\Psi=(\pi_\alpha^{2}\Psi=)\pi_\alpha^{3}\Psi=0\ .$$

($\pi_\alpha^2\Psi$ comes from $[\pi_\alpha^{-1}\mathscr{S}_n^{(1)},H^{(0)}]$ if we have a term $\hbar\pi_\alpha$ in $H_{\perp 0}$ as a quantum correction. See Eq. (3.4).)

The boundary conditions derived here are consistent with those proposed in Ref. 4 (the "no boundary" condition), namely, $\pi_\phi\Psi=e^{-\alpha}\pi_\alpha\Psi=0$ at $\alpha=-\infty$ though the second one is missing here. Moreover $\Psi(\alpha=-\infty,f_n)\propto\delta(f_n)$ is what was proposed in Ref. 2. These are boundary conditions for the wave function at $s_n=0$. As for the s_n dependence, Eq. (4.1) implies $\pi_{s_n}\Psi=0$ at $\alpha=-\infty$. This is inconsistent with the proposal $\Psi(\alpha=-\infty,s_n)\propto\delta(s_n)$ in Ref. 2. However, in the semiclassical approach in which π_α and π_ϕ are replaced with the corresponding derivatives of the action (the c numbers), we get a gauge-invariant variable[5] f_n+Ks_n ($K\equiv\pi_\phi/\pi_\alpha$), and $Ks_n\to0$ for $\alpha\to-\infty$ even in the present boundary conditions. Therefore the analysis in Ref. 2 remains valid with minor modification.

Finally we point out that the value of $\Psi(\alpha=-\infty)$ is closely related to the operator ordering. First, consider a simple minisuperspace model in which there is only one physical degree of freedom α. Then the Wheeler-DeWitt equation is

$$(\pi_\alpha^{2}+e^{4\alpha})\Psi(\alpha)=0\ .$$

However there is $O(\hbar)$ ambiguity in the above which is related to the operator-ordering problem. A possible form which has frequently appeared in the literature (e.g., Ref. 4) is

$$\left[-\hbar^2a^{2-P}\frac{\partial}{\partial a}a^{P}\frac{\partial}{\partial a}+a^4\right]\Psi(a)=0\ ,\tag{4.13}$$

where $a=e^\alpha$ is the scale factor of the Universe. Its solution near $a=0$ is

$$\Psi\simeq a^0\ \text{ or }\ a^{1-P}\ .$$

The situation becomes more complicated when there are other degrees of freedom. Their back reaction gives rise to a multiplicative factor which is singular at $a=0$ (Ref. 6). In order to get $\Psi(\alpha=-\infty)<\infty$ such a factor should be canceled by adjusting the operator ordering of the type (4.13).

V. DISCUSSIONS

In this paper we discussed consistency of the superspace quantization of gravity coupled to the minimal scalar field in the lowest nontrivial order of the cosmological perturbations. We showed that the consistency imposes nontrivial constraints on the boundary conditions for the physical state (the wave function of the Universe) at $a=e^\alpha=0$. In particular we got the two classes of the solutions: (i) the "singularity-free" condition and (ii) a modified version of the one in Refs. 2 and 4.

Obviously the present analysis should be extended to higher order or hopefully to all order. We hope that such analysis will give us a consistent quantum gravity in which both the operator ordering and the boundary conditions are fixed. The relation of our analysis to the previous studies[7] of the closedness of the algebra of H^i and $H_\perp$ should also be clarified.

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APPENDIX

We present the explicit form of $H^{(0)}$, $H_n^{(1)}$, and $H_n^{(2)}$ in Eq. (4.4). We assume that the contribution of the last term of Eq. (4.8) is subtracted by a counterterm:

$$
\begin{aligned}
H^{(0)} = & -\pi_\alpha{}^2+\pi_\phi{}^2+m^2e^{6\alpha}\phi^2-e^{4\alpha} \\
& +\sum_n \left[\left[1+\frac{3}{n^2-4}\frac{\pi_\phi{}^2}{\pi_\alpha{}^2} \right] \pi_{f_n}{}^2 - 6\frac{n^2-1}{n^2-4}\frac{\pi_\phi{}^2}{\pi_\alpha{}^2} f_n \pi_{f_n} - (n^2-1) f_n{}^2 \pi_\alpha{}^2 + \frac{(n^2-1)(n^2+5)}{n^2-4} f_n{}^2 \pi_\phi{}^2 \right. \\
& \quad + m^2 e^{6\alpha} \left\{ f_n{}^2 + (n^2-1)\phi^2 f_n{}^2 + \frac{6}{n^2-4}\frac{1}{\pi_\alpha{}^2}\phi\pi_\phi f_n \pi_{f_n} \right. \\
& \quad \left. \left. - 6\frac{n^2-1}{n^2-4}\frac{\phi\pi_\phi}{\pi_\alpha} f_n{}^2 + \frac{3m^2e^{6\alpha}}{n^2-4}\frac{\phi^2}{\pi_\alpha{}^2} f_n{}^2 \right\} \right] ,
\end{aligned}
$$

$$
\begin{aligned}
H_n^{(1)} = & -4\pi_{f_n}\pi_\phi - 2\frac{n^2+5}{n^2-4}\frac{\pi_\phi{}^3}{\pi_\alpha{}^2}\pi_{f_n} - 2\frac{n^2-1}{n^2-4}[(n^2-4)\pi_\alpha{}^2-(n^2+5)\pi_\phi{}^2]\frac{\pi_\phi}{\pi_\alpha} f_n \\
& + 2m^2e^{6\alpha}\left[-\frac{\phi^2\pi_\phi}{\pi_\alpha{}^2}\pi_{f_n} + 4\phi f_n - \frac{n^2+5}{n^2-4}\frac{\phi\pi_\phi{}^2}{\pi_\alpha{}^2} f_n + (n^2-1)\frac{\phi^2\pi_\phi}{\pi_\alpha} f_n - m^2e^{6\alpha}\frac{\phi^3}{\pi_\alpha{}^2} f_n \right] ,
\end{aligned}
$$

$$
\begin{aligned}
H_n^{(2)} = & \tfrac{2}{3}(n^2-4)\pi_\alpha{}^2 - (n^2-7)\pi_\phi{}^2 + \frac{1}{3}\frac{(n^2+5)^2}{n^2-4}\frac{\pi_\phi{}^4}{\pi_\alpha{}^2} \\
& + m^2e^{6\alpha}\left[-\frac{n^2-4}{3}\phi^2 + \frac{n^2-4}{3} m^2 e^{6\alpha}\frac{\phi^3}{\pi_\alpha{}^2} + \tfrac{2}{3}(n^2+5)\frac{\phi^2\pi_\phi{}^2}{\pi_\alpha{}^2} \right] .
\end{aligned}
$$

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