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Cosmic strings and black holes

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The metric for a Schwarzschild black hole with a cosmic string passing through it is discussed. The thermodynamics of such an object is considered, and it is shown that $S=\frac{1}{4}A$, where S is the entropy and A is the horizon area. It is noted that the Schwarzschild mass parameter M, which is the gravitational mass of the system, is no longer identical to its energy. A solution representing a pair of black holes held apart by strings is discussed. It is nearly identical to a static, axially symmetric solution given long ago by Bach and Weyl. It is shown how these solutions, which were formerly a mathematical curiosity, may be given a more physical interpretation in terms of cosmic strings.

It has been recognized that certain gauge theories allow the possibility of topological defects, such as strings, and that these defects represent objects which might have been created in the very early Universe.[1] Cosmic strings are strands of matter which could be created in a cosmological phase transition. In this paper we shall be concerned with strings in the presence of black holes. The gravitational field of a straight string in flat spacetime has the rather peculiar property that the Newtonian potential vanishes, yet there are nontrivial gravitational effects.[2] The space outside of a string is locally flat but has the topology of a cone. Thus a test particle is neither attracted to nor repelled by a string; yet the conical nature of the space outside of a string produces observable effects such as light deflection. The metric outside the string can be written in cylindrical coordinates as

$$ds^2=-dt^2+d\rho^2+dz^2+b^2\rho^2d\phi^2 , \tag{1}$$

where $0\leq\phi<2\pi$. Equivalently, one may let $\phi'=b\phi$ with $0\leq\phi'<2\pi b$. Let μ be the mass per unit length of the string. If $|\mu| \ll 1$, then

$$b=1-4\mu . \tag{2}$$

A cosmic string possesses a positive energy density so $\mu>0$ and $b<1$. The fact that the region outside of the string is a conical space described by Eq. (1) is true for all μ, although the relation between b and μ may be modified if μ is of order unity. Let S be a two-dimensional surface that intersects the string and ${}^{(2)}g_{ij}$ be the metric in this surface. The four-dimensional Riemann tensor ${}^{(4)}R_{ijlm}$ is nonzero inside the string but vanishes outside it. The Gauss-Bonnet theorem leads to the relation[3]

$$1-b=\frac{1}{8\pi}\int {}^{(2)}g^{ik\,(2)}g^{jl\,(4)}R_{ijkl}({}^{(2)}g)^{1/2}d^2x , \tag{3}$$

where the integration is over the surface S. Particular models for the interiors of strings have been discussed by several authors.[4]

The unusual gravitational field of a string is due to negative pressure. The equation of state of the matter in the interior of the string is

$$p_1=-\rho_E,\quad p_2=p_3=0 , \tag{4}$$

where ρ_E is the energy density, p_1 is the pressure along the axis of the string, and p_2 and p_3 are the pressures perpendicular to this axis, averaged over a cross section. The source for the Newtonian potential in linearized gravity theory is $\rho_E+p_1+p_2+p_3$; the vanishing of this source for strings explains the absence of Newtonian gravitational effects. This conclusion applies only to straight strings. A curved infinite string or a closed loop does have a nonzero gravitational potential.

We now wish to consider examples of curved spacetimes containing strings. An example is a Schwarzschild black hole with a straight string passing through it. The metric for such an object can be written as

$$ds^2=-\left(1-\frac{2M}{r}\right)dt^2+\left(1-\frac{2M}{r}\right)^{-1}dr^2 + r^2(d\theta^2+b^2\sin^2\theta\, d\phi^2) , \tag{5}$$

where $0\leq\phi<2\pi$. This is a black hole of mass M with a string extending along the $\theta=0$ and $\theta=\pi$ axes. By going to a locally inertial frame near the string, one may transform the metric into a form equivalent to Eq. (1), where the gravitational field is locally transformed away but the conical singularity on the axis remains. The metric (5) is locally identical to the Schwarzschild metric, so as in flat spacetime the motions of test particles are locally unchanged by the presence of the string. A black hole with strings emanating from it could be formed in a phase transition. One possible type of string consists of a flux tube of a confined gauge field. Above the transition temperature a black hole containing a nonzero magnetic charge will possess a spherically symmetric Coulomb-type field. If this system were then cooled below the transition temperature, the field would become confined and strings emanating from the black hole would be formed. Gauss's theorem requires the total flux carried by the strings to equal the net flux across the black hole's horizon before the transition. Fields such as the electromagnetic and

Yang-Mills fields that are associated with a Gauss's law do not violate the dictum that "black holes have no hair." Nonetheless a black hole may have "hairs" in the form of strings.

The same trick which was used in writing Eq. (5) could be employed to construct an infinite family of spaces containing strings. One needs only a spacetime with a symmetry axis. If one then cuts a wedge out of this space, which is done by requiring the azimuthal angle around the axis to run over a range other than 0 to 2π, one has a space with a string along this axis.

We now turn to the consideration of the problem of two black holes held in equilibrium by strings. The appropriate solution of Einstein's equations is closely related to one given many years ago by Bach and Weyl.[5] The latter solution may be interpreted as representing a pair of black holes held in static equilibrium by a "strut" extending between the black holes. This strut is a conical singularity on the symmetry axis which is identical to a string of negative energy density.

The most general static, axially symmetric vacuum solution can be written in the form

$$ds^2=-f^2dt^2+f^{-2}[e^{2\gamma}(d\rho^2+dz^2)+\rho^2d\phi^2]\ , \tag{6}$$

where f and γ are functions of ρ and z. If $f=e^{\psi}$, then ψ must be a solution of the flat-space Laplace equation, $\nabla^2\psi=0$; and γ is a solution of

$$\frac{\partial\gamma}{\partial\rho}=\rho\left[\left(\frac{\partial\psi}{\partial r}\right)^2-\left(\frac{\partial\psi}{\partial z}\right)^2\right]$$

and

$$\frac{\partial\gamma}{\partial z}=2\rho\frac{\partial\psi}{\partial\rho}\frac{\partial\psi}{\partial z}\ . \tag{7}$$

The Schwarzschild solution is obtained by taking ψ to be the potential of a finite rod. Similarly, solutions containing multiple black holes arise if ψ is the potential produced by several finite rods all lying along the symmetry axis.[6] A two black-hole solution is obtained if we let

$$f^2=\left(\frac{r_1+r_2-2m}{r_1+r_2+2m}\right)\left(\frac{r_3+r_4-2m'}{r_3+r_4+2m'}\right) \tag{8}$$

and

$$e^{2\gamma}=K^2\frac{(r_1+r_2)^2-4m^2}{4r_1r_2}\frac{(r_3+r_4)^2-4m'^2}{4r_3r_4}\times\left[\frac{(m'+d)r_1+(m+m'+d)r_2-lr_4}{(m+d)r_2+r_1d-mr_3}\right]. \tag{9}$$

Here

$$r_i{}^2=\rho^2+(z-z_i)^2,\quad i=1,\ldots,4\ . \tag{10}$$

The constants z_i satisfy the conditions

$$z_1-z_2=2m,\quad z_2-z_3=2d,\quad z_3-z_4=2m'\ . \tag{11}$$

This solution represents a pair of black holes of masses[7] m and m' separated by a (z-coordinate) distance $2d$. The interval $z_2<z<z_3$ is the portion of the symmetry axis which lies between the black holes. The intervals $z>z_1$ and $z<z_4$ are the portions of this axis which extend from infinity to the black hole of mass m and from infinity to that of mass m', respectively. The constant K determines the location and nature of the conical singularities on the z axis. If we let

$$K=K_{\rm BW}=\frac{d}{m'+d}\ , \tag{12}$$

then the Bach-Weyl solution results. Here the intervals $z>z_1$ and $z<z_4$ are free of conical singularities, but the interval $z_2<z<z_3$ contains a conical singularity with parameter b defined in Eq. (1) given by

$$b=\frac{(m+d)(m'+d)}{d(m+m'+d)}\ . \tag{13}$$

Here $b>1$ so this is a pair of black holes held apart by a strut, or a string of negative energy density. If we let

$$K=K_s=\frac{m+d}{m+m'+d}\ , \tag{14}$$

then we obtain the solution for a pair of black holes held apart by cosmic strings extending to infinity in opposite directions. The region between the black holes, $z_2<z<z_3$, is free of conical singularities, but for $z>z_1$ and $z<z_4$ there is a singularity with

$$b=\frac{d(m+m'+d)}{(m+d)(m'+d)}\ . \tag{15}$$

Here $b<1$, so this singularity could be produced by a physical string with positive energy density. If we let $d\gg m,m'$, then

$$\mu\simeq\tfrac{1}{4}(1-b)\approx\frac{mm'}{(2d)^2}\ . \tag{16}$$

This is just the tension required to hold a pair of bodies of masses m and m' at a separation of $2d$ in the Newtonian limit.

For general values of K, there are conical singularities representing strings or struts on all segments of the z axis. For $z>z_1$ and $z<z_4$,

$$b=b_1=\frac{d}{K(m'+d)} \tag{17}$$

and for $z_3<z<z_2$,

$$b=b_2=\frac{m+d}{K(m+m'+d)}\ . \tag{18}$$

If $b_1<1$ and $b_2<1$, there are strings on both sides of each black hole, with the difference in string tensions being that required to produce equilibrium. Solutions containing black holes held apart by strings are more physical than those containing struts in that one can imagine a physical process which generates such a solution. However, all of these solutions can be expected to be unstable to small perturbations. Because the tension in a string is constant, this system is unstable against perturbations which tend to cause the black holes either to fall together or fly apart.

Let us now consider the limiting cases of these solutions in which only one black hole appears. Set $m'=0$ in the metric of Eq. (6) and let

$$k = \lim_{m' \to 0} K \; . \tag{19}$$

Define new coordinates r and θ given by

$$r = k\left[\tfrac{1}{2}(r_1 + r_2) + m\right] \tag{20}$$

and

$$\cos\theta = \frac{r_1 - r_2}{2m} \; . \tag{21}$$

Then the metric becomes

$$ds^2 = -\left[1 - \frac{2M}{r}\right]dt^2 + \left[1 - \frac{2M}{r}\right]^{-1} d\rho^2 + r^2(d\theta^2 + k^{-2}\sin^2\theta\, d\phi^2) \; , \tag{22}$$

where $M = km$. This is just the metric Eq. (5) for a black hole of mass M with a string of $b = k^{-1}$ passing through it. If K is given by either Eq. (12) or Eq. (14), then $k = 1$. As expected, the limiting case of either of these solutions is simply a black hole without a string.

Now consider the asymptotic form of the metric when one black hole is removed to a very large, but not infinite, distance. We restrict our attention to the case of the solution of Eq. (14), a pair of black holes held apart by strings extending to infinity in opposite directions. Let $d \gg m, m', z$ and again introduce the coordinates defined by Eqs. (20) and (21). The metric in the region $z > z_1 = m$ now takes the approximate form

$$ds^2 = -\left[1 - \frac{2m}{r}\right](1 + 2gz)dt^2 + (1 - 2gz)\left[\left[1 - \frac{2m}{r}\right]^{-1} dr^2 + r^2 d\theta^2 + r^2\left[1 - \frac{mm'}{d^2}\right]^2 \sin^2\theta\, d\phi^2\right] \; , \tag{23}$$

where $g = m'/(2d)^2$ and $z = (r - m)\cos\theta$. In the region $z < z_2 = -m$, $|z| \ll d$, the metric is of the above form except without the conical singularity factor of $(1 - mm'/d^2)^2$. This is the metric in the region above the black hole of mass m in the limit where the black hole of mass m' is far away. This metric can be interpreted as representing a black hole suspended by a string in an approximately uniform gravitational field. If $2gz \ll 2m/r$, this is the Schwarzschild metric with a string of tension $\mu = mm'/(2d)^2$ on the positive z axis. If $r \gg 2m$, it is the metric for a weak uniform gravitational field for which the Newtonian potential is $\Phi = gz$.

There is no static solution for a single black hole with a single string (i.e., only along the $z > 0$ axis) attached to it in the absence of an external field. Such a black hole must undergo uniform acceleration, being towed by the string. The corresponding metric can be obtained from Eq. (23) by a coordinate transformation.

The final issue which we wish to discuss is the thermodynamics of a black hole with a string passing through it. We can see from Eq. (5) that the horizon area of such a black hole is

$$A = 16\pi^2 b M^2 \; . \tag{24}$$

Because A depends upon b, it is not immediately obvious that the entropy is simply related to the horizon area, as it is for ordinary black holes. The Hawking temperature of a black hole with a string is unchanged and is given by the usual relation

$$T = \frac{1}{8\pi M} \; . \tag{25}$$

This may be seen by rewriting Hawking's original derivation[8] of black-hole radiance for the present case. The mode functions for a quantum field propagating on the Schwarzschild background are of the form

$$R_l(r) P_{lm}(\cos\theta) e^{im\phi} \; ,$$

where P_{lm} is an associated Legendre function. In the presence of a string, the mode functions are still of the above form, although l and m are no longer integers. The temperature of the black-hole radiation is independent of the form of the angular functions and of the l dependence of $R_l(r)$; it is determined only by the behavior of the radial functions on the horizon and at infinity, which is l independent. Thus, one must obtain the same temperature when a string is present as one does in the case when it is absent. The reflection coefficients which describe the effect of scattering of the outgoing radiation by spacetime curvature will, however, be modified.

The entropy is determined by the relation

$$dS = \frac{dE}{T} \; , \tag{26}$$

where E is the energy of the black hole as measured by an observer at infinity. Without a string present, $E = M$. However, with a string the space is no longer asymptotically Minkowskian and we cannot assume that energy and mass at infinity are identical. Their relationship may be found by the following argument. Let $T_{\mu\nu}$ be the stress tensor for some matter field propagating on the Schwarzschild background; it could represent either Hawking radiation or classical matter being thrown into the black hole. Let $\xi^\mu = (1,0,0,0)$ be the timelike Killing vector for the Schwarzschild metric; then $\xi^\mu T_{\mu\nu}$ is a covariantly conserved vector current and the rate of flow of energy in or out of the black hole may be written as

$$\dot{E} = \int \xi^\mu T_{\mu\nu} d\Sigma^\nu \; , \tag{27}$$

where the surface integral can be taken over the horizon. The metric for a black hole with slowly changing mass is of the form of Eq. (5) with

$$M = M(t) = M_0 + \dot{M}t \; , \tag{28}$$

where M_0 and $\dot{M}$ are constants. The Einstein tensor for this metric is

$$G_{\mu\nu} = G^{(0)}_{\mu\nu} + G^{(1)}_{\mu\nu} = G^{(1)}_{\mu\nu} + O(\dot{M}^2) \; , \tag{29}$$

where $G^{(0)}_{\mu\nu} = 0$ is the Einstein tensor for the Schwarzschild metric and $G^{(1)}_{\mu\nu}$ is of first order in $\dot{M}$. The Einstein equations, $G_{\mu\nu} = 8\pi T_{\mu\nu}$, then lead to

$$\dot{E} = \frac{1}{8\pi} \int G^{(1)}_{\mu\nu} \xi^\mu d\Sigma^\nu$$
$$= \frac{b}{8\pi} \int G^{(1)}_{tr} r^2 \sin\theta \, d\theta \, d\phi \ . \quad (30)$$

Using

$$G^{(1)}_{tr} = \frac{2\dot{M}}{r^2} \ , \quad (31)$$

this yields

$$\dot{E} = b\dot{M} \ . \quad (32)$$

Thus in the presence of a string, the energy at infinity E and the Schwarzschild mass parameter M are not identical. To give another argument leading to Eq. (32), consider a spherical shell of matter falling from rest at $r \to \infty$ onto a black hole of mass M. If there is no string passing through the black hole, then the energy of the shell at $r \to \infty$ is equal to its mass. A string can now be introduced, as before, by changing the range of the azimuthal angle from $0 \le \phi < 2\pi$ to $0 \le \phi < 2\pi b$. This does not change the form of the solution and hence does not affect the masses of the black hole and of the shell, but the energy of the in-falling matter is changed by a factor of b (simply because of the change in the volume of the shell). This leads immediately to Eq. (32).

If we now use $dE = b\, dM$ and Eq. (25) in Eq. (26), we find that the entropy is given by

$$S = 4\pi b M^2 = \tfrac{1}{4} A \ . \quad (33)$$

The relation between S and A is unaltered by the presence of a string. The relation $S = \frac{1}{4} A$ seems to be very general and applies not only to black holes but also to situations where a cosmological event horizon is present.[9]

One might also raise the question of what happens when a black hole and a string merge. Hawking's area theorem[10] prevents the area of an event horizon from decreasing provided that $T_{\mu\nu} l^\mu l^\nu \ge 0$ for every null vector l^μ. This energy condition is marginally satisfied by cosmic strings. Because a black hole with a string has a smaller area than a black hole of the same mass M without a string, the mass must increase when a black hole and string coalesce. As discussed above, there is no Newtonian force between a straight string and a test mass m. There is however a higher-order force which has been calculated by Smith[11] to be

$$\mathbf{F} = -\frac{\pi}{16}(1-b)\frac{m^2}{\rho^2}\hat{\rho} \ , \quad (34)$$

where ρ is the separation of the mass and the string. This expression is valid if $\rho \gg m$ and $|1-b| \ll 1$. Because this force is attractive, there must be a repulsive interaction at shorter distances which is such that the net work required to bring a string and a black hole together is positive.

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[1]For a recent review, see A. Vilenkin, Phys. Rep. **121**, 263 (1985).
[2]A. Vilenkin, Phys. Rev. D **23**, 852 (1981).
[3]L. H. Ford and A. Vilenkin, J. Phys. A **14**, 2353 (1981).
[4]D. Garfinkle, Phys. Rev. D **32**, 1323 (1985); J. R. Gott, Astrophys. J. **288**, 422 (1985); B. Linet, Gen. Relativ. Gravit. **17**, 1109 (1985).
[5]R. Bach and H. Weyl, Math. Z. **13**, 134 (1922).
[6]W. Israel and K. A. Khan, Nuovo Cimento **23**, 331 (1964). Numerous authors have discussed related solutions containing conical singularities; for example, W. Israel, Phys. Rev. D **15**, 935 (1977); D. D. Sokolov and A. A. Starobinsky, Dok. Akad. Nauk SSSR **234**, 1043 (1977) [Sov. Phys. Dokl. **22**, 312 (1977)].
[7]Although we will loosely refer to m and m' as masses, there are some subtleties involved. If we let $m' \to 0$, then we obtain a Schwarzschild metric of mass m only if the conical singularity disappears in this limit; otherwise the mass differs by the factor k given in Eq. (19). This is related to the fact that the Schwarzschild mass parameter M for a black hole and string [Eq. (5)] is not the energy measured at infinity. [See Eq. (32).]
[8]S. W. Hawking, Commun. Math. Phys. **43**, 199 (1975).
[9]S. W. Hawking and G. W. Gibbons, Phys. Rev. D **15**, 2738 (1977); P. C. W. Davies, D. N. Page, and L. H. Ford, Phys. Rev. D **34**, 1700 (1986).
[10]S. W. Hawking and G. F. R. Ellis, *The Large Scale Structure of Space-Time* (Cambridge University Press, Cambridge, England, 1973), p. 318.
[11]A. G. Smith, Tufts University Report No. TUTP-86-11 (unpublished).