Relativistic effects in local inertial frames

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The concept of a generalized Fermi frame is introduced with the aim of describing the relativistic effects due to a third, distant body (such as the Sun) upon the motion of an Earth satellite. This extends Fermi's construction of a local inertial frame to the case in which there are local gravitating masses. This is done in the slow-motion, weak-field approximation by splitting the metric into an external part and a local part; Fermi's construction of local inertial coordinates defined with respect to the external metric is then used to transform the complete metric. The results show that the main relativistic effects on an Earth satellite are due to the nonlinear correction in the Earth's own Schwarzschild field. There are much smaller relativistic corrections in the tidal field of the Sun, and an Earth-Sun interaction term. The spatial axes of the local frame also undergo geodetic precession. Particular care must be taken with respect to the definition of the time coordinate in the generalized Fermi frame in order that the unit of time be consistent with readings of reasonable physical clocks on Earth's surface. Also discussed more rigorously is the generalized Fermi frame for a system of two bodies revolving in circular orbits around a common barycenter.

I. INTRODUCTION

In Newtonian mechanics the motion of a test body under the influence of a nearby mass m and a distant mass M is most aptly described by means of coordinates relative to m; then the force exerted by M is a tidal force, corresponding to a potential energy quadratic in the relative coordinates. One could say, this is the Newtonian version of the principle of equivalence: dynamical effects (forces) due to the distant body manifest themselves only by virtue of the test body's displacement relative to m, and tend to zero as the distance from m tends to zero.

In general relativity (GR) a precise formalism is available to describe the relative motion of two test bodies (equation of geodesic deviation[1]); there tidal forces appear in a much more general way through the curvature tensor. The principle of equivalence is quantitatively described by means of the construction of a Fermi frame of reference[2,3] using local inertial coordinates in the neighborhood of one test body. In local inertial coordinates the time variable $x^0=s$ is the proper time on a freely falling clock on this body and x^i ($i=1,2,3$) are spatial coordinates invariantly defined. As is well known, the metric in this Fermi frame differs from the Minkowskian form only by terms which to leading order are quadratic in the spatial coordinates x^i. One could say, in this frame the metric is "as Minkowskian as possible" and there are no gravitational forces acting on the test body at the Fermi frame's origin. That is, in local inertial coordinates the Christoffel symbols of the second kind vanish at the frame's origin.

While the relativistic treatment is much more general than the Newtonian treatment, the latter has the important advantage of being applicable also to the case in which the central body has a finite mass and produces a finite gravitational field. Because of the nonlinear character of Einstein's field equations, the gravitational fields due to local and distant bodies cannot be separated and the rigorous construction of a Fermi frame, in which the distant bodies act only through their tidal forces, is an unsolved problem.

In Newtonian celestial mechanics it would be rather inappropriate to describe local motion in a frame of reference where the center of gravity of the whole system is at rest. Relative coordinates, referred to the center of gravity of the local system (or to the center of Earth if in the model under consideration the only local body is Earth) should be used. Much simpler equations are obtained in this way and their physical consequences are easy to interpret. In the analysis of relativistic effects on Earth-orbiting satellites this has been done only in a partially satisfactory way.[4] The starting point is the appropriate generalization of the Newtonian N-body equations to GR using the slow-motion, weak-field (SMWF) approximation;[5] the approximate equations of motion exhibit post-Newtonian corrections to the forces acting on each body in the barycentric frame of reference. These corrections have been incorporated in numerical orbit computations, currently used[6] to analyze not only planetary motion, but also the dynamics of natural and artificial Earth satellites.[7] This approach has two drawbacks. First, the coordinates X^μ used in the SMWF approximation are not uniquely defined because of the gauge invariance of GR, and they do not have a precise physical meaning. As a consequence, comparison with observations cannot be done at the level of the field equations, but must be done using appropriate, invariantly defined observables. Second, the equations of motion themselves are very complicated. For example, relativistic corrections to the acceleration of a test body in the fields of both local and distant masses have several dozen different types of terms.[8]

In this paper we address the following question: can one define, within the SMWF approximation, a generalized Fermi frame even when local massive bodies are present? A case of particular interest is that in which the local massive body is Earth, and the distant bodies are the Sun and the other planets. Since the transformation to a local Fermi frame essentially reduces the effects of distant bodies to their tidal forces (nonlinear as well as linear), this Fermi frame must be defined using an appropriately chosen "external" metric, with respect to which one of the local bodies moves along a geodesic. One then applies the coordinate transformation so obtained to the whole metric, obtaining a transformed metric in which the local gravitational forces are referred to the origin of the local coordinates and the distant bodies appear only through the curvature tensor of the external metric.

The appropriate way to choose the external metric is indicated by results of Einstein, Infeld, and Hoffman,[9] Eddington and Clark,[10] and Bertotti[11] (see also Misner, Thorne, and Wheeler[12]). When only one local body is present, the essential result is that its motion is a geodesic for a "renormalized" or external metric, obtained from the complete metric by dropping all the divergent or undefined terms (in particular, the infinite self-interactions). It is this renormalized metric which must be used to construct the Fermi frame. It is a remarkable result, that the transformation constructed using this renormalized metric reduces the complete metric to a form in which the primary effect of the external bodies appears in forces which are like tidal forces. Numerous cancellations occur, so that to lowest-order forces of all other types are reduced to zero. One thus obtains a solution of the field equations in which the effect of external bodies is felt only through tidal terms. Local sources produce both linear and nonlinear contributions in the masses, as one would expect.

The general problem of matching a local solution of the field equations—e.g., a Schwarzschild field—to a global solution has been dealt with by means of two different expansions[13–15] with respect to two suitable small parameters, similar to the boundary-layer theory of fluid dynamics. This method has found its most important applications in the theory of gravitational radiation and the motion of compact bodies.[16] Our primary aim here is more limited and practical; it is the construction of the Fermi frame in the presence of nearby masses, in the SMWF approximation. A more general, rigorous construction of a Fermi frame would be very useful and interesting.

Another purpose of the present work is to give a detailed justification of the results of Ashby and Bertotti[4] (AB). There the final SMWF metric in the local frame was constructed heuristically, exploiting its assumed properties and solving the field equations for the nonlinear part of the interaction between Earth and the Sun. The detailed construction of the local coordinates given herein and the analysis of the transformation of the metric in such a complicated model problem as this is, however, essential for the full understanding of the problem.

The bulk of our work is developed in Secs. II—IV, where the full metric in the local frame is constructed. The problem of selection of a physical time coordinate, actually used in experiments, is discussed in Sec. V. Section VI completes the proof of the results presented in AB (Ref. 4) and Sec. VII develops in detail a particular model of the Solar System involving two bodies in circular orbits around a common barycenter.

Notation. In this paper we use capital letters (e.g., X^μ) to denote coordinates or other quantities expressed in barycentric coordinates. Small letters (e.g., x^μ) are used for local Fermi coordinates. Greek indices run from 0 to 3; latin indices from 1 to 3. The signature of the metric tensor, and other sign conventions used to defined the Ricci tensor $R_{\mu\nu}$ and the Christoffel symbols of the second kind are as in Weber.[1] This treatment is given entirely within the framework of GR, a special case of the parametrized-post-Newtonian (PPN) metric for which all PPN parameters are zero except for $\gamma=\beta=1$. It is possible to extend the construction discussed here to include relativity parameters γ, β, ζ_1, and ζ_2 (Ref. 17), but to do so a substantial separate work is required.

II. THE GENERALIZED FERMI FRAME

Fermi coordinates $x^\mu=(x^0,x^k)$ are constructed geometrically as follows. Referring to Fig. 1, $X^\mu(s)$ specifies a world line G, which is a solution of the geodesic equations of motion of the mass M_E in the external metric. $\Lambda^\mu_{(0)}=dX^\mu/ds$ is the tangent vector to the geodesic. Given a point $P(X^\mu)$ near G, a spacelike geodesic S is constructed which passes through P and intersects G orthogonally at the point P_0. Let the unit tangent vector to S at P_0 pointing in the direction of P be denoted by T^μ, and let the proper distance along S from P_0 to P be denoted by r. Then the coordinate time x^0 is defined to be the proper time elapsed on a standard clock falling along G from some chosen reference point 0. If we introduce three additional mutually orthonormal vectors $\Lambda^\mu_{(i)}(s)$, which are obtained by parallel transport from the reference point 0 to P_0, then the spatial coordinates in the local Fermi frame are defined by

$$x^i=rT_\mu\Lambda^\mu_{(i)}(P_0)\ . \tag{1}$$

The direction cosines of T^μ at P_0 are $\alpha^i=T_\mu\Lambda^\mu_{(i)}(P_0)$, measured with respect to the spatial axes $\Lambda^\mu_{(i)}(P_0)$. If there were no mass M_E falling along at the origin of the local frame, then the realization of the definition of x^0 given above would be obvious. In the present work, however, in which the mass M_E is placed at the origin, the potential singularity so introduced creates difficulty in defining a coordinate time in this way; later on we shall handle this by defining a coordinate time scale s' appropriate for a clock a small distance away from the apparent singularity. Such a scale change will not change the locally inertial character of the Fermi coordinates.

We next must specify the approximation scheme used in these calculations. The gravitational bodies are divided into two groups: an external group with characteristic mass M at typical distance R (mainly the Sun) and a local body or group of bodies with characteristic mass $m \ll M$

(mainly Earth). The local metric will be computed in a region, near the local bodies, of size $r \ll R$. In the case of interest, the parameters m/r and M/R are small and of the same order of magnitude as the square of a typical velocity. Here we use symbols M,m to denote masses measured in terms of their Schwarzschild radii so m/r is dimensionless; also velocities are measured in units of c so V^2 is dimensionless. Our small expansion parameter is thus V, and

$$O(m/r)=O(M/R)=O(V^2)\ .$$

Besides this small quantity, an expansion in r/R will be used to an order which will be specified later.

In the SMWF approximation, as is discussed by Eddington and Clark[10] and used by AB (Ref. 4), the metric tensor components G_{00} must be determined to an accuracy of $O(V^4)$; G_{0i} and G_{ij} are needed to accuracies of $O(V^3)$ and $O(V^2)$, respectively. We shall treat all the masses as point masses, labeled by Latin indices A, B, or E (for Earth). The position of the Ath mass M at coordinate time X^0 will be denoted by $\mathbf{X}_A(X^0)=\mathbf{X}_A$. The field point or observation point is denoted by $\mathbf{X}=(X^1,X^2,X^3)$. Then the point-mass form of the Eddington-Clark metric[10] before modification is

$$G_{00}=-1+2\sum_A \frac{M_A}{|\mathbf{X}-\mathbf{X}_A|}-2\left[\sum_A \frac{M_A}{|\mathbf{X}-\mathbf{X}_A|}\right]^2+4\sum_A \frac{M_A {V_A}^2}{|\mathbf{X}-\mathbf{X}_A|}-2\sum_A \frac{M_A}{|\mathbf{X}-\mathbf{X}_A|}\sum_B{}' \frac{M_B}{|\mathbf{X}_A-\mathbf{X}_B|}$$
$$-\sum_A \frac{M_A[(\mathbf{X}-\mathbf{X}_A)\cdot\mathbf{A}_A]}{|\mathbf{X}-\mathbf{X}_A|}-\sum_A \frac{M_A[(\mathbf{X}-\mathbf{X}_A)\cdot\mathbf{V}_A]^2}{|\mathbf{X}-\mathbf{X}_A|^3}\ , \tag{2}$$

$$G_{0i}=-4\sum_A \frac{M_A V_A^i}{|\mathbf{X}-\mathbf{X}_A|},\quad G_{ij}=\delta_{ij}\left[1+\sum_A \frac{2M_A}{|\mathbf{X}-\mathbf{X}_A|}\right]\ , \tag{3}$$

where $\mathbf{V}_A$ and $\mathbf{A}_A$ are the velocity and acceleration of the Ath mass, and a prime on a summation symbol means that undefined terms—or terms which are indefinitely large—are to be omitted. In the following discussion we shall make use of the abbreviation $R_{AB}=|\mathbf{X}_A-\mathbf{X}_B|$, which is a function of X^0 because both $\mathbf{X}_A$ and $\mathbf{X}_B$ depend on X^0. We introduce the "renormalized" or external part of the metric by defining, for example, the negative of the potential in the neighborhood of the mass M_E due to external sources:

$$U^{(e)}=\sum_{A\neq E}\frac{M_A}{|\mathbf{X}-\mathbf{X}_A|}=\sum_{A\neq E}M_A/R_A\ . \tag{4}$$

The modified metric tensor component $G_{00}^{(e)}$ is then written as $-1+H_{00}^{(e)}$, where

$$H_{00}^{(e)}=2U^{(e)}-2(U^{(e)})^2+4\sum_{A\neq E}\frac{M_A {V_A}^2}{|\mathbf{X}-\mathbf{X}_A|}-2\sum_{A\neq E}\frac{M_A}{|\mathbf{X}-\mathbf{X}_A|}\sum_B{}'\frac{M_B}{|\mathbf{X}_A-\mathbf{X}_B|}$$
$$-\sum_{A\neq E}\frac{M_A[(\mathbf{X}-\mathbf{X}_A)\cdot\mathbf{A}_A]}{|\mathbf{X}-\mathbf{X}_A|}-\sum_{A\neq E}\frac{M_A[(\mathbf{X}-\mathbf{X}_A)\cdot\mathbf{V}_A]^2}{|\mathbf{X}-\mathbf{X}_A|^3}\ . \tag{5}$$

In the fourth term of the above expression, the sum over B includes the term $B=E$. The remaining components of the modified metric tensor are

$$G_{0i}^{(e)}=-4\sum_{A\neq E}\frac{M_A V_A^i}{|\mathbf{X}-\mathbf{X}_A|}\ , \tag{6}$$

$$G_{ij}^{(e)}=\delta_{ij}(1+2U^{(e)})\ . \tag{7}$$

We shall also use the abbreviation $R_A=|\mathbf{X}-\mathbf{X}_A|$ for the distance between the field point $X=(X^1,X^2,X^3)$ and the source point, the position of mass M_A. (See note added in proof.)

The above metric is determined to within gauge transformations of the following type:

$$X^\mu \rightarrow X^\mu+\xi^\mu\ , \tag{8}$$

where ξ^0 is $O(V^3)$ and ξ^k is $O(V^2)$; this assignment of the orders of magnitude of the gauge functions is necessary in order that the form of the Newtonian potential in the metric be unchanged.

We may now construct the transformation to local Fermi coordinates as follows: when P and P_0 (see Fig. 1) are close together, x^i will be small compared to R and one may obtain the following expression for the coordinates X^μ of P in terms of a Taylor expansion about the point P_0:

$$X^\mu(P)=X^\mu(P_0)+\Lambda^\mu_{(i)}x^i-\tfrac{1}{2}\Gamma^\mu_{\alpha\beta}(P_0)\Lambda^\alpha_{(i)}\Lambda^\beta_{(j)}x^ix^j$$
$$-\tfrac{1}{6}\Gamma^\mu_{\alpha\beta,\gamma}(P_0)\Lambda^\alpha_{(i)}\Lambda^\beta_{(j)}\Lambda^\gamma_{(k)}x^ix^jx^k+O(x^4)\ . \tag{9}$$

The coefficients of x^i in this expression are evaluated at P_0 and are therefore functions of s (or x^0) only. The coefficient of the third term in Eq. (9) is obtained from the equation of the spacelike geodesic S. The coefficient of the fourth term is obtained from the derivative of this equation:

$$\frac{d^3X^\mu}{d\Lambda^3}+\Gamma^\mu_{\alpha\beta,\gamma}(P_0)\frac{dX^\alpha}{d\Lambda}\frac{dX^\beta}{d\Lambda}\frac{dX^\gamma}{d\Lambda}$$
$$+2\Gamma^\mu_{\alpha\beta}(P_0)\frac{d^2X^\alpha}{d\Lambda^2}\frac{dX^\beta}{d\Lambda}=0\ . \tag{10}$$

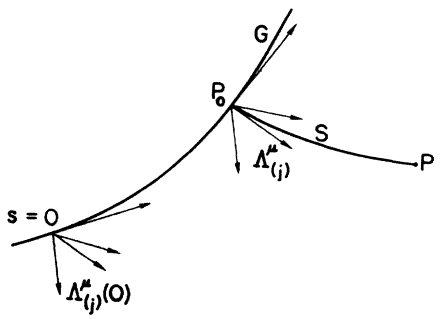

FIG. 1. Diagram showing how local inertial coordinates are constructed. G is the base geodesic, a freely falling timelike orbit. The spacelike geodesic S from the observation point P to G intersects G orthogonally at point P_0. The spatial direction of S at P_0 is characterized by direction cosines α^i. If r is the proper distance from P_0 to P, then $P(x^\mu)=(s,\alpha^1 r,\alpha^2 r,\alpha^3 r)$.

The last term in Eq. (10) is negligible as the terms in the Taylor expansion need be calculated only to $O(V^3)$ whereas the Christoffel symbols are all $O(V^2)$ or smaller and $d^2X^\mu/d\Lambda^2$ is $O(V^2)$. In Eq. (9), the coefficients are computed using the renormalized external metric, Eqs. (5)–(7).

Upon separating the coordinate transformations into time and space parts, it is easy to see that the coefficients in the X^0 transformation are of odd order in V and the coefficients in the X^k transformation are of even order in V. In applying the tensor transformation law to a metric tensor to obtain the metric in the local Fermi frame, the component to be computed to highest accuracy [to $O(V^4)$] is

$$g_{00}=G_{00}\left[\frac{\partial X^0}{\partial x^0}\right]^2+2G_{0i}\frac{\partial X^0}{\partial x^0}\frac{\partial X^i}{\partial x^0}+G_{ij}\frac{\partial X^i}{\partial x^0}\frac{\partial X^j}{\partial x^0}\ . \tag{11}$$

The leading term in G_{00} is -1; therefore, X^0 must be computed to $O(V^3)$ in order to find $\partial X^0/\partial x^0$ to $O(V^4)$. X^k need be computed only to $O(V^2)$, however, as the leading contribution to $\partial X^i/\partial x^0$ will be of $O(V)$ and the next higher contributions to this will give g_{00} to the required order.

We note also that the basis vectors $\Lambda^\mu_{(\alpha)}$ are mutually orthonormal with respect to the external metric defined above:

$$\Lambda^\mu_{(\alpha)}\Lambda^\nu_{(\beta)}G^{(e)}_{\mu\nu}=\eta_{\alpha\beta}\ , \tag{12}$$

where $\eta_{\alpha\beta}=\mathrm{diag}(-1,1,1,1)$ is the Minkowski metric.

III. THE EXTERNAL METRIC IN THE FERMI FRAME

The coordinate transformation discussed in the preceding section is constructed in detail in the Appendix; here we apply it to the computation of the complete metric in the Fermi frame:

$$g_{\mu\nu}=\eta_{\mu\nu}+h_{\mu\nu}=G_{\alpha\beta}\frac{\partial X^\alpha}{\partial x^\mu}\frac{\partial X^\beta}{\partial x^\nu}$$

$$=\eta_{\mu\nu}+h^{(e)}_{\mu\nu}+h^{(l)}_{\mu\nu}\ . \tag{13}$$

Note that the renormalized external metric is not a solution of the field equations. Nevertheless, since Earth's world line is a geodesic of the given external metric it is entirely appropriate to define and construct a Fermi frame in the usual way based on this world line. The construction is invariant and does not depend upon the particular gauge used. The difference between the full metric and the external metric defines the local metric perturbations:

$$H^{(l)}_{\mu\nu}=G_{\mu\nu}-G^{(e)}_{\mu\nu}\ . \tag{14}$$

The local metric is still undetermined to within gauge transformations of the appropriate order; the gauge functions ξ^μ vary with local spatial and temporal scales. Equation (11) then naturally splits into two parts. The contribution to $g_{\mu\nu}$ coming from the external metric

$$g^{(e)}_{\mu\nu}=G^{(e)}_{\alpha\beta}\frac{\partial X^\alpha}{\partial x^\mu}\frac{\partial X^\beta}{\partial x^\nu} \tag{15}$$

differs from the Minkowski metric $\eta_{\mu\nu}$ by terms of order $(x^k)^2$. The absence of linear terms in x^k is an expression of the principle of equivalence and results from a large number of cancellations; this has been checked in detail in the SMWF approximation for the present case. However, since it is a consequence of the rigorous theorem by Fermi,[2] it is not necessary to demonstrate these details here. Similarly, the part of $g^{(e)}_{\mu\nu}$ which is quadratic in the curvature tensor can be found in the literature.[3,18] This completes the first step in the calculation.

In the second step we compute the (small) local perturbations,

$$h^{(l)}_{\mu\nu}=H^{(l)}_{\alpha\beta}\frac{\partial X^\alpha}{\partial x^\mu}\frac{\partial X^\beta}{\partial x^\nu}\ , \tag{16}$$

a relatively easy task, except for the term $h^{(l)}_{00}$ which has to be evaluated to $O(V^4)$. Later we shall discuss one additional step, that of changing the time scale to one which represents more realistically the physical time scale used on Earth, but which affects only the time-time component of the metric. The splitting (14) requires particular care for the terms of $O(V^4)$ in G_{00}, because nonlinear interactions between local and external sources contribute to these terms.

Consider the case of a single local body of mass M_E at $\mathbf{X}_E(X^0)$ and a number of other distant pointlike bodies with trajectories $\mathbf{X}_A(X^0)$. The external part of G_{00} is given by Eq. (5). Even in this external metric the mass M_E enters in two ways. First, it reduces the effective mass of each of the external bodies, which can be written as

$$M_A\left[1-\sum_B{}'M_B/R_{AB}-M_E/R_{AE}\right]$$

[which includes the term in Eq. (5) involving a double summation]; second, in the last few terms of Eq. (5), the mass M_E will affect the determination of the accelerations, velocities, and positions of each of the external bodies.

Only the time-time component of the local metric is at all difficult to calculate. It is

$$H_{00}^{(l)}=2U^{(l)}\left[1-\sum_A{}'\frac{M_A}{R_{AE}}\right]-2U^{(l)2}-4U^{(l)}U^{(e)}+\frac{M_E V_E{}^2}{R_E}-\frac{M_E(\mathbf{X}-\mathbf{X}_E)\cdot\mathbf{A}_E}{R_E}-\frac{M_E[(\mathbf{X}-\mathbf{X}_E)\cdot\mathbf{V}_E]^2}{R_E{}^3}\,, \tag{17}$$

where

$$U^{(l)}=M_E/|\mathbf{X}-\mathbf{X}_E|=M_E/R_E \tag{18}$$

is the negative of the gravitational potential of Earth and $\mathbf{V}_E, \mathbf{A}_E$ are Earth's velocity and acceleration with X^0 the independent time variable.

Synge[18] has developed a systematic approximation scheme for computation of the external contributions from $G_{\mu\nu}^{(e)}$, in terms of the expansion parameter κr^2 where κ is a typical value of the curvature tensor. The metric is expressed in terms of integrals along the space geodesic S from P_0 to P. If λ is a dimensionless parameter varying from zero at P_0 to unity at P, then in Fermi coordinates the equations of the geodesic S are

$$\xi^i=x^i\lambda=\lambda r T_\mu \Lambda_{(i)}^\mu(P_0)\,. \tag{19}$$

Using this notation, Synge has shown that, for example,

$$g_{00}^{(e)}=-1-2x^m x^n\int_0^1 d\lambda(1-\lambda)r_{0m0n}(s,\lambda x^i)+O(\kappa^2 r^4)\,, \tag{20}$$

where

$$r_{0m0n}(s,\lambda x^i)=R_{\alpha\beta\gamma\delta}(\lambda)\Lambda_{(0)}^\alpha(\lambda)\Lambda_{(m)}^\beta(\lambda)\Lambda_{(0)}^\gamma(\lambda)\Lambda_{(n)}^\delta(\lambda) \tag{21}$$

are the intrinsic components of the curvature tensor computed on S at $\xi^i=x^i\lambda$. The orthonormal tetrad $\Lambda_{(\mu)}^\alpha(\lambda)$ used here is obtained from the tetrad defined on G, by parallel transport along S from P_0. The components of the two sets of vectors $\Lambda_{(\mu)}^\alpha(\lambda)$ and $\Lambda_{(\mu)}^\alpha(P^0)$ however differ at most by integrals involving the Christoffel symbols along S. Such integrals, of order V^2r/R, would contribute terms of order $V^2(r/R)^2V^2(r/R)$ which we neglect in this calculation as we are only carrying terms quadratic in r in the metric.

Furthermore in performing the integrals indicated in Eq. (20), there are simplifications. In the first-order Newtonian part of the tensor arising from $U^{(e)}$, one must be careful to perform the integral to the required accuracy. But in the contributions to the curvature tensor which are of $O(V^2/R^2)$, the curvature tensor may be taken to be a constant, equal to the value it has at P_0 where $\lambda=0$. The change along S would produce contributions to the metric of order $V^2(r/R)^3$ to $g_{\mu\nu}$ which we neglect.

The lowest-order contribution to r_{0m0n} is easily seen to be $-U_{,mn}^{(e)}$. Therefore, let us define

$$r_{0m0n}=-U_{,mn}^{(e)}+\delta r_{0m0n}\,, \tag{22}$$

where all contributions to δr_{0m0n} are $O(V^4/R^2)$. The contribution to Eq. (20) from the first-order part of Eq. (21) may easily be performed by integrating by parts:

$$\begin{aligned}2x^m x^n\int_0^1 d\lambda(1-\lambda)U_{,mn}^{(e)}(s,\lambda x^i)&=2\int_0^1 d\lambda(1-\lambda)\frac{d^2U^{(e)}(s,\lambda x^i)}{d\lambda^2}\\&=2\left[U^{(e)}(s,x^i)-U^{(e)}(s,0)-\left.\frac{dU^{(e)}}{d\lambda}\right|_{\lambda=0}\right]\\&=2[U^{(e)}(s,x^i)-U^{(e)}(s,0)-x^m U_{,m}^{(e)}(s,0)]\,.\end{aligned} \tag{23}$$

Only when P and P_0 are very near to each other does this reduce to the ordinary expression $x^m x^n U_{mn}^{(l)}(s,0)$. This generalization of the classical tidal potential is an interesting result in itself.

The remaining contributions can now be expressed directly in terms of curvature tensor components:[3,19]

$$g_{00}^{(e)}=-1+2[U^{(e)}(s,x^i)-U^{(e)}(s,0)-x^m U_{,m}^{(e)}(s,0)]-\delta r_{0m0n}x^m x^n\,, \tag{24}$$

$$g_{0i}^{(e)}=-\tfrac{2}{3}r_{0min}x^m x^n\,, \tag{25}$$

$$g_{ij}^{(e)}=\delta_{ij}-\tfrac{1}{3}r_{imjn}x^m x^n\,. \tag{26}$$

When P is sufficiently close to P_0 that it is necessary to retain only quadratic terms, Eq. (24) becomes

$$g_{00}^{(e)}=-1+U_{,mn}^{(e)}x^m x^n-\delta r_{0m0n}x^m x^n\,. \tag{27}$$

IV. THE COMPLETE METRIC IN THE LOCAL FRAME

An advantage of the splitting (14) is that in the tensor transformation, Eq. (16), the coordinate transformations are not needed to high accuracy. Equation (16) for $\mu=\nu=0$ reads, explicitly,

$$h_{00}^{(l)}=H_{00}^{(l)}\left[\frac{\partial X^0}{\partial x^0}\right]^2+2H_{0i}^{(l)}\frac{\partial X^0}{\partial x^0}\frac{\partial X^i}{\partial x^0}+H_{ij}^{(l)}\frac{\partial X^i}{\partial x^0}\frac{\partial X^j}{\partial x^0}\,. \tag{28}$$

Noting that

$$H_{0i}^{(I)}=-\frac{4M_E V_E^i}{|\mathbf{X}-\mathbf{X}_E|}=-4M_E V_E^i/R_E \;, \tag{29}$$

$$H_{ij}^{(I)}=\delta_{ij}\frac{2M_E}{|\mathbf{X}-\mathbf{X}_E|}=\delta_{ij}2M_E/R_E \;, \tag{30}$$

it is easy to see that the transformation coefficient $\partial X^0/\partial x^0$ is needed to $O(V^2)$ in the first term of Eq. (28), while in all the other terms only the leading contributions to $\partial X^i/\partial x^0$ are needed. These transformation coefficients are computed in the Appendix, and to the required order are

$$\frac{\partial X^0}{\partial x^0}=K\left[1-\sum_A{}'\frac{M_A\rho_A}{R_{EA}{}^2}\right], \tag{31}$$

where

$$\rho_A\equiv(X_E^i-X_A^i)x^j\delta_{ij}/R_{EA}$$

and K is given by Eq. (A7). Then

$$\frac{\partial X^0}{\partial x^i}=V_E^i,\quad \frac{\partial X^i}{\partial x^0}=V_E^i,\quad \frac{\partial X^i}{\partial x^j}=\delta_{ij} \;. \tag{32}$$

Also, the functions $H_{\mu\nu}^{(I)}$ must be expressed in terms of the new coordinates. This must be done with great care in the lowest-order contribution, the Newtonian part of $H_{00}^{(I)}$; it will be done in detail here for the distance function $R_E=|\mathbf{X}-\mathbf{X}_E(X^0)|$. In this expression, the independent variables are X^k and X^0, arguments of the position of the source, $X_E^k(X^0)$. We need to evaluate it in terms of the Fermi coordinates x^μ by means of the coordinate transformations, Eq. (9). These transformations are evaluated in detail in the Appendix, and to the order required here are

$$X^0=\int K\,dx^0+\mathbf{V}_E\cdot\mathbf{r}=X^0(P_0)+\mathbf{V}_E\cdot\mathbf{r} \;, \tag{33}$$

$$X^k=X_E^k[X^0(P_0)]+x^k(1-U_e-\mathbf{A}_E\cdot\mathbf{r}-\tfrac{1}{6}U_{e,mn}x^m x^n)+\tfrac{1}{2}V_E^k(\mathbf{V}_E\cdot\mathbf{r})+\Omega^{kj}\delta_{jm}x^m+\tfrac{1}{2}r^2A_E^k \;. \tag{34}$$

Equation (33) describes the different synchronization of events in the two different frames; in Eq. (34) the term $\frac{1}{2}V_E^k(\mathbf{V}_E\cdot\mathbf{r})$ is a manifestation of the Lorentz contraction. The term $-U_e x^k$ in Eq. (34) corresponds to an isotropic rescaling of length due to the external potential.

The difference $X^k-X_E^k(X^0)$ can be written as

$$\begin{aligned}X^k-X_E^k(X^0)&=X^k-X_E^k[X^0(P_0)]-V_E^k(\mathbf{V}_E\cdot\mathbf{r})\\&=x^k(1-U_e-\mathbf{A}_E\cdot\mathbf{r}-\tfrac{1}{6}U_{e,mn}x^m x^n)\\&\quad-\tfrac{1}{2}V_E^k(\mathbf{V}_E\cdot\mathbf{r})+\Omega^{kj}\delta_{jm}x^m+\tfrac{1}{2}r^2A_E^k \;,\end{aligned} \tag{35}$$

and, therefore, to $O(V^2)$,

$$[\mathbf{X}-\mathbf{X}_E(X^0)]^2=r^2(1-2U_e-\mathbf{A}_E\cdot\mathbf{r}-\tfrac{1}{3}U_{e,mn}x^m x^n)-(\mathbf{V}_E\cdot\mathbf{r})^2 \tag{36}$$

which leads to

$$|\mathbf{X}-\mathbf{X}_E(X^0)|^{-1}=\frac{1}{r}[1+U_e+\tfrac{1}{2}\mathbf{A}_E\cdot\mathbf{r}+\tfrac{1}{6}U_{e,mn}x^m x^n+\tfrac{1}{2}(\mathbf{V}_E\cdot\mathbf{r})^2/r^2] \;. \tag{37}$$

Collecting these results and substituting them into Eq. (28), many cancellations occur. The incorporation of both the synchronization correction, Eq. (33), and the Lorentz contraction correction, included in Eq. (32), play a crucial role in ensuring these simplifications. All velocity-dependent terms cancel. Also, interaction terms of the type $U_e M_E/r$ and $U_{e,i}x^i(M_E/r)$ disappear; the latter terms are associated with acceleration of the local source and arise from retardation. There are two contributions of the type $U_{e,mn}x^m x^n(M_E/r)$ which add up with a net coefficient of $-\frac{5}{3}$. The results are

$$h_{00}^{(I)}=\frac{2M_E}{r}-\frac{2M_E{}^2}{r^2}-\frac{5}{3}\frac{M_E}{r}U_{e,mn}x^m x^n \;. \tag{38}$$

A similar calculation shows that

$$h_{0i}^{(I)}=0,\quad h_{ij}^{(I)}=\frac{2M_E}{r}\delta_{ij} \;. \tag{39}$$

These results may now be combined with the external contributions, Eqs. (24)–(27), to yield the following expressions for the full metric tensor in the Fermi frame:

$$\begin{aligned}g_{00}=&-1+2[U^{(e)}(s,x^i)-U^{(e)}(s,0)-x^m U^{(e)}_{,m}(s,0)]\\&-\delta r_{0m0n}x^m x^n+\frac{2M_E}{r}-\frac{2M_E{}^2}{r^2}\\&-\frac{5}{3}\frac{M_E}{r}U_{e,mn}x^m x^n \;,\end{aligned} \tag{40}$$

$$g_{0i}=-\tfrac{2}{3}r_{0min}x^m x^n \;, \tag{41}$$

$$g_{ij}=\delta_{ij}\left[1+\frac{2M_E}{r}\right]-\tfrac{1}{3}r_{imjn}x^m x^n \;. \tag{42}$$

Equations (40)–(42), in particular the expression for g_{00}, accomplish our main aim, to show that in the generalized Fermi frame the external bodies affect the local metric only in two ways: first through generalized tidal forces of order Mr^2/R^3, with relativistic corrections, and through a nonlinear interaction proportional to the product of the local Newtonian potential, M_e/r and the Newtonian tidal potential, $U_{e,mn}x^m x^n/2$. The external potential term has disappeared due to rescaling of lengths and times; also the gradients of the external potential have disappeared as a result of the transformation to the freely falling system. The gravitational field of Earth, which in this frame is at rest, is described by the Schwarzschild solution in the appropriate approximation.

Such conclusions are very plausible on physical grounds; on such grounds in AB we wrote the local metric in the form of Eqs. (40)–(42), except for the nonlinear interaction term [the last term in Eq. (38)] which was computed by solving the field equations to the required order. There it was assumed that the only external body was the Sun, so the nonlinear interaction term took the form

$$\delta h_{00}=-\frac{10}{3}\frac{M_E}{r}\frac{M}{R}\frac{3(\hat{\mathbf{R}}\cdot\mathbf{r})^2-r^2}{2R^2} \;, \tag{43}$$

where $\hat{\mathbf{R}}$ is a unit vector, in barycentric coordinates, from the Sun to Earth (see Fig. 2). Damour has kindly pointed out to the authors that this interaction term may also be obtained from the results of Manasse[15] who constructed normal Fermi coordinates on a test body falling radially inward toward the source of a Schwarzschild field.

V. CHOICE OF THE COORDINATE TIME SCALE

In order to have a model which is as realistic as possible, it is desirable to introduce one further transformation: a change in coordinate time scale. Consider an equipotential surface at mean sea level (the geoid) on the rotating Earth,[20] and let us choose a standard clock at rest on this surface, having time s', as a reference. Then relabel the surfaces $s=\text{const}$ with the label s'. This entails the coordinate transformation $(s,x^i)\to(s'(s),x'^i=x^i)$ and generates

$$g'_{00}=g_{00}\left[\frac{ds}{ds'}\right]^2\equiv g_{00}(1+\Delta)\ ,\qquad g'_{0i}=g_{0i}\left[\frac{ds}{ds'}\right]=g_{0i}\left[1+\frac{\Delta}{2}\right]\ . \tag{44}$$

The correction factor Δ is, as we shall see, of $O(V^2)$; then the vector component g_{0i} is effectively unchanged. Let

$$v'=\omega r\sin\theta \tag{45}$$

be the proper velocity of the clock with respect to s'; θ is the colatitude and ω the angular velocity of rotation. The demand that s' be the time coordinate x'^0 means

$$g_{00}(1+\Delta)+g_{0i}v'^i+g_{ij}v'^iv'^j=-1\ . \tag{46}$$

In the determination of the correction factor Δ we shall neglect terms of $O(V^4)$; hence Eq. (46) simplifies to

$$\Delta=h_{00}+(v')^2=h_{00}+\omega^2r^2\sin^2\theta\ . \tag{47}$$

Strictly speaking we should use here the metric of a rotating body, e.g., Kerr's solution; but in our linear approximation the net result is that the local potential is corrected by including quadrupole and higher multipole corrections.

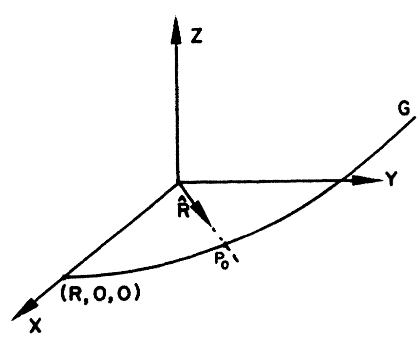

FIG. 2. Computation of curvature tensor using symmetry arguments. X,Y,Z are isotropic rectangular coordinates. $\hat{\mathbf{R}}$ is a unit vector in the XY plane pointing along the line of sight from the Sun to a point on the base geodesic. $\mathbf{V}$ is the coordinate velocity.

The local potential $U^{(l)}$ is therefore of the form

$$U^{(l)}=\frac{M_E}{r}[1-J_2(a_1/r)^2P_2(\cos\theta)+\cdots]\ , \tag{48}$$

where $J_2\approx10^{-3}$ is the quadrupole moment coefficient of Earth, a_1 is Earth's equatorial radius, and P_2 is the Legendre polynomial of degree 2 (Ref. 21). Such angular functions are evaluated in a frame rotating with the angular velocity ω.

We then get from Eqs. (47) and (27)

$$\Delta=2U^{(l)}+\omega^2r^2\sin^2\theta+\frac{M}{R}(r/R)^2P_2(\cos\alpha)\ , \tag{49}$$

where α is the angle between the radius vector to the clock and the direction of the Sun. The last term in Eq. (49) is due to the solar tidal potential; this is a small correction of order 10^{-17} and can be neglected. Equation (49) has to be evaluated on an equipotential surface, taking into account also the centrifugal potential:

$$U^{(l)}+\tfrac{1}{2}\omega^2r^2\sin^2\theta\equiv U_0=\text{const} \tag{50}$$

so that

$$\Delta=2U_0\ . \tag{51}$$

The correction factor Δ is constant and equal to $2U_0\simeq1.5\times10^{-9}$.

We now discuss the International Astronomical Union's definition of barycentric dynamical time[22] (BDT). We have so far two physical times: the coordinate time X^0, measured by a standard clock at rest at infinity, and the proper time s', measured by a standard clock at rest on the rotating geoid. To obtain the ratio of their rates to $O(V^2)$, let us write the line element ds' of the geoid clock in the barycentric frame of reference. In this frame, to the required order, the velocity of this clock is the sum of the velocity $\mathbf{V}_E$ of Earth, and the velocity $\mathbf{v}'$ of the clock with respect to Earth's center [Eq. (45)]. Using also the value of Δ [Eq. (51)] and neglecting tidal corrections we get

$$\frac{dX^0}{ds'}=1+U_e+\tfrac{1}{2}V_E{}^2+\tfrac{1}{2}\Delta+\mathbf{V}_E\cdot\mathbf{v}'\ . \tag{52}$$

We see that the main contributions to this ratio are comprised mainly of the sum of a constant part, a part which varies with the period of nearly a day, and a part with the period of a year (because of the eccentricity of Earth's orbit). BDT is obtained from s' by suppressing, with a long-time average, these periodic terms:

$$\frac{dX^0}{dX^0_{\text{BDT}}}=1+\langle U_e+V_E{}^2/2+\Delta/2+\mathbf{V}_E\cdot\mathbf{v}'\rangle\ . \tag{53}$$

This new time X^0_{BDT} differs from the asymptotic time X^0 only by a constant rate difference, which to a good approximation is given by

$$\langle U_e+V_E{}^2/2+\Delta/2\rangle\simeq1.55\times10^{-8}\ . \tag{54}$$

This gives, in seconds of the asymptotic clock, the duration of a BDT second. Since in cosmic physics distances are measured by means of transit times with a value of light speed fixed by convention, the unit of length is not independent. This difference must be kept in mind when a physical quantity, such as GM_E/c^2, can be measured both in the local frame through its effect on satellites, or in the barycentric frame through solar-system dynamics. In the first case it will be measured in SI meters, the SI standard of length being in free fall. In the second case the BDT meter is used and in such units GM_E has a slightly smaller numerical value. This situation has been discussed by Hellings.[6]

VI. CURVATURE TENSOR CONTRIBUTIONS FOR A TWO-BODY SYSTEM

To complete the evaluation of Eqs. (40)—(42) for a simplified solar-system model, we compute the curvature tensor at the instantaneous position of Earth. This will be done assuming only one external body—the Sun—is present. In this section the mass of Earth is treated as sufficiently small that the Sun may be assumed to remain at rest at the origin. In terms of the usual Schwarzschild metric, with independent variables $X^0=T$, R, Θ, and Φ,

$$-ds^2=-(1-2M/R)(dX^0)^2+\frac{dR^2}{1-2M/R}+R^2d\Omega^2 , \tag{55}$$

the nonvanishing components of the curvature tensor are[3]

$$\begin{aligned} &R_{TRTR}=2M/R^3,\quad R_{T\Theta T\Theta}=-(M/R)(1-2M/R) ,\\ &R_{R\Theta R\Theta}=(M/R)/(1-2M/R) ,\\ &R_{T\Phi T\Phi}=-(M\sin^2\Theta/R)(1-2M/R) ,\\ &R_{R\Phi R\Phi}=(M\sin^2\Theta/R)/(1-2M/R) ,\\ &R_{\Theta\Phi\Theta\Phi}=-2MR\sin^2\Theta . \end{aligned} \tag{56}$$

It is first necessary to transform these to isotropic coordinates, in which a new radial coordinate R' is defined by[23]

$$R=R'(1+M/2R')^2=R'+M+O(M^2/R') . \tag{57}$$

Keeping only terms which are linear and quadratic in M, and dropping primes, we have

$$\begin{aligned} &R_{TRTR}=(2M/R)(1-3M/R) ,\\ &R_{T\Theta T\Theta}=(-M/R)(1-3M/R) ,\\ &R_{T\Phi T\Phi}=(-M\sin^2\Theta/R)(1-3M/R) ,\\ &R_{R\Theta R\Theta}=(M/R)(1+M/R) ,\\ &R_{R\Phi R\Phi}=(M\sin^2\Theta/R)(1+M/R) ,\\ &R_{\Theta\Phi\Theta\Phi}=-2MR\sin^2\Theta(1+M/R) . \end{aligned} \tag{58}$$

Referring to Fig. 2, let X,Y,Z be rectangular (isotropic) Cartesian coordinates, and consider the point whose coordinates are $(R,0,0)$. At this point, in terms of these isotropic coordinates the values of the curvature tensor specialize to

$$\begin{aligned} &R_{TXTX}=(2M/R^3)(1-3M/R) ,\\ &R_{TYTY}=R_{TZTZ}=(-M/R^3)(1-3M/R) ,\\ &R_{XYXY}=R_{XZXZ}=(M/R^3)(1+M/R) ,\\ &R_{YZYZ}=(-2M/R^3)(1+M/R) . \end{aligned} \tag{59}$$

The curvature tensor at an arbitrary point in the XY plane can then be obtained by the following symmetry arguments. If $\hat{\mathbf{R}}=(X,Y,0)/R$ is a unit vector pointing from the origin (where the Sun is located) to the point $(X,Y,0)$ on Earth's world line, then R_{TiTj} can only have the form

$$R_{TiTj}=\alpha\delta_{ij}+\beta\hat{\mathbf{R}}_i\hat{\mathbf{R}}_j , \tag{60}$$

where α and β are scalar quantities. At the particular value $\hat{\mathbf{R}}=(1,0,0)$, we have

$$\begin{aligned} &R_{TXTX}=\alpha+\beta=(2M/R^3)(1-3M/R) ,\\ &R_{TYTY}=\alpha=(-M/R^3)(1-3M/R) . \end{aligned} \tag{61}$$

These equations can be solved for α and β and hence R_{TiTj} must read

$$\begin{aligned} R_{TiTj}=&(-M/R^3)(1-3M/R)\delta_{ij}\\ &+(3M/R^3)(1-3M/R)\hat{\mathbf{R}}_i\hat{\mathbf{R}}_j . \end{aligned} \tag{62}$$

The only other nonzero components of the curvature tensor in the XY plane are R_{ijpq}. Using the known symmetry properties of the curvature tensor and the fact that the only available vector is $\hat{\mathbf{R}}$, it can be seen that the only possible form for R_{ijpq} is

$$R_{ijpq}=(M/R^3)(1+M/R)[\gamma(\delta_{ip}\delta_{jq}-\delta_{jp}\delta_{iq})+\delta(\delta_{ip}\hat{\mathbf{R}}_j\hat{\mathbf{R}}_q-\delta_{jp}\hat{\mathbf{R}}_i\hat{\mathbf{R}}_q+\delta_{jq}\hat{\mathbf{R}}_i\hat{\mathbf{R}}_p-\delta_{iq}\hat{\mathbf{R}}_j\hat{\mathbf{R}}_p)] , \tag{63}$$

where γ and δ are scalars. At the pont $(R,0,0)$, the curvature tensor components from Eq. (63) specialize to the values

$$R_{XYXY}=(M/R^3)(1+M/R)(\gamma+\delta),\quad R_{YZYZ}=(M/R^3)(1+M/R)\gamma . \tag{64}$$

Upon comparing with Eq. (59), we find $\gamma=-2$, $\delta=3$, and thus

$$R_{ijpq}=-(M/R^3)(1+M/R)[2(\delta_{ip}\delta_{jq}-\delta_{jp}\delta_{iq})-3(\delta_{ip}\hat{\mathbf{R}}_j\hat{\mathbf{R}}_q-\delta_{jp}\hat{\mathbf{R}}_i\hat{\mathbf{R}}_q+\delta_{jq}\hat{\mathbf{R}}_i\hat{\mathbf{R}}_p-\delta_{iq}\hat{\mathbf{R}}_j\hat{\mathbf{R}}_p)] . \tag{65}$$

We now transform to local inertial coordinates, with the transformation coefficients along G given by Eq. (9):

$$\left.\frac{\partial X^\mu}{\partial x^\nu}\right|_G=\Lambda^\mu_{(\nu)} . \tag{66}$$

We need to evaluate the quadrupole sums in Eq. (21). We shall drop the subscript E on V_E in the remainder of this section. Upon introducing the abbreviations

$$V_R=\mathbf{V}\cdot\hat{\mathbf{R}},\ \ \rho_R=\mathbf{r}\cdot\hat{\mathbf{R}},\ \ \tau_V=\mathbf{r}\cdot\mathbf{V}/V\ , \tag{67}$$

we find, after some calculation,

$$r_{0m0n}=\frac{M}{R^3}\left[(-\delta_{mn}+3\hat{\mathbf{R}}_m\hat{\mathbf{R}}_n)\left[1-\frac{3M}{R}+V^2\right]+V^2(-2\delta_{mn}+3\hat{\mathbf{R}}_m\hat{\mathbf{R}}_n)+3(\delta_{mn}V_R{}^2+V_mV_n)\right.$$
$$\left.-\tfrac{9}{2}V_R(\hat{\mathbf{R}}_mV_n+V_m\hat{\mathbf{R}}_n)+3(\hat{\mathbf{R}}_m\hat{\mathbf{R}}_i\Omega^{in}+R_nR_i\Omega^{im})\right]\ . \tag{68}$$

Equation (21) may now be used to evaluate $g_{00}^{(e)}$. The result is

$$g_{00}^{(e)}=-1+\frac{M}{R^3}\left[\left[1-\frac{3M}{R}+V^2\right](3\rho_R{}^2-r^2)+V^2(-2r^2+3\rho_R{}^2+3\tau_V{}^2)\right.$$
$$\left.-9VV_R\rho_R\tau_V+3V_R{}^2r^2-6R_i\Omega^{im}x^j\delta_{jm}\right]\ . \tag{69}$$

The quantities ρ_R and τ_V used above differ from the quantities ρ and τ to be used in Sec. V because the reference directions of the XYZ coordinate system do not undergo geodetic precession. This is the source of the term involving Ω^{ij} in Eq. (69), above.

In a similar manner, the remaining components of the external contributions to the metric tensor in the local frame are obtained by performing the sums in Eqs. (25) and (26). The results are

$$g_{0i}^{(e)}=-\frac{2M}{R^3}[V_i(r^2-2\rho_R{}^2)+x_i(V_R\rho_R-V\tau_V)+\hat{\mathbf{R}}_i(2V\rho_R\tau_V-3V_Rr^2)]\ , \tag{70}$$

$$g_{ij}^{(e)}=\delta_{ij}\left[1-\frac{M}{3R^3}(2r^2-3\rho_R{}^2)\right]+\frac{M}{3R^3}[2x_ix_j+3r^2\hat{\mathbf{R}}_i\hat{\mathbf{R}}_j-3\rho_R(x_i\hat{\mathbf{R}}_j+\hat{\mathbf{R}}_ix_j)]\ . \tag{71}$$

The local contributions and Earth-Sun interaction terms are as given in Eqs. (38) and (39). The metric tensor we have obtained may thus be expressed in the following final form:

$$g_{00}=g_{00}^{(e)}+\frac{2m}{r}-\frac{2m^2}{r^2}-\frac{5}{3}\frac{m}{r}\frac{M}{R^3}(3\rho_R{}^2-r^2)\ , \tag{72}$$

$$g_{0i}=g_{0i}^{(e)}\ , \tag{73}$$

$$g_{ij}=g_{ij}^{(e)}+\delta_{ij}\frac{2m}{r}\ . \tag{74}$$

This completes the formal proof of the metric expression already given in AB (Ref. 4).

Effect of other nearby bodies. The treatment given above has regarded Earth as a point mass and has ignored the effects of the planets. These may be taken into account to a first approximation by replacing the solar tidal potential by the sum of the tidal potentials due to all such bodies. Since these effects are small, it is only necessary to work in a classical approximation to correct the tidal part of $h_{00}^{(e)}$ for such effects.

VII. SPECIAL CASE—TWO BODIES IN CIRCULAR ORBITS

As a special case of the Fermi frame construction we consider a model two-body system consisting of masses m and M (the Sun) revolving around a common barycenter in circular orbits. This model problem admits of a solution for the components of the vierbein which is valid for all times, thus it is not necessary to expand the expressions for the vierbein into symmetric and antisymmetric parts, the latter representing rotation. The discussion is therefore not limited to a number of orbital revolutions not larger than $1/(V_E)^2$, as in the previous sections; moreover, the ratio m/M is here kept arbitrary. The material of this section may thus be useful in other contexts.

The first step in the calculation is to obtain a solution of the geodesic equations of motion, in barycentric coordinates, for the world line G of the center of the freely falling body m, the origin of local inertial coordinates. The equations to be solved are

$$\frac{d^2X^\mu}{ds^2}+\Gamma^\mu_{\alpha\beta}\frac{dX^\alpha}{ds}\frac{dX^\beta}{ds}=0\ , \tag{75}$$

where the Christoffel symbols of the second kind are computed neglecting Earth's self-interaction terms which would be infinite for a point Earth. As discussed previously, such infinite terms are omitted in the construction of the coordinate transformations.

An approximate solution to these equations is obtained by assuming the Sun and Earth travel in circles around a mutual center of mass. It is found that, correct to $O(V^4)$, the barycenter is at the location which would have been expected classically. Thus if R is the distance between the masses, the position of mass m in the orbital plane (X^1,X^2), is given by

$$\mathbf{X}_E=\frac{MR}{M_T}(\cos\Omega X^0,\sin\Omega X^0,0)\ , \tag{76}$$

where we use the symbol M_T for the total mass, $M_T=M+m$, and where the Sun-Earth line has been assumed to lie along the X^1 axis at the initial instant. The coordinate angular rotation frequency Ω obtained by solving Eq. (75) is

$$\Omega^2 = M_T(1-3M/R-2m/R-m^2/RM_T)/R^3 \ . \tag{77}$$

Equation (77) expresses Kepler's third law of motion in the coordinate system used here.

One further result is needed, which is the expression for $X^0(s)$, the elapsed coordinate time in terms of the elapsed proper time s on a standard clock falling along the world line G. We find, to $O(V^4)$,

$$\frac{dX^0}{ds} = \text{const} \equiv K \ , \tag{78}$$

where

$$K^2 = 1 + \frac{3M}{R} - \frac{Mm}{RM_T} + \frac{6M^2}{R^2} - \frac{2Mm}{R^2} + \frac{3Mm^2}{R^2M_T} \ . \tag{79}$$

This result may be obtained conveniently from Eq. (A7), or from the solutions given in Eqs. (76) and (77) above and the expression for ds^2 in terms of the metric tensor in barycentric coordinates, Eqs. (2) and (3), in which the sum over A has only one term, corresponding to $M_A=M$ for the Sun.

The second step is the construction of the vierbein, which is carried by parallel transport along the geodesic G. The equations to be solved are

$$\frac{d\Lambda^\mu_{(i)}}{ds} + \Gamma^\mu_{\alpha\beta}\Lambda^\alpha_{(i)}\frac{dX^\beta}{ds} = 0 \ . \tag{80}$$

The vectors of the vierbein are basis vectors at the origin of the local inertial frame. The vector $\Lambda^\mu_{(0)}$, chosen to be tangent to the base geodesic G, is

$$\Lambda^\mu_{(0)} = \frac{dX^\mu}{ds} = K[1, -(R\Omega M/M_T)\sin\Omega X^0, (R\Omega M/M_T)\cos\Omega X^0, 0] \ . \tag{81}$$

At $x^0=0$ the other three vectors point radially outward along the Sun-Earth line, normally to the Sun-Earth line in the orbital plane, and normally to the orbital plane, respectively. It is convenient to express these solutions in terms of the proper time s given by Eqs. (78) and (79), to be used as the time coordinate x^0 in the local inertial frame. If we then write $nx^0=\Omega X^0$ at P_0, we have

$$n^2 = \Omega^2K^2 = M_T(1-3m/R)/R^3 \ . \tag{82}$$

Defining

$$k^2 = M_T(1-3M_T/R-3Mm/RM_T) \ , \tag{83}$$

the solutions for the other three members of the vierbein can be obtained for arbitrary x^0 and written in component form as follows:

$$\Lambda^\mu_{(1)}(x^0) = \left[-\frac{R\Omega M}{M_T}(1+7M/2R+4m/R-Mm/2RM_T)\sin kx^0, C_1\cos nx^0\cos kx^0 + C_2\sin nx^0\sin kx^0 \ , \right. \left. C_1\sin nx^0\cos kx^0 - C_2\cos nx^0\sin kx^0, 0 \right] , \tag{84}$$

$$\Lambda^\mu_{(2)}(x^0) = \left[\frac{R\Omega M}{M_T}(1+7M/2R+4m/R-Mm/2RM_T)\cos kx^0, -C_2\sin nx^0\cos kx^0 + C_1\cos nx^0\sin kx^0 \ , \right. \left. C_2\cos nx^0\cos kx^0 + C_1\sin nx^0\sin kx^0, 0 \right] , \tag{85}$$

$$\Lambda^\mu_{(3)}(x^0) = (0,0,0,C_1) \ , \tag{86}$$

where the constants C_1 and C_2 are given by

$$C_1 = 1-M/R, \quad C_2 = 1-M/2R-Mm/2RM_T \ . \tag{87}$$

These vectors satisfy the orthogonality and normalization conditions given by Eq. (12).

The vectors of the vierbein undergo slow geodetic precession with respect to fixed directions in the barycentric reference frame, at a rate determined by the difference between k and n:

$$2\pi(n-k)/n = 3\pi M(1+m/M_T)/R \simeq 19\times 10^{-3} \text{ arcsec/yr} \tag{88}$$

for axes centered at Earth.

The coefficients C_1 and C_2 differ from unity because of a rescaling of coordinate distances in the barycentric reference frame which is necessary in order that x^k measure proper lengths. C_2 differs from C_1 because of the Lorentz contraction of lengths oriented parallel to the direction of the orbital velocity of the mass m.

In the third step of the calculation, construction of the transformations to locally inertial coordinates, it is necessary to solve the equations for the geodesic S by means of the Taylor expansion, Eq. (9). Cubic terms in x^k are necessary in order to compute the transformation coefficients correctly including quadratic terms; quartic terms are needed for any application involving the field equations. The Taylor expansion coefficients in Eq. (9) above are needed to $O(V^3)$ for $\mu=0$, but only to $O(V^2)$ for $\mu=1,2,3$. The resulting coordinate transformations are most compactly expressed in terms of the quantities

$$\rho = x^1\cos kx^0 + x^2\sin kx^0 \ , \tag{89}$$

$$\tau = -x^1\sin kx^0 + x^2\cos kx^0 \ . \tag{90}$$

These linear combinations of the Fermi coordinates correspond to Cartesian components of the proper distance along S from the origin to the observation point. The distance ρ is measured along a line radially outward from the Sun, and τ is measured normal to this line in the orbital plane of Earth. The quantity k rather than n enters into these definitions because of geodetic precession of the vierbein.

The resulting coordinate transformations are as follows up to quartic order:

$$X^0=Kx^0+\frac{R\Omega M}{M_T}(1+7M/2R+4m/R-Mm/2RM_T)\tau-\frac{R\Omega M}{R^2}\rho\tau(1+m/M_T)$$
$$+\frac{R\Omega M}{3R^3}\tau[(1+m/M_T)(3\rho^2-r^2)+mr^2/2M_T]+\frac{R\Omega M}{4R^4}\rho\tau[3r^2-5\rho^2+(2r^2-5\rho^2)m/M_T]\,, \quad (91)$$

$$X^1=\frac{RM}{M_T}\cos nx^0+C_1\rho\cos nx^0-C_2\tau\sin nx^0+\frac{M}{2R^2}[(2\rho^2-r^2)\cos nx^0-2\rho\tau\sin nx^0]$$
$$+\frac{M}{6R^3}[x^1(r^2-6\rho^2)+3r^2\rho\cos nx^0]+\frac{M}{8R^4}[r^2(r^2-5\rho^2)\cos nx^0-2x^1\rho(2r^2-5\rho^2)]\,, \quad (92)$$

$$X^2=\frac{RM}{M_T}\sin nx^0+C_1\rho\sin nx^0+C_2\tau\cos nx^0+\frac{M}{2R^2}[(2\rho^2-r^2)\sin nx^0+2\rho\tau\cos nx^0]$$
$$+\frac{M}{6R^3}[x^2(r^2-6\rho^2)+3r^2\rho\sin nx^0]+\frac{M}{8R^4}[r^2(r^2-5\rho^2)\sin nx^0-2x^2\rho(2r^2-5\rho^2)]\,, \quad (93)$$

$$X^3=C_1x^3+M\rho x^3/R^2+Mx^3(r^2-6\rho^2)/6R^3+M[-x^3\rho(2r^2-5\rho^2)]/4R^4\,. \quad (94)$$

Having obtained coordinate transformations by the above prescription, gauge freedom allows one to apply them to the computation of the metric tensor in the new coordinate system without reference to the construction procedure. The metric so obtained by the standard tensor transformation law, such as Eq. (11), must then be properly interpreted according to the conventions of GR.

It is now a straightforward but tedious calculation to obtain the transformation coefficients $\partial X^\alpha/\partial x^\mu$ and carry out the summations required to evaluate $g_{\mu\nu}$. In performing this calculation, it is also necessary to express the metric coefficients $G_{\alpha\beta}$ in terms of the new coordinates x^μ and to expand the resulting functions in powers of the small parameters M/R, m/r, r/R.

In the following equations we list the resulting components of the metric tensor, using transformation coefficients derived from Eqs. (91)–(94). Terms of $O(V^2)$, $O(V^4)$, $O(V^2r^2/R^2)$, and $O(V^4r^2/R^2)$ are retained in g_{00}:

$$g_{00}=-1+\frac{2m}{r}-\frac{2m^2}{r^2}+\frac{2Mr^2}{R^3}P_2-\frac{2Mr^3}{R^4}P_3$$
$$-\frac{10}{3}\frac{Mmr}{R^3}P_2+\frac{7}{2}\frac{Mmr^2}{R^4}P_3+\frac{Mmr^2}{R^4}P_2$$
$$-\frac{3(M^2+Mm)}{R^4}(\rho^2-\tau^2)\,, \quad (95)$$

where P_2 and P_3 are Legendre polynomials,[21] of degrees 2 and 3, which are functions of ρ/r. Higher-order Newtonian solar tides are also present and should be included, as was done in Sec. III. The leading contributions to g_{0i} and g_{ij} are given by

$$g_{01}=\frac{2M}{R^3}R\Omega[(x^3)^2\sin nx^0-(x^2)\rho]\,, \quad (96)$$

$$g_{02}=\frac{2M}{R^3}R\Omega[-(x^3)^2\cos nx^0+(x^1)\rho]\,, \quad (97)$$

$$g_{03}=\frac{2M}{R^3}R\Omega[(x^3)\tau]\,, \quad (98)$$

$$g_{11}=1+\frac{2m}{r}+\frac{M}{3R^3}[(x^2)^2+(x^3)^2(1-3\sin^2 nx^0)]$$
$$-\frac{M\rho}{2R^4}[(x^2)^2+(x^3)^2(1-5\sin^2 nx^0)]\,, \quad (99)$$

$$g_{12}=\frac{M}{3R^3}[-x^1x^2+3(x^3)^2\cos nx^0\sin nx^0]$$
$$+\frac{M\rho}{2R^4}[x^1x^2-5(x^3)^2\cos nx^0\sin nx^0]\,, \quad (100)$$

$$g_{22}=1+\frac{2m}{r}+\frac{M}{3R^3}[(x^1)^2+(x^3)^2(1-3\cos^2 nx^0)]$$
$$-\frac{M\rho}{2R^4}[(x^1)^2+(x^3)^2(1-5\cos^2 nx^0)]\,, \quad (101)$$

$$g_{13}=\frac{2}{3}\frac{Mx^1x^3}{R^3}-\frac{Mx^3\rho\cos nx^0}{R^3}$$
$$+\frac{M}{2R^4}(5x^3\rho^2\cos nx^0-4x^1x^3\rho)\,, \quad (102)$$

$$g_{23}=\frac{2}{3}\frac{Mx^2x^3}{R^3}-\frac{Mx^3\rho\sin nx^0}{R^3}$$
$$+\frac{M}{2R^4}(5x^3\rho^2\sin nx^0-4x^2x^3\rho)\,, \quad (103)$$

$$g_{33}=1+\frac{2m}{r}+\frac{M}{3R^3}(\rho^2-2\tau^2)-\frac{M\rho}{2R^4}(\rho^2-4\tau^2)\,. \quad (104)$$

It can be proved independently by substitution into the field equations, that Eqs. (95)—(104) provide approximate solutions to the field equations in the Fermi frame.

VIII. SUMMARY AND CONCLUSIONS

In this paper we have studied the relationship between barycentric coordinates and local inertial coordinates with origin at the center of one of the bodies of interest for a system of mass points, by explicitly constructing the coordinate transformation required to change one's point of view from one frame to the other. A very large number of cancellations among relativistic effects occurs upon transforming to the local frame. The results may be summarized by saying that, in the local frame, the presence of the local mass m whose center is at the origin may be described adequately by the Schwarzschild solution including a nonlinear term in the square of m/r which gives rise to precession of the perigee of satellite orbits. The axes of the local frame undergo geodetic precession. In addition to Newtonian tidal forces due to distant masses, there are also much smaller relativistic tidal corrections and nonlinear interaction terms in g_{00} which are of the form of products of Earth's gravitational potential and the tidal potential.

The best tracked artificial satellite is the Laser Geodynamics Satellite (LAGEOS), whose orbit can be determined to within a few cm accuracy corresponding to a perturbation δH in the Hamiltonian of order $10^{-9}m/r$. At this level the only relevant relativistic correction to the orbit is the secular advance of the perigee; its observability is made difficult by the small eccentricity of LAGEOS, but a long-term orbital integration should bring this secular perturbation out of the noise. The relativistic precession of the spatial axes of the local inertial frame of reference should also be observable, if directions determined by LAGEOS are checked against distant sources by means of very long baseline interferometry.

As explained in AB (Ref. 4), the other relativistic corrections can be classed in three groups according to their orders of magnitude dependence on the distance. Nonlinear interactions ($\delta H \approx Mmr/R^3$) and "magnetic" terms due to the motion of the Sun [$\delta H \approx (Mr^2/R^3)(Mm/Rr)^{1/2}$] do affect the lunar motion by a few cm. Such perturbations will be difficult to observe because they occur at the same frequencies as the ordinary tidal perturbations. We hope to discuss these effects in a future paper.

We did not consider in our work the effect of the moon on the local Fermi frame. More generally, one would like to have a theory of the motion of several local bodies. We expect that in this case the central world line G may be placed at the center of any one of the masses of interest.

We conclude by suggesting that it would be useful to develop appropriate software for description of the motion of Earth's satellites in a generalized Fermi frame as discussed here, with relativistic corrections included. Because of the invariant character of these coordinates and the great simplicity of the relativistic corrections in this formulation the physical interpretation of the results of a numerical integration will be much easier.

Note added in proof. It is interesting that the Eddington-Clark form of the metric can be written in a simple form by introducing a modified mass

$$M_A' = M_A\left[1+3V_A{}^2/2-\sum{}' M_B/R_{AB}\right]$$

and a modified retarded potential

$$U'=\sum_A M_A'/R_A\,|_{\rm ret} = M_A'\left[1-\frac{1}{2}R_A\frac{\partial^2 R_A}{(\partial X^0)^2}\right]\Big/ R_A\ .$$

The retarded value includes, of course, the velocity contraction factor familiar from electron theory. With these definitions we can write

$$G_{00} = -1+2U'-2(U')^2+O(V^6)\ .$$

The external part of G_{00} is then obtained by replacing U' with

$$U'^{(e)} = \sum_{A\neq E} M_A'/R_A\,|_{\rm ret}\ .$$

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APPENDIX

In this appendix we give the explicit construction of a transformation of coordinates which can be interpreted as the transformation to a local Fermi frame for an arbitrary point-mass metric. The origin of this reference frame falls along a base geodesic G which is the solution of the geodesic equations of motion of a freely falling test particle in a suitably chosen "renormalized" external metric. Specifically, this "external" metric contains all the terms in the complete metric except those which would become undefined or singular when the metric is evaluated at the position of the local mass M_E, at whose center we wish to construct the Fermi frame's origin. The modified metric includes nonlinear interaction terms depending on M_E, but which are well defined.

The external part of the metric is given in Eqs. (5)—(7). The geodesic equations of motion of the mass M_E in this metric can be written in the following form when X^0 is used as the independent variable:

$$\frac{d^2X^k}{(dX^0)^2}+\Gamma^k_{00}+2\Gamma^k_{0i}V_E^i+\Gamma^k_{ij}V_E^iV_E^j - V_E^k(\Gamma^0_{00}+2\Gamma^0_{0i}V_E^i)=0\ , \quad (A1)$$

where the Christoffel symbols are calculated using the external metric. Only terms of $O(V^4)$ are retained. Using $\mathbf{X}_{EA}=\mathbf{X}_E-\mathbf{X}_A$, this leads to the following equations of motion:

$$A_E^k=-\sum_A{}'\frac{M_A X_{EA}^k}{R_{EA}{}^3}\left[1-4U_e+V_E{}^2-\sum_B{}'\frac{M_B}{R_{BA}}+2V_A{}^2-4\mathbf{V}_E\cdot\mathbf{V}_A-\frac{3}{2}\frac{(\mathbf{X}_{EA}\cdot\mathbf{V}_A)^2}{R_{EA}{}^2}-\tfrac{1}{2}\mathbf{X}_{EA}\cdot\mathbf{A}_A\right]$$
$$+4\sum_A{}'\frac{M_A(\mathbf{X}_{EA}\cdot\mathbf{V}_E)(V_E^k-V_A^k)}{R_{EA}{}^3}-3\sum_A{}'\frac{M_A(\mathbf{X}_{EA}\cdot\mathbf{V}_A)(V_E^k-V_A^k)}{R_{EA}{}^3}+\frac{7}{2}\sum_A{}'\frac{M_A A_A^k}{R_{EA}}\ , \tag{A2}$$

where

$$U_e=U^{(e)}\Big|_{\mathbf{R}=\mathbf{R}_E} \tag{A3}$$

is the negative of the gravitational potential due to external sources, evaluated at the position of M_E on G. These equations, and similar equations of motion for the other bodies, have been derived by Moyer,[8] Estabrook,[24] Will,[25] and others. They are needed here for calculation of $\mathbf{A}_E\cdot\mathbf{r}$ to $O(V^4)$.

We may suppose that solutions to such equations of motion have been found—e.g., by numerical integration. Then the positions, velocities, and accelerations will be known as functions of coordinate time. These functions determine the base geodesic G, along which an orthonormal tetrad of basis vectors $\Lambda^\mu_{(\alpha)}$ (a vierbein) is generated by parallel transport. If we set $K\equiv dX^0/ds$ on G, then to quadratic order in the spatial coordinates x^k of the Fermi frame we have

$$X^0=\int_{s_0}^{s}K\,ds+\Lambda^0_{(i)}x^i-\tfrac{1}{2}\Gamma^0_{\alpha\beta}\Lambda^\alpha_{(i)}\Lambda^\beta_{(j)}x^ix^j\ , \tag{A4}$$

$$X^k=X_E^k[X^0(P_0)]+\Lambda^k_{(i)}x^i-\tfrac{1}{2}\Gamma^k_{\alpha\beta}\Lambda^\alpha_{(i)}\Lambda^\beta_{(j)}x^ix^j\ . \tag{A5}$$

The coordinate time x^0 in the local frame is taken to be $x^0=s$, the proper time elapsed from some reference point, along the base geodesic G of the external metric. Of course if the mass M_E were actually present, such a proper time would be impossible to realize. This indicates that, as discussed in Sec. V, an additional time transformation will have to be applied in order to obtain a local coordinate time which corresponds to physical reality.

We now proceed with the computation of the various terms in the coordinate transformations, Eqs. (A4) and (A5). A self-consistent calculation of the vierbein vectors requires that $\Lambda^0_{(0)}$ be calculated to $O(V^4)$, $\Lambda^k_{(0)}$ and $\Lambda^0_{(k)}$ to $O(V^3)$, and $\Lambda^k_{(j)}$ to $O(V^2)$. The vector $\Lambda^\mu_{(0)}$ is defined in terms of the timelike four-velocity of the base geodesic G:

$$\Lambda^\mu_{(0)}=\frac{dX^\mu}{ds}=K[1,V_E^k] \tag{A6}$$

and K is evaluated to $O(V^4)$ on G:

$$K^{-2}=ds^2/(dX^0)^2=1-2U_e+2(U_e)^2-4\sum_A{}'\frac{M_AV_A{}^2}{R_{EA}}+2\sum_A{}'\frac{M_A}{R_{EA}}\sum_B{}'\frac{M_B}{R_{AB}}+\sum_A{}'\frac{M_A(\mathbf{X}_{EA}\cdot\mathbf{A}_A)}{R_{EA}}$$
$$+\sum_A{}'\frac{M_A(\mathbf{X}_{EA}\cdot\mathbf{V}_A)^2}{R_{EA}{}^3}+8\sum_A{}'\frac{M_A\mathbf{V}_E\cdot\mathbf{V}_A}{R_{EA}}-V_E{}^2-2U_eV_E{}^2\ . \tag{A7}$$

Thus the member of the vierbein which is tangent to G is expressed to the required order in terms of quantities which may be assumed known.

The components $\Lambda^0_{(k)}$ may be obtained from the orthogonality condition, Eq. (10), leading to

$$\Lambda^0_{(k)}=-G^{(e)}_{0k}\,|_G+(1+4U_e)V_E^i\Lambda^j_{(k)}\delta_{ij}\ . \tag{A8}$$

To complete this calculation we need $\Lambda^j_{(k)}$. The equation of parallel transport, using coordinate time as the independent variable, may be expressed as

$$\frac{d\Lambda^j_{(k)}}{dX^0}+\Gamma^j_{00}\Lambda^0_{(k)}+\Gamma^j_{0i}\Lambda^0_{(k)}V_E^i+\Gamma^j_{0i}\Lambda^i_{(k)}+\Gamma^j_{il}\Lambda^i_{(k)}V_E^l=0\ . \tag{A9}$$

Each term in this equation must be calculated to $O(V^3)$. It is straightforward to verify that the equations may be solved by setting

$$\Lambda^j_{(k)}=\delta^j_k(1-U_e)+\tfrac{1}{2}V_E^jV_E^k+\Omega^{jk}\ , \tag{A10}$$

where Ω^{jk} satisfies the differential equation

$$\frac{d\Omega^{jk}}{dX^0}=\Omega^{jk}_{,0}=\tfrac{1}{2}(G^{(e)}_{0j,k}-G^{(e)}_{0j,k})+\tfrac{3}{2}(A_E^jV_E^k-A_E^kV_E^j)\ , \tag{A11}$$

with $G^{(e)}_{0j,k}$ evaluated at $\mathbf{X}=\mathbf{X}_E$ on G. The computation of the coordinate transformation equations may now be performed, with the following results:

$$X^0=\int K\,ds+\mathbf{V}_E\cdot\mathbf{r}(1+3U_e+\tfrac{1}{2}V_E{}^2)+G^{(e)}_{0k}x^k$$
$$+V_E^j\Omega^{km}x^n\delta_{jk}\delta_{mn}+(\mathbf{V}_E\cdot\mathbf{r})(\mathbf{A}_E\cdot\mathbf{r})$$
$$+2\sum_A{}'\frac{M_A(\mathbf{V}_A\cdot\mathbf{r})(\mathbf{X}_{EA}\cdot\mathbf{r})}{R_{EA}{}^3}+\frac{r^2}{2}\sum_A{}'\frac{M_A\mathbf{V}_A\cdot\mathbf{X}_{EA}}{R_{EA}{}^3}\ , \tag{A12}$$

$$X^k=R_E^k+x^k(1-U_e-\mathbf{A}_E\cdot\mathbf{r}-\tfrac{1}{6}U_{e,mn}x^mx^n)$$
$$+\tfrac{1}{2}V_E^k(\mathbf{V}_E\cdot\mathbf{r})+\Omega^{kj}\delta_{jm}x^m+\tfrac{1}{2}r^2A_E^k\ . \tag{A13}$$

The coordinate transformations in Eqs. (A12) and (A13), given to cubic order in r/R, permit the evaluation of transformation coefficients $\partial X^\mu/\partial x^\nu$ to quadratic order in

x^k. Then they would allow one to find by direct computation the metric $g_{\mu\nu}$ to quadratic order. A consequence of such computations is that one may verify in detail that no terms linear in x^k will appear in the metric of the local frame. Although we have verified this result, we shall not give details here because, for the external contributions to the metric, the disappearance of such terms is a consequence of Fermi's theorem[2] and is to be expected. In Sec. III we showed how these coordinate transformations, applied to the local part of the metric, give the expected contributions from local sources in the approximate Schwarzschild form.

Although the coordinate transformation given in Eqs. (A12) and (A13) was constructed in a manner which depended on the external part of the metric under consideration, in one sense this is irrelevant. As GR is a generally covariant theory under arbitrary coordinate transformations, upon applying the given coordinate transformation to a metric tensor which is a solution of the field equations, one must obtain another solution to the field equations. The new solution obtained in this way is particularly simple to interpret as it is close to a local inertial frame with a mass M_E at the origin. A test particle placed near the origin will be acted on by gravitational forces—both linear and nonlinear—due to M_E itself, but the effect of distant bodies is to be found only in the tidal forces.

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