Simple model for the inflationary expansion of the early Universe

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The evolution of the early Universe is studied under the following assumptions. (1) The large-scale structure of the Universe is homogeneous and isotropic and governed by the Robertson-Walker metric in flat space with a spatial scale factor $a(t)$. (ii) Matter can be described by a quantum-mechanical N-vector model with a single- or double-well potential. (iii) The system is initially in thermal equilibrium at some high temperature T relative to the Planck temperature. Quantitative progress is possible in the large-N limit where the phase structure of the N-vector model is known exactly. The evolution of the system is characterized by three time regimes. For the earliest times the main effect of gravity is to rapidly decrease the kinetic energy of the system, while the potential energy and the local "order"-parameter fluctuation $S(t)$ change very little. Thus the system is driven far from equilibrium. There is rapid growth of $a(t)$ during this period. One enters the intermediate time period after $a(t)$ is sufficiently large that the spectrum of the order-parameter correlation function "freezes" due to red-shifting. By this time the energy density ϵ is dominated by the potential energy and one enters a de Sitter phase where, approximately, the pressure $P=-\epsilon$ and $a(t)$ is growing exponentially. In this regime the Hubble constant and $S(t)$ are decaying exponentially with precisely the same slow rate. The amount of inflation depends most strongly on the initial temperature [$\ln a \approx (T/T_P)^2$] where T_P is the Planck temperature. In the final stage the system crosses over to behavior in agreement with the standard model. Except for the "critical" case where the quadratic part of the potential vanishes, the system appears to be "matter" dominated: $a(t) \approx t^{2/3}$ and the system grows the ordered state associated with a spontaneously broken symmetry if the quadratic part of the potential is negative. In the critical case the system appears to be radiation dominated and $a(t) \approx t^{1/2}$.

I. INTRODUCTION

The evolution of matter driven by gravity after the big bang raises intriguing but demanding questions. Assuming that the large-scale properties of the Universe are homogeneous and isotropic,[1] the structure of Einstein's equation becomes relatively simple. If, in addition, one ignores spatial fluctuations in the stress-energy tensor and assumes local equilibrium, one can postulate various equations of state which serve to close the set of equations governing the evolution of the early Universe. Assuming a "matter-" or "radiation"-dominated universe, one is led to a version of the "standard model."[2] In a very important paper, Guth[3] pointed out that if one weakens the assumption of local equilibrium and allows for the possibility of a phase transition and associated metastability as the effective temperature is lowered, one could be led to an inflationary regime [exponential growth of the Robertson-Walker (RW) scale factor] which helps to explain several puzzles associated with the standard model. While there were certain problems with the inflationary scenario developed by Guth, it opened up a new way of thinking about these problems. Efforts to improve upon Guth's work have led to the so-called "new" inflationary scenario.[4] These works, however, were built upon several untested hypothesis: the ignoring of spatial fluctuation at early times, the metastable nature of the dynamics and the establishment of local equilibrium. These assumptions were questioned by Mazenko, Unruh, and Wald.[5] The new inflationary scenario also leads to energy density perturbations much larger than is compatible with known constraints inferred from galaxy formation.[6]

It has subsequently been realized by several groups[7-9] that it is unnecessary to argue about "scenarios" if one goes back to the initial-value problem coupling matter to Einstein's equation. In principle this direct approach will answer questions concerning the role of fluctuations, establishment of equilibrium, nature of the spontaneous symmetry breaking, etc. Thus one can test the assumptions made in the standard model and in the inflationary scenarios. This type of approach is complicated by the necessity of treating a field theory in a strongly nonequilibrium situation. Thus, while one would like to treat the case of gauge field theories, such as SU(5), coupled to gravity, this is, at present, not within our capabilities. A more modest but still constructive first step is to argue, as suggested by several authors, that the important coupling is between gravity and the Higgs field which is, in turn, coupled to the gauge fields and as a first approximation one can average over the gauge fields and treat an effective Hamiltonian for the Higgs field. The point of view developed here is that one wants to study the simplest field theory which is robust enough to include the possibility of a phase transition and symmetry breaking. The large-N limit of the N-vector model satisfies these conditions. The model can be solved exactly in thermal equili-

brium and has a nontrivial phase structure. This model, when coupled to gravity, leads to a dynamical system which is tractable enough to be treated numerically and, in certain limits, analytically.

The model developed here was discussed earlier[8] in a somewhat simplified form. In the analysis here the system is quantized, while the earlier work was carried out using a classical cutoff chosen to simulate the associated quantum-mechanical problem. It was suggested there that this procedure was probably adequate for estimating the most important features of the problem. It is shown here that this is indeed true and the explicit role of quantum effects will be pointed out. The method of quantization used here is the simplest and most direct procedure which ignores the existence of horizons.

The assumption will be made throughout this paper that the system is initially in thermal equilibrium. This is not a necessary assumption within the context of the model, nor is it the physically most satisfying assumption. It does seem, however, to be a logical starting point. If the initial temperature is very high relative to any other energy scale, then the equilibrium state, in the sense of the most random state, is not unreasonable. Assumptions of local equilibrium with an initial temperature comparable to some ordering temperature, and the associated long-range ordering, does not make good physical sense, although it can be treated within the context of the model. From a formal point of view the assumption of initial equilibrium allows one to avoid some technically difficult questions associated with treating initial states not generated by a Lagrangian.[10] Nonetheless it is interesting and important to consider other possible initial states. Indeed the "ball rolling down the hill" scenario advocated by a number of authors does not result in our model except in the case where the system is prepared in a nonequilibrium initial state where the level of order-parameter fluctuations are constrained to be much less than in the associated thermal state. As yet there is not apparent motivation for such constraints.

The main results of this paper are in accord with those of Ref. 8. The analysis in Ref. 8 was based primarily on the numerical solution of the dynamical equations of motion satisfied by the various correlation functions entering the theory. In the present work most of the conclusions are supported by analytical calculations as well as numerical analysis. The dynamical evolution of the system, assumed to be initially in thermal equilibrium at some very high temperature T, can be divided into three distinct regimes. The dynamics of the earliest time regime are dominated by the existence of two distinct and quite different characteristic times. One characteristic time τ_g governs the rate of decay of the kinetic energy and is inversely proportional to the initially quite large Hubble "constant" $H(t)$. The second time scale τ_ϕ governs the rate of decay of "order-parameter" or matter fluctuations and is very large relative to τ_g: $\tau_\phi/\tau_g \approx (T/T_P)^2$ where T_P is the Planck temperature. Thus in the earliest time regime the main effect of gravity is to rapidly decrease the kinetic energy of the system, while the potential energy and the local order-parameter fluctuation, $S(t)$, change very little. During this point the energy density $\epsilon(t)$ changes from being dominated by kinetic energy to being primarily potential energy Thus while the Hubble constant drops rapidly during this regime, it is still sufficiently large to drive a very rapid increase in the RW scale factor $a(t)$.

The second time regime is entered (i) after there has been sufficient red-shifting [$a(t)$ is large enough] that the momentum distributions of the various correlation functions "freeze" and (ii) the energy density is dominated by the potential energy. These effects occur at about the same time and define the crossover region between regime one and two. While the crossover region must be treated numerically, once in regime two the analysis can be carried out analytically if the initial temperature is large relative to the critical temperature. In that case one finds that the Hubble constant is proportional to $S(t)$, and both are decaying exponentially with a rate proportional to $\sqrt{U}/T$ were U is the quartic coupling in the potential. This regime is a de Sitter phase in the following sense. For progressively larger values of T one obtains longer periods over which the potential dominates kinetic energy and the pressure $P=-\epsilon$, to a good approximation. One has therefore, to the same approximation, as shown below, exponential growth of $a(t)$. Since, however, $H(t)$ is slowly decaying exponentially, one does not quite have exponential growth of the scale factor. One does, however, obtain the same qualitative result—a regime where the scale factor can change many e foldings over a short period of time. One finds that the number of e foldings is proportional to $(T/T_P)^2$—a new result. The behavior described above is essentially independent of the choice of parameters of the potential with the following reservations: If the coefficient of the quadratic term in the potential is positive and the quadratic coefficient is small or zero then $H^2(t)$ is proportional to $S(t)$. In this case the kinetic energy is constant during this phase of the evolution, while for the case discussed above, the kinetic energy is proportional to $S(t)$. In both cases one obtains $P=-\epsilon$ and considerable inflation.

At some point the order-parameter fluctuations drop to a level where the potential and kinetic energy are again comparable—the system begins to come to local equilibrium. This part of the evolution depends on the choice of the potential. If the potential has a double-well structure then the system will have undergone a first-order phase transition and the initial continuous symmetry of the problem will be broken—the system will have grown domains of the new ordered phase. The characteristic size of the domains goes as $a(t)$ and the value of the order parameter within each domain takes on the value associated with long-range order: $\langle\phi\rangle^2=-r/u$ where r is the negative quadratic coefficient in the potential and u is the positive quartic coefficient. In this longest-time regime one can again solve the model analytically. If r is not equal to zero, the long-time dynamics corresponds to a matter-dominated standard model: $a(t)\approx t^{2/3}$ and $P\approx 0$ (actually P oscillates sinusoidally with time and it is the time average which is zero). For the "critical" case, $r=0$, one is led to a radiation-dominated long-time regime where $a(t)\approx t^{1/2}$ and $P/\epsilon=\frac{1}{3}$. In all cases the energy spectrum is governed by essentially a Planck distribution.

II. MODEL STUDIED

A. General development

The basic model studied is Einstein's equation[11]

$$G_{ab}=8\pi G T_{ab} \tag{2.1}$$

which couples the metric tensor g_{ab} to the stress-energy tensor T_{ab}, which depends on matter fields ϕ_i. On the left-hand side of (2.1),

$$G_{ab}=R_{ab}-\tfrac{1}{2}Rg_{ab} \tag{2.2}$$

is the Einstein tensor, R_{ab} is the Ricci tensor, and

$$R=g^{ab}R_{ab} \tag{2.3}$$

is the curvature scalar. On the right-hand side of (2.1), G is the gravitational constant and the stress-energy density tensor T_{ab} assumed to be of the form

$$T_{ab}=\nabla_a\phi_i\nabla_b\phi_i-\tfrac{1}{2}g_{ab}[\nabla_c\phi_i\nabla^c\phi_i+2V(\phi)] , \tag{2.4}$$

where the index i labels the N components of the matter field ϕ_i, and ∇_a indicates the covariant derivative. $V(\phi)$ is a potential function specified below.

The equation of motion governing the fields is given by the conservation law

$$\nabla^a T_{ab}=0 . \tag{2.5}$$

One obtains immediately that

$$\nabla^a\nabla_a\phi_i=\partial V(\phi)/\partial\phi_i . \tag{2.6}$$

The invariant d'Alembertian can be evaluated in the form

$$\nabla^a\nabla_a\phi_i=g^{ab}\frac{\partial^2\phi_i}{\partial x^a\partial x^b}-\Gamma^a\frac{\partial\phi_i}{\partial x^a} , \tag{2.7}$$

where

$$\Gamma^a=g^{bc}\Gamma^a_{bc} \tag{2.8}$$

and Γ^a_{bc} is the affine connection.

This set of equations, supplemented with initial conditions and a quantization prescription, form a closed set. However, they are completely intractable. Progress can be made if one assumes that one is interested in the largest-scale properties of the Universe and that the initial conditions can be specified in terms of some average distribution for the fields ϕ_i. With this assumption one can replace the stress-energy tensor T_{ab} with its average $\langle T_{ab}\rangle$ in Einstein's equation (2.1). If the initial probability distribution is homogeneous and isotropic, so also will $\langle T_{ab}\rangle$ be homogeneous and isotropic. It then follows, using standard arguments, that the metric tensor can be expressed in the Robertson-Walker form (for flat space)[12]

$$g_{ab}=\delta_{ab}[-\delta_{a0}+a^2(t)\delta_{ai}] , \tag{2.9}$$

where 0 indicates the timelike index, i is one of the three spacelike indices, and $a(t)$ is the RW scale factor. In this metric it is easy to work out the various quantities entering equations (2.1)—(2.8):

$$\Gamma^a=3H(t)\delta_{a0} , \tag{2.10}$$

where the Hubble "constant" is defined by

$$H(t)=\dot{a}(t)/a(t) , \tag{2.11}$$

$$R_{ab}=\delta_{ab}[2(\dot{a}^2+a\ddot{a})\delta_{ai}-3(\dot{a}/a)\delta_{a0}] , \tag{2.12}$$

$$R=6\dot{a}/a+6H^2 , \tag{2.13}$$

$$G_{00}=3H^2 , \tag{2.14}$$

$$G_{ii}=12a\ddot{a}-\dot{a}^2 , \tag{2.15}$$

and the off-diagonal components of G_{ab} vanish. The stress-energy tensor in this case can be set in the general form

$$\langle T_{ab}\rangle=\delta_{ab}(\epsilon\delta_{a0}+Pa^2\delta_{ai}) , \tag{2.16}$$

where ϵ is the average energy density and P is the pressure. Einstein's equation immediately reduces to the set

$$3H^2=8\pi G\epsilon , \tag{2.17}$$

$$-2a\ddot{a}-\dot{a}^2=8\pi GPa^2 . \tag{2.18}$$

It is straightforward to show that (2.18), using (2.17), can be recast in the form

$$\dot{\epsilon}=-3(\dot{a}/a)(\epsilon+P) \tag{2.19}$$

or

$$\frac{d}{dt}(\epsilon a^3)=-P\frac{d}{dt}a^3 \tag{2.20}$$

which is in the form of the first law of thermodynamics. This last equation is redundant as long as the conservation law (2.5) is satisfied.

The matter field ϕ_i evolves according to the equation of motion given by (2.6), with the invariant d'Alembertian given for the RW metric by

$$\nabla^a\nabla_a\phi_i=-\ddot{\phi}_i+a^{-2}\nabla^2\phi_i-3H\dot{\phi}_i . \tag{2.21}$$

One can imagine setting the solution of the equation of motion for ϕ_i as a function of time back into T_{ab} which is then averaged over the initial distribution. The resulting $\langle T_{ab}\rangle$ is then inserted into Einstein's equation (2.17) to determine $H(t)$. This is a highly nonlinear process since $a(t)$ and $H(t)$ enter into the field equation (2.6) for ϕ_i.

In principle one could go around this loop one more time by allowing for fluctuations in the metric tensor away from the RW form. This would lead to correction terms for the Ricci tensor as well as generating new terms in the equation of motion for the matter fields. Such a development seems possible, but takes one well beyond the current understanding of the problem even in the case of the RW metric.

Given the expression for the stress-energy tensor (2.4) and suppressing the index i which is summed over, one easily finds that the energy density and pressure are given by

$$\epsilon=\langle T_{00}\rangle=\tfrac{1}{2}\langle(\dot{\phi})^2\rangle+\tfrac{1}{2}a^{-2}\langle(\nabla\phi)^2\rangle+\langle V(\phi)\rangle \tag{2.22}$$

and

$$P=\frac{1}{3a^2}\left\langle\sum_{i=1}^{3}T_{ii}\right\rangle \tag{2.23}$$

$$= \tfrac{1}{2}\langle(\dot{\phi})^2\rangle - \frac{1}{6a^2}\langle(\nabla\phi)^2\rangle - \langle V(\phi)\rangle \ , \tag{2.24}$$

and the sum of the energy density and pressure is given in terms of the average gradient and kinetic energies

$$P+\epsilon = \frac{1}{3a^2}\langle(\nabla\phi)^2\rangle + \langle(\dot{\phi})^2\rangle \ . \tag{2.25}$$

With these definitions it is easy to show that the "first law" (2.20) is satisfied by ϵ, P, and a.

The development thus far is valid for a general N-component self-interacting field theory. Assume that the potential term is of the form[13]

$$V(\phi) = \frac{1}{4uN}\left[rN + u\sum_{i=1}^{N}\phi_i^2(\mathbf{x})\right]^2 , \tag{2.26}$$

where u is assumed positive and r may be positive or negative. For r negative this is the double-well potential which has served as the prototypical example for demonstrating spontaneous symmetry breaking in a wide variety of circumstances. By allowing the number of components of the field N to be variable one is able to treat a wide variety of models with "order parameters" with different topological structures. Within the context of critical phenomena,[14] field theories with different values of N fall into different universality classes. $N=1$ corresponds to the Ising-type universality class, $N=2$ the x-y universality class, and $N=3$ the Heisenberg universality class.

Given this potential, the equation of motion satisfied by the field is given by

$$\ddot{\phi}_i = a^{-2}\nabla^2\phi_i - 3H\dot{\phi}_i - \left[r + \frac{u}{N}\sum_{j=1}^{N}\phi_j^2\right]\phi_i \ . \tag{2.27}$$

Thus far the matter field ϕ_i has been treated classically. It is not very difficult to quantize the theory (within the RW metric). Define the second quantized "Hamiltonian" operator

$$\mathcal{H}(t) = \int d^3x\left[\sum_{i=1}^{N}\left[\frac{1}{2\alpha(t)}\tilde{\pi}_i^2(\mathbf{x}) + \frac{\alpha(t)}{2a^2(t)}[\nabla\phi_i(\mathbf{x})]^2\right] + \alpha(t)V(\phi)\right] , \tag{2.28}$$

where ϕ_i is now an operator and $\tilde{\pi}_i$ is the canonically conjugate momentum density. ϕ and $\tilde{\pi}$ satisfy the equal-time canonical commutation relations

$$[\tilde{\pi}_i(\mathbf{x},t),\phi_j(\mathbf{x}',t)] = -i\hbar\delta_{ij}\delta(\mathbf{x}-\mathbf{x}') \ . \tag{2.29}$$

$\mathcal{H}$ then generates the equations of motion

$$\dot{\phi}_i = (i/\hbar)[\mathcal{H},\phi_i] = \tilde{\pi}_i/\alpha(t) \tag{2.30}$$

and

$$\dot{\tilde{\pi}}_i = (i/\hbar)[\mathcal{H},\tilde{\pi}_i] = \alpha(t)\left[a^{-2}(t)\nabla^2\phi_i - \left[r + \frac{u}{N}\sum_{j=1}^{N}\phi_j^2\right]\phi_i\right] . \tag{2.31}$$

Choosing the factor $\alpha(t)$ such that

$$\frac{\dot{\alpha}}{\alpha} = 3H$$

or

$$\alpha(t) = a^3(t) \ , \tag{2.32}$$

and eliminating $\tilde{\pi}_i$ between (2.30) and (2.31), one regains (2.27). It is then easy to see, with the expression for the stress-energy tensor given by (2.4), evaluated in the RW metric, that

$$\langle\mathcal{H}(t)\rangle = \int a^3 d^3x\langle T_{00}\rangle \ . \tag{2.33}$$

Introducing the auxiliary quantity

$$\pi_i(\mathbf{x},t) = \tilde{\pi}_i(\mathbf{x},t)/a^3(t) \ , \tag{2.34}$$

the equations of motion can be set into the form

$$\dot{\phi}_i = \pi_i \tag{2.35}$$

and

$$\dot{\pi}_i = -3H\pi_i + a^{-2}(t)\nabla^2\phi_i - \left[r + \frac{u}{N}\sum_{j=1}^{N}\phi_j^2\right]\phi_i \ . \tag{2.36}$$

This set of equations is identical in form to those studied in Ref. 8. Note also that the energy component of the fluctuating stress-energy tensor can be set in the form

$$\epsilon(\mathbf{x},t) = T_{00} = \tfrac{1}{2}\pi_i^2 + \tfrac{1}{2}a^{-2}(\nabla\phi_i)^2 + V(\phi) \ . \tag{2.37}$$

One must now begin to deal with the questions of cutoffs and renormalization. This takes one immediately into the questions of initial and boundary conditions. These questions are difficult to sort out and can lead to enormous technical complications. Consider first a well-posed set of assumptions, and then one can consider how these might be generalized and extended. Assume that the system is initially in equilibrium with a probability distribution (density matrix) of the Boltzmann form

$$P[\phi] \approx e^{-\beta\mathcal{H}[\phi]} \tag{2.38}$$

where β is the inverse temperature characterizing the initial state. All of the averages $\langle\ \rangle$ discussed above are assumed to be over $P[\phi]$. From a technical point of view this is a very convenient choice since the initial state is constructed in terms of the same Hamiltonian (and Lagrangian) that drives the subsequent dynamics. Technically speaking the advantage here is that the construction of perturbation theory in powers of the nonlinear coupling u can be carried out simultaneously for the initial state and the subsequent dynamical evolution; in particular the graphical structure of the equilibrium theory will be identical to the dynamical analysis. Treatment of more general initial conditions is clearly possible, but one must be very careful. Even in the much less complicated cases of

dynamic evolution from a general initial state in the absence of gravity, one finds a number of technical difficulties not found in standard field theory.[10] In particular one is forced to treat many-body interactions generated by arbitrary initial conditions which are not found in the case discussed above. This point is worthy of further study.

If one is restricted to an initial state which is thermal, then one mechanism for renormalizing the system is immediately apparent. It is well known that the equilibrium properties of the system governed by $\mathscr{H}$ can be regularized by normal ordering. As is spelled out in more detail below, this leads one to a Planck distribution in the noninteracting case ($u=0$, $r>0$). It is clear (at least in the large-N limit), if one regulates the initial conditions and then follows the subsequent dynamic evolution generated by the equations of motion, that one generates no new singularities. The normal ordering is assumed only for the initial thermal state. Except for the reservation pointed out below, this seems a natural solution to the regularization problem and this prescription will be followed in all of the work below.

The reservation referred to above many be a serious one. Thus far boundary conditions have been ignored. There may be very important constraints put on the theory which are associated with the existence of horizons and fundamental constraints on the degrees of freedom "participating" in the theory. Such constraints will not be enforced here, but this may not be the most physical approach.

The problem of interest is now well posed and one can proceed to develop the appropriate perturbation theory expansion. In general this requires introducing Green's functions of the type[15]

$$G_{ij}(\mathbf{x}t,\mathbf{x}'t')=\langle\tau[\phi_i(\mathbf{x},t)\phi_j(\mathbf{x}',t')]\rangle\ , \tag{2.39}$$

where τ indicates a properly chosen time-ordering operator, and the other required Green's functions involve the fields π_i. This general development is formally well understood, but, because of the lack of time translational invariance, technically very involved. Fortunately, for the case of interest here, this general development is not needed.

B. The large-N limit

For reasons which will become clear below, the analysis can be restricted to the study of equal-time correlation functions of the form

$$C_{ij}(\mathbf{x},\mathbf{x}',t)=\langle\phi_i(\mathbf{x},t)\phi_j(\mathbf{x}',t)\rangle\ , \tag{2.40}$$

$$D_{ij}(\mathbf{x},\mathbf{x}',t)=\langle\pi_i(\mathbf{x},t)\phi_j(\mathbf{x}',t)\rangle\ , \tag{2.41}$$

$$\tilde{D}_{ij}(\mathbf{x},\mathbf{x}',t)=\langle\phi_i(\mathbf{x},t)\pi_j(\mathbf{x}',t)\rangle\ , \tag{2.42}$$

$$G_{ij}(\mathbf{x},\mathbf{x}',t)=\langle\pi_i(\mathbf{x},t)\pi_j(\mathbf{x}',t)\rangle\ . \tag{2.43}$$

A key point to realize is that the energy density can be "almost" expressed in terms of these objects. The average of (2.37) can be written in the form

$$\epsilon(t)=\tfrac{1}{2}\sum_{i=1}^{N}G_{ii}(\mathbf{x},\mathbf{x},t)+\tfrac{1}{2}\sum_{i=1}^{N}[\nabla\cdot\nabla' C_{ii}(\mathbf{x},\mathbf{x}',t)]\,|_{\mathbf{x}=\mathbf{x}'}+\langle V(\phi)\rangle \tag{2.44}$$

and from (2.26)

$$\langle V(\phi)\rangle=\frac{Nr^2}{4u}+\frac{1}{2}r\sum_{i=1}^{N}C_{ii}(\mathbf{x},\mathbf{x},t)+\frac{u}{4N}\sum_{i=1}^{N}\sum_{j=1}^{N}\langle\phi_i{}^2(\mathbf{x},t)\phi_j{}^2(\mathbf{x},t)\rangle \tag{2.45}$$

and it is only the term of $O(u)$ in (2.45) which cannot be expressed in terms of the two-point correlation functions.

Without approximations or developing perturbation theory in u one can go no further. As mentioned above it will not be necessary to develop the full perturbation theory. The reason is that the analysis will be restricted to the large-N limit. This type of approximation has played an important role in "cracking" a number of technically difficult problems. Examples in condensed matter physics abound and are growing: See the work of Wilson,[16] Ma,[17] and Abe[18] for critical phenomena, Oppermann and Wegner[19] in the theory of localization, Coleman[20] in the theory of the Kondo problem, and Mazenko and Zannetti[21] in the theory of growth kinetics. They key points are the following. (i) Problems are, in many cases, solvable in this limit (gauge field theories[22] is an exception). (ii) The resulting theory shows a nontrivial phase structure—a phase transition with spontaneous symmetry breaking, Nambu-Goldstone modes, etc. (iii) Corrections to the leading behavior can be systematically worked out.[23,24] A main deficiency of the model is that the defect structure which may be important (vortices, etc.) are lost in the analysis. It will be assumed in the rest of this work that the limiting theory is sensible.

In taking the large-N limit one must organize things properly if the limit is to be well behaved. This organization has already been carried out in the way the potential $V(\phi)$ has been defined. The assumption there is that $\langle V\rangle$ is of $O(N)$ if the bare quartic coefficient (u/N) is of $O(N^{-1})$. This factor of $1/N$ was included explicitly in defining the quartic coupling. If the theory is to be sensibly defined in the large-N limit, one must make the same type of assumption concerning the other coupling in the problem G. The point is that the energy density will be of $O(N)$ for large N and, just as in the case of the quartic term, one wants to take the large-N limit in such a way that the energy per internal component is fixed. This can be organized in the following way. Suppose that the physical system of interest has N_0 components and a gravitational coupling G. One then has the combination $G\epsilon(N_0)$ entering Einstein's equation. This can be rewritten in the form $G(N_0/N_0)\epsilon(N_0)$. One can then generalize this model to the case of N components through the choice $G(N_0/N)\epsilon(N)$. Thus one has an effective gravitational constant which goes as $1/N$ for large N and the limit $N\to\infty$ is well posed. One can then rewrite Einstein's equation in the form

$$3H^2=8\pi\tilde{G}\epsilon/N\ , \tag{2.46}$$

where $\tilde{G}=GN_0$ is assumed to be independent of N in the large-N limit.

Before proceeding one needs a bit of further development. If $P[\phi]$ and $\mathscr{H}[\phi]$ are isotropic and the system is symmetric with respect to the internal space labeled by the index i, then one has the result

$$C_{ij}(\mathbf{x},\mathbf{x}',t)=\delta_{ij}C(\mathbf{x},\mathbf{x}',t)\ , \tag{2.47}$$

where

$$C(\mathbf{x},\mathbf{x}',t)=\langle\phi_i(\mathbf{x},t)\phi_i(\mathbf{x}',t)\rangle \tag{2.48}$$

is independent of the particular value of i unless the internal symmetry is broken (more about this below). Similarly since one has translational invariance,

$$C(\mathbf{x},\mathbf{x}',t)=C(\mathbf{x}-\mathbf{x}',t)\ . \tag{2.49}$$

It will be convenient to define

$$S(t)=C(\mathbf{x},\mathbf{x},t) \tag{2.50}$$

in terms of Fourier transforms

$$C(\mathbf{q},t)=\int d^3x e^{i\mathbf{q}(\mathbf{x}-\mathbf{x}')}C(\mathbf{x}-\mathbf{x}',t) \tag{2.51}$$

and

$$S(t)=\int\frac{d^3q}{(2\pi)^3}C(\mathbf{q},t)\ . \tag{2.52}$$

Defining the analogous transforms of D, $\tilde{D}$, and G, it is convenient to express the kinetic energy (to within a factor of $\frac{1}{2}$) as

$$G(t)=\int\frac{d^3q}{(2\pi)^3}G(\mathbf{q},t) \tag{2.53}$$

and, the gradient energy as

$$K(t)=\int\frac{d^3q}{(2\pi)^3}q^2C(\mathbf{q},t)$$
$$=[\nabla\cdot\nabla' C(\mathbf{x},\mathbf{x}',t)]\,|_{\mathbf{x}=\mathbf{x}'}\ . \tag{2.54}$$

The nature of the large-N limit can be understood by an analysis of the graphical structure of the perturbation theory expansion in powers of u. Each vertex introduces a factor of $1/N$, while each closed loop generates a factor of N due to the internal summation. Thus, the self-energy graph in Fig. 1(a) is of $O(1)$, while those in Fig. 1(b) are all of $O(1/N)$. One can develop an algebraic approach[24] to obtain a systematic expansion in powers of $1/N$. At $O(1)$ the basic point is that correlation functions factorize into products of two-point correlation functions. A key result for the purposes here is that

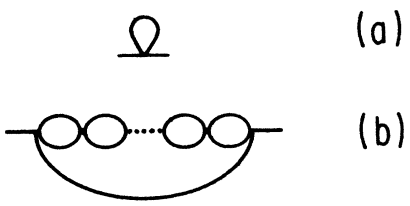

FIG. 1. Feynman graphs contributing to the self-energy. The graph labeled (a) is of $O(1)$ and gives the leading contribution as $N\to\infty$. Graphs of the type labeled (b) are of $O(1/N)$ and give the first-order correction.

$$\sum_{i=1}^{N}\sum_{j=1}^{N}\langle\phi_i{}^2(\mathbf{x},t)\phi_j{}^2(\mathbf{x}',t)\rangle=N^2C(\mathbf{x},\mathbf{x},t)C(\mathbf{x}',\mathbf{x}',t)$$
$$=N^2S^2(t) \tag{2.55}$$

plus corrections of higher order in $1/N$. One then has from (2.44) that

$$\lim_{N\to\infty}\epsilon(t)/N=\tfrac{1}{2}G(t)+\tfrac{1}{2}K(t)+v(t)\ , \tag{2.56}$$

where

$$v(t)=\lim_{N\to\infty}\langle V(\phi)\rangle/N=\frac{1}{4u}[r+uS(t)]^2\ . \tag{2.57}$$

One then wants to determine $G(t)$, $K(t)$, and $S(t)$ using the "equations of motion" for $C(\mathbf{q},t)$ and $G(\mathbf{q},t)$. For general N one has, using (2.35) and (2.36), that

$$\dot{C}_{ij}(\mathbf{x},\mathbf{x}',t)=D_{ij}(\mathbf{x},\mathbf{x}',t)+\tilde{D}_{ij}(\mathbf{x},\mathbf{x}',t)\ , \tag{2.58}$$

$$\dot{\tilde{D}}_{ij}(\mathbf{x},\mathbf{x},t)=-3H(t)\tilde{D}_{ij}(\mathbf{x},\mathbf{x}',t)+(a^{-2}\nabla^2-r)C_{ij}(\mathbf{x},\mathbf{x}',t)-(u/N)\sum_{k=1}^{N}\langle\phi_k{}^2(\mathbf{x},t)\phi_i(\mathbf{x},t)\phi_j(\mathbf{x}',t)\rangle+G_{ij}(\mathbf{x},\mathbf{x},t)\ , \tag{2.59}$$

$$\dot{D}_{ij}(\mathbf{x},\mathbf{x}',t)=-3H(t)D_{ij}(\mathbf{x},\mathbf{x}',t)+(a^{-2}\nabla'^2-r)C_{ij}(\mathbf{x},\mathbf{x}',t)-(u/N)\sum_{k=1}^{N}\langle\phi_i(\mathbf{x},t)\phi_k{}^2(\mathbf{x},t)\phi_j(\mathbf{x}',t)\rangle+G_{ij}(\mathbf{x},\mathbf{x},t)\ , \tag{2.60}$$

and

$$\dot{G}_{ij}(\mathbf{x},\mathbf{x}',t)=-6H(t)G_{ij}(\mathbf{x},\mathbf{x}',t)+(a^{-2}\nabla'^2-r)D_{ij}(\mathbf{x},\mathbf{x}',t)+(a^{-2}\nabla^2-r)\tilde{D}_{ij}(\mathbf{x},\mathbf{x}',t)$$
$$-(u/N)\sum_{k=1}^{N}\langle\phi_k{}^2(\mathbf{x},t)\phi_i(\mathbf{x}',t)\pi_j(\mathbf{x}',t)\rangle-(u/N)\sum_{k=1}^{N}\langle\pi_i(\mathbf{x},t)\phi_k{}^2(\mathbf{x}',t)\phi_j(\mathbf{x}',t)\rangle\ . \tag{2.61}$$

Again in the large-N limit the higher-order correlation functions can be factorized to obtain after Fourier transformation, the equations of motion:

$$\dot{C}(q,t)=D(q,t)+\tilde{D}(q,t)\ , \tag{2.62}$$

$$\dot{D}(q,t)=-3H(t)D(q,t)-\Gamma(q,t)C(q,t)+G(q,t)\ , \tag{2.63}$$

$$\dot{\tilde{D}}(q,t)=-3H(t)\tilde{D}(q,t)-\Gamma(q,t)C(q,t)+G(q,t)\ , \tag{2.64}$$

$$\dot{G}(q,t)=-6H(t)G(q,t)-\Gamma(q,t)[D(q,t)+\tilde{D}(q,t)]\ , \tag{2.65}$$

$$\Gamma(q,t)=r+uS(t)+q^2/a^2 , \tag{2.66}$$

and the facts that the system is isotropic and that the correlation functions depend only on the magnitude of q have been used. These equations together with Einstein's equation (2.46) and the appropriate initial conditions form a closed-coupled set. The first step in analyzing these equations is to look at the set of initial conditions resulting from the assumption of equilibrium.

III. EQUILIBRIUM PROPERTIES

In order to specify the initial conditions and also to gain some feeling for the model, it is useful to work out the various equilibrium properties. The development from a more formal point of view is given by Dolan and Jackiw.[25] The analysis given here will be a bit more schematic, but also more direct. The basic point is that in equilibrium the effective equations of motion in the large-N limit can be written in the simplified form

$$\dot{\phi}_i=\pi_i \tag{3.1}$$

and

$$\dot{\pi}_i=-(r+uS-\nabla^2)\phi_i , \tag{3.2}$$

where S is the equilibrium limit of the quantity $S(t)$ introduced above,

$$S=\langle\phi_i^2\rangle , \tag{3.3}$$

which is independent of time due to the time-translational invariance of the equilibrium ensemble. The equation of motion for π can then be written, after Fourier transformation, in the form

$$\dot{\phi}_i(q,t)=\pi_i(q,t) \tag{3.4}$$

and

$$\dot{\pi}_i(q,t)=-\omega^2(q)\phi_i(q,t) , \tag{3.5}$$

where, with the assumed initial condition $a(0)=1$, $\omega^2(q)=r+uS+q^2$ is just the equilibrium limit of $\Gamma(q,t)$ defined by (2.66). With these results it is straightforward to work out the equilibrium quantities in the system. At this stage it will not lead to confusion if the index i on the operators is supressed. Introduce the creation and annihilation operators $a^\dagger(q)$ and $a(q)$ via

$$\pi(q)=i\left[\frac{\hbar\omega(q)}{2}\right]^{1/2}[a^\dagger(q)-a(q)] \tag{3.6}$$

and

$$\phi(q)=\left[\frac{1}{2\hbar\omega(q)}\right]^{1/2}[a^\dagger(q)+a(q)] \tag{3.7}$$

or

$$a^\dagger(q)=\left[\frac{1}{2\hbar\omega(q)}\right]^{1/2}[-i\pi(q)+\omega(q)\phi(q)] , \tag{3.8}$$

$$a(q)=\left[\frac{1}{2\hbar\omega(q)}\right]^{1/2}[i\pi(q)+\omega(q)\phi(q)] . \tag{3.9}$$

a and $a^\dagger$ satisfy the commutation relations

$$[a(q),a^\dagger(q')]=\delta(\mathbf{q}-\mathbf{q}') . \tag{3.10}$$

The equations of motion for a and $a^\dagger$, which follow from (3.4) and (3.5) are

$$\dot{a}(q)=-i\omega(q)a(q) \tag{3.11}$$

and

$$\dot{a}^\dagger(q)=+i\omega(q)a^\dagger(q) \tag{3.12}$$

and can, of course, be integrated immediately.

The theory in equilibrium is regulated in the standard way: normal order products of a and $a^\dagger$ and drop constant terms which are temperature independent and which would lead to infinities in various physical quantities. For example, for the correlation function $\langle\phi(q)\phi(q')\rangle$, one has

$$\begin{aligned}\langle\phi(q)\phi(q')\rangle&=-\frac{\hbar}{2\omega^{1/2}(q)\omega^{1/2}(q')}\\&\quad\times\langle[a^\dagger(q)-a(q)][a^\dagger(q')-a(q')]\rangle\\&=\frac{\hbar}{2\omega^{1/2}(q)\omega^{1/2}(q')}\langle a^\dagger(q)a(q')+a(q)a^\dagger(q')\rangle\\&=\frac{\hbar}{2\omega^{1/2}(q)\omega^{1/2}(q')}\langle a^\dagger(q')a(q)\rangle ,\end{aligned} \tag{3.13}$$

and the vacuum term was dropped. Similarly one finds that

$$\begin{aligned}\langle\pi(q)\pi(q')\rangle&=\tfrac{1}{2}\hbar\omega^{1/2}(q)\omega^{1/2}(q')\\&\quad\times\langle[a^\dagger(q)a(q')+a^\dagger(q')a(q)]\rangle\end{aligned} \tag{3.14}$$

and

$$\langle\pi(q)\phi(q')\rangle=\langle\phi(q)\pi(q')\rangle=0 . \tag{3.15}$$

Using the equations of motion (3.11) and (3.12), one easily finds (with time $-i\beta\hbar$)

$$\begin{aligned}a^\dagger(q,\beta)&=e^{-\beta\mathcal{H}}a^\dagger(q)e^{\beta\mathcal{H}}\\&=e^{-\beta\hbar\omega(q)}a^\dagger(q) .\end{aligned} \tag{3.16}$$

If one then considers

$$\mathrm{Tr}e^{-\beta\mathcal{H}}a^\dagger(q)a(q')=\mathrm{Tr}a^\dagger(\beta,q)e^{-\beta\mathcal{H}}a(q') , \tag{3.17}$$

one can use (3.16) and the cyclic invariance of the trace to obtain

$$\mathrm{Tr}e^{-\beta\mathcal{H}}a^\dagger(q)a(q')=e^{-\beta\hbar\omega(q)}\mathrm{Tr}e^{-\beta\mathcal{H}}a(q')a^\dagger(q) . \tag{3.18}$$

Using the commutation relations for a and $a^\dagger$ and regrouping terms, one is led to the familiar result

$$\langle a^\dagger(q)a(q')\rangle=n(q)\delta(\mathbf{q}+\mathbf{q}') , \tag{3.19}$$

where $n(q)$ is just the Planck distribution

$$n(q)=\frac{1}{e^{\beta\hbar\omega(q)}-1} . \tag{3.20}$$

The correlation functions of interest are given by

$$\langle\phi(\mathbf{q})\phi(\mathbf{q}')\rangle=C(q)\delta(\mathbf{q}+\mathbf{q}') , \tag{3.21}$$

$$\langle \pi(\mathbf{q})\pi(\mathbf{q}')\rangle = G(q)\delta(\mathbf{q}+\mathbf{q}') , \tag{3.22}$$

where

$$C(q) = \frac{\hbar}{\omega(q)} n(q) , \tag{3.23}$$

$$G(q) = \hbar\omega(q) n(q) , \tag{3.24}$$

and

$$D(q) = \tilde{D}(q) = 0 . \tag{3.25}$$

These equations appear to be simply those for a harmonic oscillator with frequency $\omega(q)$. Things are not quite so simple. Since $\omega^2(q) = r + uS + q^2$ and

$$S = \int \frac{d^3q}{(2\pi)^3} \frac{\hbar}{\omega(q)} n(q) , \tag{3.26}$$

S must be determined self-consistently. Once S is known then the correlation functions $C(q)$ and $G(q)$ are determined and the thermodynamic quantities like the energy density and pressure can be determined using

$$\epsilon/N = \frac{1}{2}\int \frac{d^3q}{(2\pi)^3}[G(q)+q^2C(q)] + \frac{1}{4u}(r+uS)^2 \tag{3.27}$$

and

$$P/N = \int \frac{d^3q}{(2\pi)^3}\left[\frac{1}{2}G(q) - \frac{q^2}{6}C(q)\right] - \frac{1}{4u}(r+uS)^2 . \tag{3.28}$$

Focus, for the moment, on the determination of S as a function of temperature.

Introducing the dimensionless variables

$$S_0 = \hbar\beta^2 S , \tag{3.29}$$

$$U = \hbar u , \tag{3.30}$$

$$R = \hbar^2\beta^2 r , \tag{3.31}$$

and the dimensionless momentum

$$x = \beta\hbar q \tag{3.32}$$

inside the integral in (3.26), one obtains the equation for S_0,

$$S_0 = \int \frac{d^3x}{(2\pi)^3} \frac{1}{\Omega} \frac{1}{e^{\Omega}-1} , \tag{3.33}$$

where

$$\Omega^2 = x^2 + R + US_0 . \tag{3.34}$$

One sees from inspection that S_0 is a function of the two independent parameters R and U. Note that temperature enters only through the parameter R.

There are three basic cases of interest.

(i) $r>0$. If r is positive then one has a single-well potential and the behavior of $S_0(R,U)$ as a function of R and U is rather bland. For fixed r and u one can look at the variation of S_0 with temperature ($R \approx r/T^2$). For sufficiently high temperatures one has that $S_0 = S_0(0,U)$. The numerical solution of the resulting equation for S_0 as a function of U is given in Fig. 2. One has the analytic result for small U:

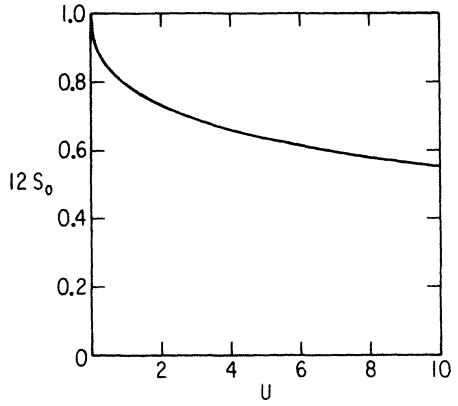

FIG. 2. Equilibrium value of S_0 for $r_1 = 0$ plotted vs U.

$$S_0(0,U) = \frac{1}{12} - \frac{\sqrt{U/3}}{8\pi} + O(U) , \tag{3.35}$$

while for large U,

$$S_0(0,U) = (\ln^2 U/U)[1 + O(1/\ln U)] \tag{3.36}$$

and $S_0(0,U)$ falls off very slowly with increasing U. For sufficiently low temperatures $R \gg US_0$ and

$$S_0(R,U) = \left[\frac{\sqrt{R}}{(2\pi)^3}\right]^{1/2} e^{-\sqrt{R}} . \tag{3.37}$$

Thus for a given U, S_0 starts at high temperatures at $S_0(0,U) \lesssim \frac{1}{12}$ and falls off to zero as the temperature is lowered. A not very dramatic behavior.

(ii) $r=0$. This is the case of a massless bare theory and a flat potential. In this case $S_0 = S_0(0,U)$ which was discussed above and shown in Fig. 2.

(iii) $r<0$. In this case one has a double-well potential as shown in Fig. 3. Fix U and r, and consider the effect of lowering the temperature from high temperatures where $|R| \ll 1$. For infinite temperature one again has

$$S_0 = S_0(0,U) < \tfrac{1}{12} . \tag{3.38}$$

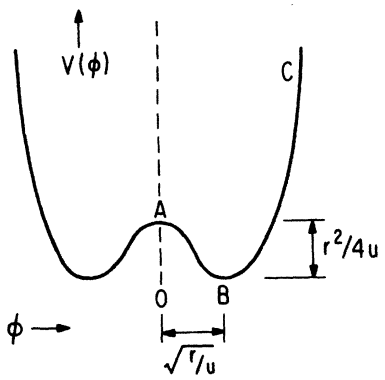

FIG. 3. Schematic of the potential $V(\phi)$ vs ϕ.

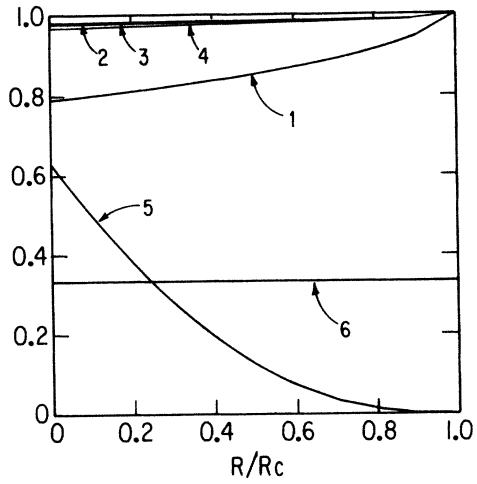

FIG. 4. Equilibrium values of various scaled quantities for $U=1$ plotted vs R/R_c ($R_c=-\frac{1}{12}$). Labeled 1 is $12S_0$, 2 is $30G_0/\pi^2$, 3 is $30\epsilon_0/\pi^2$, 4 is $30K_0/\pi^2$, 5 is $576v_0$, and 6 is P/ϵ.

As T decreases and $|R|$ increases, one finds that $S_0(R,U)$ increases relative to $S_0(0,U)$ (see Fig. 4). However, consider the inequality

$$(1/\Omega)(e^{\Omega}-1)^{-1}\lesssim(1/x)(e^x-1)^{-1} \tag{3.39}$$

which holds for $R+US_0\geq 0$. If $R+US_0<0$, the system is thermodynamically unstable for sufficiently small x. Integrating (3.39) over x, one is led to the constraint for thermodynamic stability:

$$S_0(R,U)\leq\tfrac{1}{12}\ . \tag{3.40}$$

For fixed U, as $|R|$ increases, there will be some finite value of $|R|$ where

$$S_0(R_c,U)=\tfrac{1}{12}\ . \tag{3.41}$$

At this temperature the system undergoes a phase transition. The condition determining R_c is

$$R_c+US_0(R_c,U)=0 \tag{3.42}$$

or

$$R_c+U/12=0\ . \tag{3.43}$$

The transition temperature is then given by

$$k_BT_c=\left[\frac{12\hbar|r|}{u}\right]^{1/2} \tag{3.44}$$

using the original variables.

For lower temperatures ($T<T_c$) one has spontaneous symmetry breaking. The correlation function $C(\mathbf{q})$ is given then, using the scaled momentum $x=\beta\hbar q$, in the dimensionless form[26]

$$C(\mathbf{x})=M^2(2\pi)^3\delta(\mathbf{x})+c(x)\ , \tag{3.45}$$

where M is the dimensionless spontaneous magnetization given by

$$M^2=(R_c-R)/U \tag{3.46}$$

and $c(x)$ is the remaining fluctuating part of the correlation function given by

$$c(x)=(1/x)(e^x-1)^{-1}\ . \tag{3.47}$$

In the limit of small x, $c(x)$ develops the characteristic Nambu-Goldstone mode associated with the breaking of a continuous symmetry:

$$c(x)\approx 1/x^2\ . \tag{3.48}$$

Note that the quantity S_0 is still the integral of $C(x)$, but now, because of the "Bragg peak" contribution, one has

$$S_0=M^2+\tfrac{1}{12}=-R/U\ . \tag{3.49}$$

Given the behavior for S for the various regions, one can work out the behavior of the thermodynamic quantities ϵ, P, K, G, and v. In terms of the dimensionless variables introduced earlier all of the quantities ϵ, P, K, G, and v can be written in the scaled form

$$\epsilon=\beta^{-4}\hbar^3N\epsilon_0(R,U)\ . \tag{3.50}$$

For high temperatures $|R|\ll 1$, the various thermodynamic quantities are functions of U. For very small U one finds directly that

$$\epsilon_0=G_0=K_0=\pi^2/30\ , \tag{3.51}$$

$$P^0=\epsilon_0/3\ , \tag{3.52}$$

$$v_0=\frac{U}{576}[1+O((U)^{1/2})]\ , \tag{3.53}$$

while for large U

$$G_0=\frac{\ln^4U}{U}[1+O(1/\ln U)]\ , \tag{3.54}$$

$$K_0=\frac{\ln^3U}{U}[1+O(1/\ln U)]\ , \tag{3.55}$$

$$v_0=G_0/4\ , \tag{3.56}$$

$$\epsilon_0=3G_0/4\ , \tag{3.57}$$

$$P_0=\epsilon_0/3\ . \tag{3.58}$$

For temperatures below T_c one has the simple results that the thermodynamic quantities are independent of U and given by the $R=0$ and $U=0$ results quoted above. In Fig 4 the various reduced quantities are plotted versus $R(<0)$ for fixed $U=1$. One sees that they vary rather little from their ideal values. The main temperature dependence of the thermodynamic quantities is through the explicit factors of T^4 shown in (3.50).

The properties of this system near its critical point are easy to work out. For long wavelengths $x\ll 1$ and R near R_c, Ω is small and the order-parameter correlation function can be written ($T>T_c$)

$$C(x)=1/\Omega^2=(x^2+R+US_0)^{-1} \tag{3.59}$$

which is of the Ornstein-Zernike form. As $R\to R_c$, $S_0\to\frac{1}{12}$, and $US_0\to -R_c$. The "susceptibility" is given by $C(0)$ and very near T_c goes as

$$C(0)=\left[\frac{U}{4\pi(R-R_c)}\right]^2 \tag{3.60}$$

which gives the critical index $\gamma=2$. Sitting right at T_c one obtains

$$C(x)=1/x^2 \tag{3.61}$$

for small x, which gives the critical index $\eta=0$. Finally the correlation length, ξ, can be easily identified from the Ornstein-Zernike form to be

$$\xi^{-1}=\frac{4\pi}{U}(R-R_c) \tag{3.62}$$

for $T>T_c$, and the critical index $\nu=1$.

IV. ANALYTICAL TREATMENT OF THE TIME EVOLUTION

The problem is now well defined and one can proceed [illegible] where c is the scaled equilibrium energy density given in

to look at the nonequilibrium evolution of the system away from the assumed initial equilibrium. The first thing to note is that $D(q,t)=\tilde{D}(q,t)$ since they obey the same differential equation and the same initial condition.

Just as in the case of the equilibrium behavior it is convenient to go over to a set of dimensionless variables. Just as distances were measured in units of $\hbar\beta(q=x/\hbar\beta)$, so also will all times be measured in units of $\hbar\beta(t=\tau/\hbar\beta)$ (Ref. 27). Temperatures will be measured in units of the Planck temperature $T_P=(\tilde{G}/\hbar)^{1/2}$; where $\tilde{G}$ is related to the gravitational constant by $\tilde{G}=GN_0$; so $T=\beta^{-1}T_P$. One can also introduce the dimensionless correlation functions

$$g(x,\tau)=\beta G(q,t)\ , \tag{4.1}$$

$$c(x,\tau)=C(q,t)/\beta\hbar^2\ , \tag{4.2}$$

$$d(x,\tau)=D(q,t)/\hbar\ . \tag{4.3}$$

The equations of motion can then be put into the dimensionless form

$$\dot{c}(x,\tau)=2d(x,\tau)\ , \tag{4.4}$$

$$\dot{d}(x,\tau)=-3H(\tau)d(x,\tau)-\Gamma_0(x,\tau)c(x,\tau)+g(x,\tau)\ , \tag{4.5}$$

$$\dot{g}(x,\tau)=-6H(\tau)g(q,\tau)-\Gamma_0(x,\tau)2d(x,\tau)\ , \tag{4.6}$$

where

$$\Gamma_0(x,\tau)=R+US_0(\tau)+x^2/a^2\ , \tag{4.7}$$

obtained by replacing the right-hand sides of the equations by their initial equilibrium values. This then gives one a measure of the initial instability in the system. From Eqs. (4.4)–(4.6) one immediately obtains

$$\dot{c}(x,0)=0\ , \tag{4.9}$$

$$\dot{d}(x,0)=-\Omega^2(x)c(x)+g(x)$$
$$=[-\Omega(x)+\Omega(x)]n(x)=0\ , \tag{4.10}$$

$$\dot{g}(x,0)=-6H(0)g(x)\ , \tag{4.11}$$

and

$$H(0)=(8\pi T^2\epsilon_0/3)^{1/2}\ , \tag{4.12}$$

the last section. One sees immediately that the kinetic energy, $g(x,\tau)$, responds rapidly to gravity, but $c(x,\tau)$ and $d(x,\tau)$ do not immediately respond. One must go to the second derivative to obtain the initial response of d and to the third derivative to obtain the initial response of c:

$$\ddot{d}(x,0)=-6H(0)g(x) \tag{4.13}$$

and

$$\dddot{c}(x,0)=-12H(0)g(x)\ . \tag{4.14}$$

This immediately tells one that the kinetic and potential energies are going to respond quite differently to the instability. One can define the time scales associated with this early time behavior:

$$\tau_g^{-1}=6H(0) \tag{4.15}$$

governs the initial response of the kinetic energy, and

$$\tau_c^{-3}(x)=12H(0)g(x)/c(x)=12H(0)\Omega^2(x) \tag{4.16}$$

governs the initial response of the order parameter and the potential energy. If one looks at the time evolution for the longest length scales ($x=0$), then the ratio

$$\tau_c(0)/\tau_g=[48\pi T^2\epsilon_0/(R+US_0)]^{1/3} \tag{4.17}$$

gives an estimate of the relative response of the kinetic and potential energies. Again there are three cases.

(i) $R>0$. The basic physics is clear here in the limiting case $U=0$. Then

$$\tau_c/\tau_g=\{48\pi T^2\epsilon_0(U)/[US_0(U)]^{1/2}\}^{1/3} . \quad (4.21)$$

These expressions can be simplified in the small-U limit to obtain

$$H=T\left[\frac{4\pi^3}{45}\right]^{1/2} , \quad (4.22)$$

$$\tau_c/\tau_g=(16\sqrt{3}\pi^3T^2/5U)^{1/3} . \quad (4.23)$$

Again, as one raises the temperature relative to T_P, one finds an increasing separation of time scales governing the kinetic and potential energies.

(iii) $R<0$. In the case of high temperatures this reduces to case (ii). The new feature here is that the system can support a phase transition. If the system were organized such that its initial state were in equilibrium with a temperature near T_c then

$$H(T=T_c)=\left[\frac{8\pi^3}{3U}\right]^{1/2} \quad (4.24)$$

while

$$\tau_c/\tau_g=\left[\frac{96\pi^3}{5U(R-R_c)}\right]^{1/2} . \quad (4.25)$$

H and τ_c/τ_g are plotted versus R/R_c in Fig. 5 for $U=1$. The feature that the order-parameter time scale increases as one approaches the transition is a very general physical phenomena known as critical slowing down.[29]

If the system is initially in an equilibrium state near or below the transition, then there must have been time for long-range correlations to develop—the system must anneal at that temperature for times which become increasingly long as the associated temperature is lowered toward T_c. Given the conditions corresponding to the evolution of the early Universe, it seems much more physical to assume that $T\gg T_c$. Indeed, if one chooses $T<T_c$, then one must assume the symmetry in the problem is broken from the start or (more reasonably) abandon the assumption of initial equilibrium.

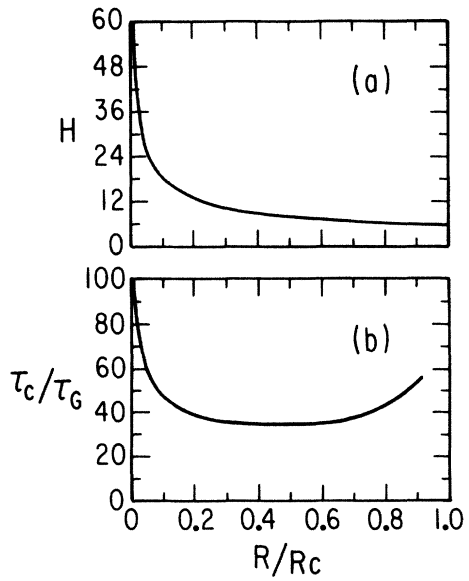

FIG. 5. (a) The initial value of the Hubble constants for $U=1$ vs R/R_c. (b) τ_c/τ_g for $U=1$ vs R/R_c.

B. Intermediate-time analysis

From the analysis above, one finds a rapid decrease in G for short times which carries with it a rapid decrease in the energy density since initially, for sufficiently high temperatures, and U of $O(1)$, $\epsilon\approx G\gg v$. After a relatively short time one finds $G\ll v$ and $\epsilon\approx v$. During this time period H remains relatively large and $a(t)$ increases very rapidly, while S changes very little. A key assumption leading to the important "intermediate-" time regime is that the momentum dependence of the correlation functions "freezes" after a relatively short time. In Sec. V it will be shown that a good measure of this phenomena is the ratio $K(\tau)/S(\tau)=\xi^{-2}(\tau)$ which gives the correlation length $\xi(\tau)$ characterizing $c(x,\tau)$. One finds that $\xi(\tau)$ is independent of time after some time τ_f which is less than 4 in dimensionless units. This freezing phenomena arises because the RW scale factor is becoming sufficiently large that the wave-number dependence in

$$\Gamma(x,\tau)=R+US(\tau)+(x/a)^2 \quad (4.26)$$

can be neglected. This assumes that the range of wave numbers contributing significantly to the matter distribution is effectively cutoff at some wave number which is not growing as rapidly with time as $a(\tau)$. It will be shown below that the effective cutoff decreases somewhat with time and one expects that one enters the regime where x/a can be neglected relatively early. A numerical study of this point will be given below.

Assuming that $\Gamma(x,\tau)$ is independent of x, the equations of motion reduce to

$$\dot{c}(x,\tau)=2d(x,\tau) , \quad (4.27)$$

$$\dot{d}(x,\tau)=-3H(\tau)d(x,\tau)-\Gamma(\tau)c(x,\tau)+g(x,\tau) , \quad (4.28)$$

$$\dot{g}(x,\tau)=-6H(\tau)g(x,\tau)-2\Gamma(\tau)d(x,\tau) . \quad (4.29)$$

Notice now that the coefficients of c, d, and g are all independent of x. One can then convert these equations into equations for the moments $G(t)$ and $S(t)$ which enter into the energy density. Defining

$$D(\tau)=\int\frac{d^3x}{(2\pi)^3}d(x,\tau) , \quad (4.30)$$

one immediately obtains

$$\dot{S}=2D , \quad (4.31)$$

$$\dot{D}=-3HD-\Gamma S+G , \quad (4.32)$$

$$\dot{G}=-6HG-2\Gamma D . \quad (4.33)$$

The conclusion is that for sufficiently large a, the quantities $c(x,\tau)$, $d(x,\tau)$, $g(x,\tau)$ are of the form

$$\phi(x,\tau)=\phi(\tau)\phi(x,,\tau^*)/\phi(\tau^*) , \quad (4.34)$$

where the x dependence is frozen for $\tau>\tau^*$.

Einstein's equation reduces in this limit to

$$H^2(\tau)=\frac{4\pi}{3}T^2[G(\tau)+\Gamma(\tau)^2/2U] . \tag{4.35}$$

Further progress is made if one recognizes that the time derivative of this equation can, using the equation of motion, but put in the simple form

$$\dot{H}=4\pi T^2 G . \tag{4.36}$$

Setting this result back into the Einstein equation and defining

$$\mu^{-1}=4\pi T^2 , \tag{4.37}$$

one obtains

$$\mu(\dot{H}+3H^2)=\Gamma^2/2U . \tag{4.38}$$

One can then eliminate D and G in (4.31)—(4.33) in terms of H and S (or equivalently Γ) to obtain

$$\ddot{S}=-3H\dot{S}-2\Gamma S-2\mu\dot{H} . \tag{4.39}$$

The dynamical problem is then reduced to studying (4.38) and (4.39).

Consider first the intermediate-time regime where $S(\tau)$ has not changed significantly from its initial value and the initial temperature is sufficiently high that

$$\Gamma(\tau)\approx US(\tau) . \tag{4.40}$$

(One should keep in mind that for sufficiently long times or small U this assumption breaks down.) In this time regime the energy density is essentially potential energy and G may be neglected in (4.35). Combining this result with (4.40) one obtains

$$H(\tau)=\left[\frac{U}{6\mu}\right]^{1/2} S(\tau) . \tag{4.41}$$

Setting this back into (4.39) and assuming that $\ddot{S}/2US^2$ and $\mu\dot{H}/US^2$ are $\ll 1$, one obtains

$$0=-3(U/6\mu)^{1/2}\dot{S}S-2US^2 \tag{4.42}$$

or

$$\dot{S}/S=-\tau_I^{-1}=-\left[\frac{2U}{3\pi T^2}\right]^{1/2} . \tag{4.43}$$

This new time scale governing the intermediate time evolution increases linearly with T for large T. Thus S and H decay exponentially during this regime with a rate going to zero as $1/T$ for large T. The inequalities listed above reduce in this case to

$$\ddot{S}/US^2=1/(3\pi T^2 SU)\ll 1 \tag{4.44}$$

and

$$\mu\dot{H}/US^2=2/(12\pi T^2 S)\ll 1 \tag{4.45}$$

which are well satisfied for large fixed T until S becomes sufficiently small. The argument is, therefore, self-consistent. Since G is proportional to $\dot{H}$ [see (4.36)], one finds that

$$G=US/(6\pi T^2) \tag{4.46}$$

which also ensures that the argument ($\epsilon=v$) is self-consistent for large enough T^2.

To the degree that $\epsilon=v$ during this time regime, $P=-\epsilon$ and one is in a de Sitter phase where H is a constant [see (2.19)] and one has exponential growth of $a(t)$. This is only approximately true since τ_I is large but not infinite. We have, instead, that

$$H=\left[\frac{2\pi UT^2}{3}\right]^{1/2} S_0 e^{-\tau/\tau_I} , \tag{4.47}$$

where S_0 is the value of $S(\tau)$ at the beginning of this regime (which will not be very different from its initial value). The RW scale factor is easily determined in this case to be given by

$$a(\tau)/a(\tau^*)=\exp[-\pi T^2 US_0(e^{-\tau/\tau_I}-e^{-\tau^*/\tau_I})] . \tag{4.48}$$

For times $\tau^*<\tau\ll\tau_I$, one obtains exponential growth

$$a(\tau)=a(\tau^*)\exp[(2\pi UT^2/3)^{1/2}S_0(t-t^*)] . \tag{4.49}$$

It is instructive to simply assume that (4.48) can be applied over the entire time regime with the initial condition $a(0)=1$ for $\tau^*=0$. One then obtains

$$a(\tau)=\exp[-\pi UT^2 S_0(e^{-\tau/\tau_I}-1)] . \tag{4.50}$$

This then allows an estimate of the amount of inflation during this era,

$$a(\infty)=e^{\pi UT^2 S_0} , \tag{4.51}$$

and the amount of inflation associated with this quasi—de Sitter phase goes exponentially with T^2. It will be shown in Sec. V that this relation holds rather well when compared to the numerical analysis.

The analysis above is valid for all r as long as U is not too small or time too long. For $r>0$ and $U=0$, for example, the analysis is modified somewhat. In this case $\Gamma=R$ and in the regime where $\epsilon=v$, one obtains the relationship

$$H^2=4\pi r_1 S/3 . \tag{4.52}$$

Going through an analysis very similar to the case directly above, one is led to the results

$$\dot{H}=-R/3 \tag{4.53}$$

and, using (4.36),

$$G=R/3\mu=r_1/(12\pi T^4) \tag{4.54}$$

is a constant. If one assumes that H can be written in the form

$$H(\tau)=H_0 e^{-\tau/\tau_{12}} \tag{4.55}$$

valid for $\tau/\tau_{12}\ll 1$, then

$$\tau_{12}=3H_0/R=T^2\left[\frac{12\pi S_0}{r_1}\right]^{1/2} . \tag{4.56}$$

Going through the same analysis as for (4.51), one obtains the amount of inflation

$$a(\infty)=e^{(4\pi T^2 S_0)} \tag{4.57}$$

which again goes as T^2 in the exponent.

C. Analysis of long-time behavior

When $S(\tau)$ has decayed to sufficiently small values one comes out of this quasi–de Sitter phase into the final longest-time regime. In this regime the kinetic and potential energies are again comparable and the system is attempting to approach equilibrium. The three cases $r>0$, $r=0$, and $r<0$ will be taken separately.

(i) $R>0$. In this case there is no phase transition and one expects the variable S to decay uneventfully to zero as time evolves. It is convenient, therefore, to eliminate Γ in favor of S to obtain the equation of motion; note that

$$V(x)=\frac{rx^2}{2}+\frac{ux^4}{4} \quad \text{for } r>0 ,$$
$$\ddot{S}=-3H\dot{S}-4RS-3US^2+6\mu H^2 . \tag{4.58}$$

For $r>0$, the role of the quartic terms proportional to U are relatively unimportant for long enough times, since, for example, $R+US\approx R$. It is therefore convenient to focus on the case $U=0$. The relevant long-time equations of motion become

$$\ddot{S}=-3H\dot{S}-2RS-2\mu\dot{H} , \tag{4.59}$$

$$\mu(\dot{H}+3H^2)=RS . \tag{4.60}$$

These equations can be solved by looking for solutions ordered by inverse powers of τ. One obtains the asymptotic solutions

$$S=(4\mu/3\omega^2\tau^2)(1-\sin\omega\tau)[1+O(1/\tau)] , \tag{4.61}$$

$$H=(\tfrac{2}{3}\tau)(1+\cos\omega\tau/\omega\tau)[1+O(1/\tau)] , \tag{4.62}$$

where the frequency is given by

$$\omega^2=2R . \tag{4.63}$$

Note that both S and H are non-negative as they must be. From (4.62), one immediately obtains the "dust" result for the RW scale factor

$$a(\tau)/a(\tau^*)=(\tau/\tau^*)^{2/3} . \tag{4.64}$$

The various "thermodynamic" quantities are given in the long-time limit by

$$G(\tau)=(2\mu/3\tau^2)(1+\sin\omega\tau) , \tag{4.65}$$

$$\epsilon=2\mu/3\tau^2 , \tag{4.66}$$

$$P/\epsilon=\sin\omega\tau , \tag{4.67}$$

plus corrections of higher order in τ^{-1}.

(ii) $r=0$. In this case the potential is very flat and there is no phase transition. Looking at the long-time limit, one again finds solutions which are ordered by inverse powers of τ:

$$H(\tau)=(\tfrac{1}{2}\tau)\{1-\ln\tau/[6\tau(2U\mu)^{1/2}+\cdots]\} , \tag{4.68}$$

$$\Gamma(\tau)=\left[\frac{U\mu}{2}\right]^{1/2}\frac{1}{\tau}-\frac{\ln\tau}{\tau^2}+\cdots , \tag{4.69}$$

$$a(\tau)/a(\tau^*)=(\tau/\tau^*)^{1/2}[1+O(\ln\tau/\tau)] , \tag{4.70}$$

$$\epsilon=\frac{3\mu}{8\tau^2}+\cdots , \tag{4.71}$$

$$P/\epsilon=\tfrac{1}{3}+\cdots . \tag{4.72}$$

(iii) $r<0$. In this case, where there is symmetry breaking, Γ goes to zero for long times and one looks for solutions for Γ and H ordered in inverse powers of τ. One finds, after some algebra, that

$$\Gamma(\tau)=(8U\mu/3)^{1/2}\cos\omega\tau/\tau + 4U\mu(\cos2\omega\tau-2)/3\omega^2\tau^2 , \tag{4.73}$$

$$H(\tau)=\tfrac{2}{3}\tau+\sin^2\omega\tau/3\omega\tau^2 \tag{4.74}$$

from which it follows that

$$a(\tau)/a(\tau^*)=(\tau/\tau^*)^{2/3} \tag{4.75}$$

and

$$\epsilon(\tau)=(2\mu/3\tau^2)(1+\sin2\omega\tau/\omega t) , \tag{4.76}$$

$$P/\epsilon=-\sin\omega\tau . \tag{4.77}$$

In summary, for long times, and for R different from zero [with a suitable choice of $V(0)$], one obtains a matter-dominated long-time behavior with $a\approx t^{2/3}$ and $P=0$ (on average). For $R=0$ one recovers the standard model for a radiation-dominated universe with $a\approx t^{1/2}$ and $P/\epsilon=\frac{1}{3}$.

V. NUMERICAL ANALYSIS

Having established analytically the long-, intermediate-, and short-time behaviors, one can proceed to connect them via a numerical solution of the equations of motion. The standard Runga-Kutta method[30] was used to forward step the equations of motion and the correlation functions were then numerically integrated to obtain the quantities $S(t)$, $K(t)$, and $G(t)$ needed to determine $\epsilon(t)$ at each time step. Because of the varying and competing time scales the integration time step had to be carefully chosen and varied over the evolution of the system.

One has three dimensionless parameters to vary: r_1, U, and T with $R=r_1/T^2$. For $r_1>0$ the quartic coupling U does not effect the short- and long-time physics of the situation in any qualitative way. As shown in the last section, U may play some role in governing the intermediate dynamics. If U is not very small, then $\Gamma\approx US$ (as held for the cases $r_1\gtrsim 0$) and the analysis goes through as for $r_1\gtrsim 0$. Since the $U=0$ case is qualitatively different, it will be treated here. For the "critical case, where $r_1=0$, one has a very flat single-well potential. One must main-

TABLE I. Parameters r_1, U, and T characterizing the five cases I–V. T_c is given by $(-12r_1/U)^{1/2}$.

Case	r_1	U	T	T/T_c
I	1	0	10	
II	0	1	10	
III	−1	1	10	$5/(3)^{1/2}$
IV	−1	1	20	$10/(3)^{1/2}$
V	−1	1	30	$15/(3)^{1/2}$

TABLE II. Equilibrium values of the scaled quantities defined by (2.53), (2.54), (2.57), (3.26)—(3.29), and (3.50).

Case	$12S_0$	$30G_0/\pi^2$	$30K_0/\pi^2$	$576v_0$	$30\epsilon_0/\pi^2$	P/ϵ
I	0.9116	0.9892	0.9869	0.2188	0.9892	0.3326
II	0.7926	0.9822	0.9690	0.6282	0.9789	0.3333
III	0.8053	0.9833	0.9717	0.4696	0.9800	0.3339
IV	0.7957	0.9825	0.9697	0.5863	0.9792	0.3335
V	0.7940	0.9824	0.9693	0.6094	0.9790	0.3334

tain $u>0$, but the particular value does not seem crucial. As in the equilibrium case, one expects the properties to be a weak function of u except for very small u. With the exception of the case of very small u, one expects the choice $U=1$ will generate the general qualitative behavior of the system.

Finally, for the case $r_1<1$, the system will order and, for consistency, one must choose $T>T_c$. Since $T_c{}^2=12r_1/U$, one must be careful in choosing U and r_1. For fixed r_1 and T there is a minimum acceptable value of U. Calculation for various choices of r_1 and U indicate that the parameter governing the qualitative behavior of the system is T. It is sensible, therefore, to focus on the set of parameters $r_1=-1$ and $U=1$ and vary T. While many different sets of parameters have been analyzed, data for the five cases shown in Table I will be given in detail. In Table II the equilibrium values of various quantities corresponding to these states are given. The states have approximately the same energy density (and therefore H), kinetic energies, gradient energies (K), and pressures. They have somewhat different values for S and the potential energy.

Before looking in detail at some of the predictions concerning the short-, intermediate-, and long-time behaviors, data, ranging over the whole time range, will be presented. Figures 6—8 show the evolution of $\ln a(\tau)$, $S_0(\tau)$, and $P(\tau)/\epsilon(\tau)$ vs τ. In all five cases there is some degree of inflation. The "amount" of inflation will be further quantified below. $S_0(\tau)$ shows the expected qualitative behavior of decaying slowly from its initially "large" value. For $r<0$ one sees that there is ordering since $S_0(\tau)$ oscillates about the value—R/U for long times. The plots of P/ϵ vs τ are particularly instructive since they show most clearly the three different time regimes. For early times P/ϵ drops very rapidly from its equilibrium value $\approx\frac{1}{3}$ to the highly nonequilibrium value -1. It stays in this quasi—de Sitter phase for a time which depends strongly on the initial temperature T. Finally, $S_0(\tau)$ approaches the "bump" value $S_0(\tau_b)=S_b$ at the time τ_b where $S_0(\tau)$ hits its first minimum. For times $\tau>\tau_b$ the system will be entering its final oscillatory phase.

One can conclude that the qualitative picture discussed in Sec. IV is obtained in the numerical analysis, and one can move on to check the more detailed results. In Fig. 9 the short-time behavior of the energy density and kinetic energy are presented. The separation of time scales discussed in Sec. IV is clearly seen since $G_0(\tau)$ changes many orders of magnitude while $S_0(\tau)$ remains near its initial value. The first requirement of the intermediate-time regime is that $G\ll v$. This is clearly satisfied in the cases treated here for dimensionless times greater than 3 or 4. The second requirement is that there be sufficient growth of $a(\tau)$ that the momentum distribution freezes, or that the gradient energy contribution to ϵ is negligible. As discussed above, the ratio K/S defines an inverse correlation length squared $[=\xi^{-2}(\tau)]$ which characterizes the width of the correlation function $c(x,\tau)$. In equilibrium, as the temperature is lowered from above T_c, ξ grows rather rapidly. Plotting K/S versus time (see Fig. 10), one[28] indeed does see a rapid increase in $\xi(\tau)$ for short times,

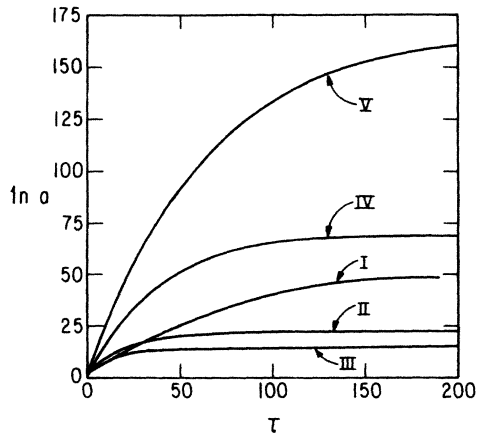

FIG. 6. $\ln a$ vs τ for the five cases listed in Table I.

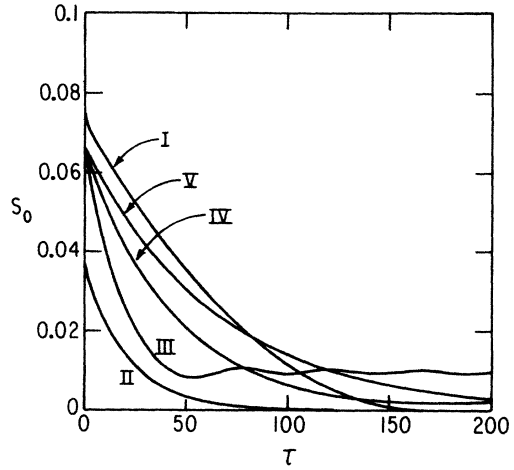

FIG. 7. S_0 vs τ for the five cases listed in Table I.

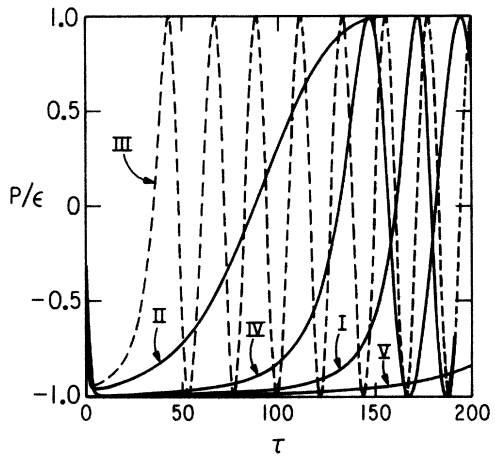

FIG. 8. P/ϵ vs τ for the five cases listed in Table I.

but for times on the order of 4, $\xi(\tau)$ saturates. In all five cases listed in Table I the freezing time is less than 5 and the ratio K/S is subsequently fixed at values evident in Fig. 10. Both requirements for entering the intermediate-time regime are satisfied for $\tau > 4$. This is clearly seen in Fig. 8 for P/ϵ where this ratio falls to near -1 for $\tau > 4$.

The intermediate-time regime is characterized for $r \lesssim 0$ by exponential decay of S and H and a constant ratio G/S. In Fig. 11 G/S is plotted versus τ and the constant region is evident. Clearly the τ range over which G/S is constant is growing with increasing T as predicted by (4.43). The value of the ratio G/S is predicted by the intermediate-time theory to be equal to $U/(6\pi T^2)$. The agreement with the numerical results for cases II–V are precise. This serves as a good check on both the analytic results as well as the numerical procedure. One can also check that $S_0(\tau)$ is decaying exponentially in this range. For case II a semilogarithmic plot of $S_0(\tau)$ vs τ in Fig. 12 shows an excellent fit to an exponential over the time range shown. Results of the linear least-square fit shown in the Fig. 12 (Ref. 31) give

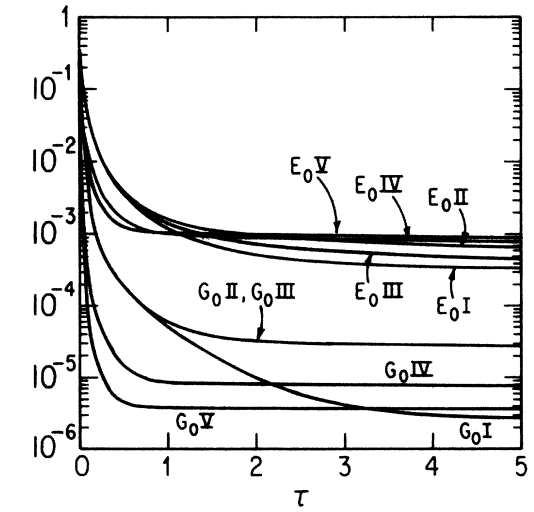

FIG. 9. G_0 vs ϵ_0 vs τ for the five cases listed in Table I.

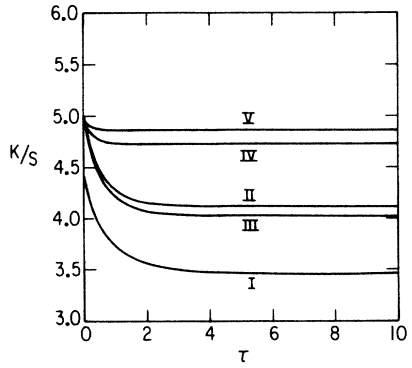

FIG. 10. K/S vs τ for the five cases listed in Table I.

$$S_0(\tau) = 0.0652 \exp(-0.046\,44\tau) . \tag{5.1}$$

The predictions from (4.43) for $\tau_I^{-1} = (2U/3\pi)^{1/2}/T$ give for $U=1$ and $T=10$ that $\tau_I^{-1} = 0.046\,07$. The agreement is extremely good. A similar check for case V leads to equally good results and the validity of this intermediate-time phase is confirmed numerically.

The intermediate behavior for $r > 0$ and $U=0$ is different from the case discussed above. In Fig. 13 $G_0(\tau)$ is plotted versus τ for case I. Clearly there is a constant region with $G_0 \approx 2.65 \times 10^{-6}$. From (4.54) one has the analytic result $G_0 = r_1/(12\pi T^4)$. For $r_1 = 1$ and $T=10$ this gives $G = 2.62 \times 10^{-6}$ and there is excellent agreement.

The analytic treatment of the intermediate-time regime gave a prediction for the "amount" of inflation. These predictions are checked more quantitatively here. The amount of inflation can be defined as the value of the RW factor at the time of the first bounce:

$$a_b = a(\tau_b) . \tag{5.2}$$

In Table III, τ_b, S_b, and $\ln a_b$ are given for the five cases in Table I. From the theory in Sec. IV the amount of inflation corresponding to the intermediate phase is given, for $r \lesssim 0$, by (4.51). For the cases of interest here $U=1$ and $\tilde{S_0}(U=1) = 0.0662$ (Ref. 32), so

$$\ln a(\infty) = 0.208 T^2 . \tag{5.3}$$

For $T=10$, which corresponds to case II, one obtains the

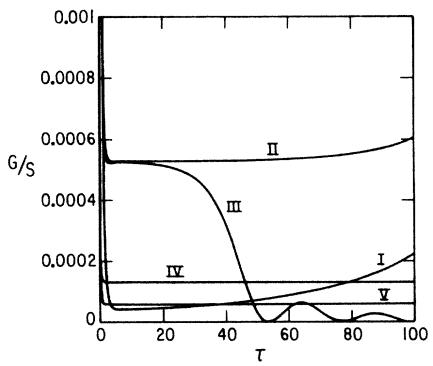

FIG. 11. G/S vs τ for the five cases listed in Table I.

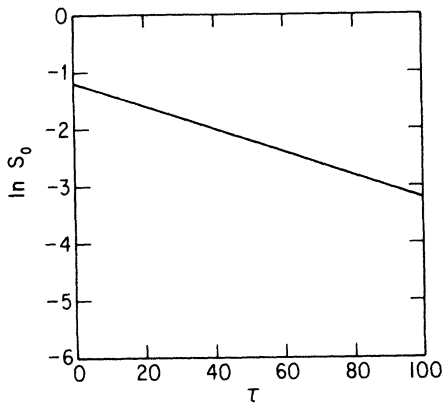

FIG. 12. Logarithmic plot of S_0 vs τ for $r_1=1$, $U=1$, $T=10$ (case II).

estimate $\ln a_b=20.8$ which is in good agreement with the numerical result taken from Table III: $\ln a_b=22.79$. For the case $r<0, \ln a_b$ is plotted versus T^2 in Fig. 14 and the fit to a straight line is essentially perfect and gives

$$\ln a_b=-5.190+0.1877T^2 \tag{5.4}$$

and the agreement between the coefficients of T^2 in (5.3) and (5.4) must be considered good.

For $r_1>0$ and $U=0$, the predicted amount of inflation given by (4.57) is $\ln a(\infty)=4\pi T^2 S_0$. For $r_1=1$ and $T=10$, $S_0=0.075\,97$ and $\ln a(\infty)=95.467$. The comparison with $\ln a_b=48.2$ given in Table III is not very good. This discrepancy is presumably because the temperature is not yet sufficiently high for the various intermediate-time approximations to become valid.

After the first bounce the system crossed over to the long-time oscillatory behavior. The numerical analysis confirms the structure of the long-term asymptotic analysis of Sec. IV.[33]

TABLE III. "Bounce" values of $S_0(\tau)$ and $\ln a(\tau)$ defined by the time τ_b where $S_0(\tau)$ hits its first minimum.

Case	τ_b	S_{0b}	$\ln a_b$
I	172.1	9.8×10^{-9}	48.2
II	175.0	3.2×10^{-10}	22.79
III	53.6	0.008 5	4.04
IV	168.5	0.002 14	68.7
V	304.5	0.000 95	163.6

VI. COMMENTS AND CONCLUSIONS

The model studied here seems to be rather rich. One finds that it can accommodate three rather different time regimes and provides smooth mechanisms into and out of inflationary eras. The model also provides some insight into some general questions associated with the problem. One of the more interesting such questions involves the role of equilibrium concepts during the time evolution of the system. Is it valid to assume that one has local equilibrium and how should one view phase transitions within this context? This leads one to question the meaning of "temperature" during the time regions where the system is dynamically evolving. While it is intuitively appealing to associate a decreasing temperature with the rapidly decreasing energy density in this system (and an effective temperature can certainly be defined in this manner), one should not confuse this quantity with temperature in the strict sense since in the intermediate-time regime the system is clearly not described by a stationary probability distribution. More specifically there is no temperature which will give the appropriate values of both potential and kinetic energies during the intermediate or inflationary regime. The inflationary regime is intrinsically a nonequilibrium regime.

Assumptions of local equilibrium can easily mislead one. An important case concerns the role of phase transitions in the evolution of the early Universe. There has been much discussion built around the notion that as the Universe cools the temperature will drop into the vicinity of a phase transition and it will then be crucial to deter-

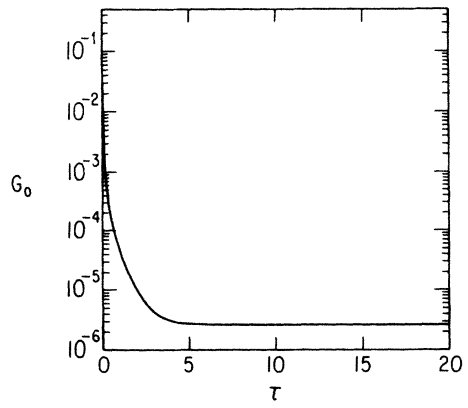

FIG. 13. G_0 vs τ for $r_1=1$, $U=0$, and $T=10$ (case I).

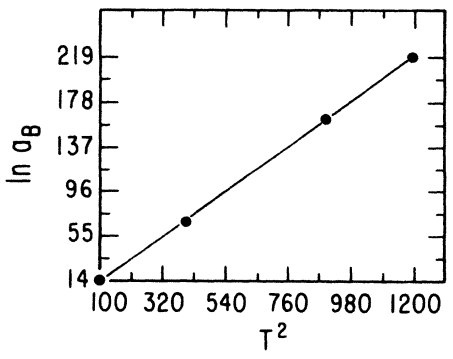

FIG. 14. $\ln a_B$ vs T^2. Dots correspond to the numerical results. The straight line is a linear fit to the four points.

mine whether this transition is first or second order since this is presumably associated with whether there will be a latent heat produced or not. The problem with this type of thinking has been discussed in Ref. 5. Unless the rate of cooling due to expansion is very slow relative to the internal rate of equilibrium of the matter the system will be very insensitive to any critical effects—effects due to the existence of a second-order phase transition. The point is that critical effects (which involve the existence of a very large correlation length) occur over a vary narrow temperature range. Outside of this range the correlation length is typically small. In order to see critical effects, therefore, one must set in this temperature range for a time sufficiently long to grow a long correlation length. However, because of critical slowing down, as the correlation length gets long the characteristic times to equilibrate the system get longer and it takes progressively more time to grow a longer correlation length. In a dynamic problem like inflation where the system is moving inexorably toward low temperatures there is no reason why the system would want to hang around the critical temperature. Therefore one expects the system to act more like a condensed-matter system where one quenches the temperature from some high temperature to some low temperature with the overall result of a finite change in the order parameter associated with a first-order phase transition—even though the quench may be directly "through" the critical point. In this case there is symmetry breaking but it is not in the temporal sense spontaneous. The new order is developed locally in the form of domains. The transition is first order in that the change in the magnitude of the amplitude of the local order parameter is not small. A competition between domains leads to a growth of the average domain size which eventually grows arbitrarily large as time evolves. Thus, while any symmetries present in the initial Lagrangian are globally preserved, an observer located in a domain of steadily increasing size will conclude that the symmetry has been broken. This qualitative picture is confirmed by the detailed calculations discussed above.

It is interesting to try and characterize the inflationary phase found here. Under the right conditions it appears to be essentially an ideal de Sitter phase where $P=-\epsilon$ and one has pure exponential growth of the RW scale factor. It must be kept in mind that the ideal nature of this state is tied to the notion of a very large initial temperature and the associated separation of time scales for the kinetic and potential energies when the system starts to evolve. For high, but finite initial temperatures, one has only an approximate de Sitter phase and the local order-parameter fluctuations and the Hubble constant decay exponentially with a rate given above. The picture given here is not of a ball rolling down a hill starting from point A in Fig. 3 and heading to point B. Rather it is of a particle with a large amount of potential energy at point C which gradually rolls to the left to point B. For reasons pointed out in Sec. IV, this rolling can be very slow if one starts at very high initial temperatures relative to the critical temperature and the Planck temperature.

The model studied here is clearly an oversimplification of the real problem. How can one improve this model and the calculation carried out here? The most obvious step is to look at higher-order corrections in powers of $1/N$. This will be rather difficult but could produce some interesting new qualitative features which may be important. Presumably at the next order one will generate a dissipative mechanism which will damp out the long-time oscillations found in leading order in $1/N$. It may, however, be more important to consider the damping effects and energy transfer associated with the coupling of the "order parameter" to other fields. There appears to be several ways of doing this stopping short of a full treatment of the full SU(5) field theory. Finally, there is the interesting question associated with the regularization of the theory, the role of horizons and the connection to the scale-invariant density perturbations studied by a number of authors. This question should probably be sorted out first.

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