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# Direct calculation of the six-gluon subprocess

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We present concise and easily programed expressions for the six-gluon hard-scattering subprocess expected to dominate four-jet production at high energies. The improved spinor inner-product technique is employed.

## I. INTRODUCTION

As high-energy accelerators attain ever larger center-of-mass energies, hard-scattering processes that produce multiple high-transverse-momentum jets become increasingly important. At the Superconducting Super Collider with $\sqrt{s}=40$ TeV these multiple-jet processes become an important source of background to new physics events. Even at currently available center-of-mass energies they provide an important new testing ground for perturbative QCD. Calculations of $2\rightarrow2$ and $2\rightarrow3$ subprocesses have yielded short and convenient expressions for the cross sections.[1,2] More recently $2\rightarrow4$ reactions are being calculated by several groups. The results for $6q$ and $4q2g$ processes appear in Refs. 3 and 4 (hereafter GK). Results for $6g$ and $4g2q$, which are the focus of this paper, have been obtained in Refs. 5 and 6. Reference 5 employs a series of supersymmetric Ward identities to reduce the $6g$ calculation to amplitudes involving the spin-0 and spin-$\frac{1}{2}$ supersymmetric partners of the gluon. Reference 6 employs the more direct approach expounded in GK and also developed in Ref. 7 [a simplification of the CALKUL (Ref. 2) techniques]. There the $4g2q$ amplitude is computed directly and the $6g$ amplitude is obtained using supersymmetric Ward identities. In this paper we report the results of a direct calculation of the $6g$ amplitude using the techniques of Refs. 3 and 4. Our expressions are concise and convenient for computer implementation. The resulting subroutine is slightly more than 10 times faster than the purely numerical program of Ref. 8.

## II. PRELIMINARIES

We mention only a few of the crucial ingredients of the calculational techniques of Refs. 3 and 4 as employed for the direct $6g$ calculation. We define all momenta to be outgoing:

$$k_1+k_2+k_3+k_4+k_5+k_6=0\ . \qquad (1)$$

The results will be expressed in terms of the massless spinor inner products defined and discussed in Refs. 3 and 4:

$$\begin{aligned}&\langle p-q\mid+\rangle\equiv\langle pq\rangle=\sqrt{p_-q_+}\,e^{i\phi_p}-\sqrt{p_+q_-}\,e^{i\phi_q}\ ,\\ &\langle q+\mid p-\rangle\equiv\langle pq\rangle^+=\langle pq\rangle^*\end{aligned} \qquad (2)$$

valid for $p_0,q_0>0$. The $\pm$ in the inner-product symbols denote the spinor helicities while $p_\pm$, $q_\mp$, and $\phi_{p,q}$ are light-cone variables given by

$$p_\pm=p_0\pm p_z,\ \ e^{i\phi_p}=(p_x+ip_y)/(p_x{}^2+p_y{}^2)^{1/2}\ . \qquad (3)$$

Inner products involving negative-energy spinors can be defined by analytic continuation according to (for $p_0,q_0>0$)

$$\begin{aligned}&\langle(-p)q\rangle=\langle p(-q)\rangle=i\langle pq\rangle\ ,\\ &\langle(-p)(-q)\rangle=-\langle pq\rangle\ ,\\ &\langle(-p)q\rangle^+=\langle p(-q)\rangle^+=i\langle pq\rangle^+\ ,\\ &\langle(-p)(-q)\rangle^+=-\langle pq\rangle^+\ .\end{aligned} \qquad (4)$$

In this paper we adopt the notation

$$\begin{aligned}&s_{ij}=\langle k_ik_j\rangle,\ \ t_{ij}=\langle k_ik_j\rangle^+\ ,\\ &d_{ij}=k_i\cdot k_j=s_{ij}t_{ij}/2\ ,\end{aligned} \qquad (5)$$

where $i$ and $j$ run from 1 to 6.

The crucial ingredient in the GK technique as applied to this problem is the form of the gluon polarization vector:

$$\epsilon_\mu^\pm(k_i,p_i)=\frac{\pm\langle k_i\pm\mid\gamma_\mu\mid p_i\pm\rangle}{\sqrt{2}\langle p_i\mp\mid k_i\pm\rangle}\ , \qquad (6)$$

where $k_i$ is the momentum of the gluon with polarization $\pm$ and $p_i$ is a reference momentum for gluon $i$ to be chosen for convenience. In the calculation of a given helicity amplitude the reference momentum of each gluon may be chosen independently of that of the others. By clever choices the calculation may be greatly simplified. Computation of the relevant Feynman diagrams involves Lorentz index contraction of the various $\epsilon_\mu$'s and momenta. Contractions between two $\epsilon$'s may be simplified by using the Fierz techniques described in GK. Contractions between an $\epsilon$ and a momentum $k_l$ are simplified using completeness,

$$\langle k_i\pm\mid\gamma_\mu\mid p_i\pm\rangle k_l^\mu=\langle k_i\pm\mid k_l\mp\rangle\langle k_l\mp\mid p_i\pm\rangle\ .$$

Clearly, by choosing each $p_i$ so as to maximize the number of zeros of the type $\langle k_i\pm\mid k_i\mp\rangle=\langle k_i\pm\mid k_j\pm\rangle=0$, the expressions for the various Feynman diagrams will take on their simplest form.

 

## III. HELICITY AMPLITUDES FOR THE SIX-GLUON SUBPROCESS

In principle there are 64 independent helicity amplitudes for the 6$g$ calculation. Amplitudes with all or five helicities equal vanish. This is easily verified if we presume the first five gluons to have the same helicity in both of the above cases, and take the momentum of the sixth gluon as reference momentum for all five. Of the remaining 50 helicity amplitudes we only need to calculate 25. The others may be obtained by parity. These 25 may be divided into two types, labeled by $N$, the number of negative-helicity gluons: 15 that have $N=2$ and 10 that have $N=3$.

In fact, it is sufficient to calculate analytically only one helicity amplitude for each of the above types. We analytically compute the helicity amplitudes specified by the helicity assignments $++-++-$ for $N=2$ and $+++---$ for $N=3$, respectively. The momenta are assigned in the order $k_1-k_6$. The remaining amplitudes of each type may be obtained by permutation of the momenta keeping the helicities fixed. Only those permutations that are independent, given Bose symmetry, need be considered. We denote these momentum permutations by $I_2^n$, $n=1-15$, for $N=2$, and by $I_3^n$, $n=1-10$, for $N=3$, respectively. The permutations are given in Table I, where the momentum labels appear in the rows and each row is labeled by the permutation number $n$. Thus $I_N^n$ is a six-component vector for each $N$ and $n$. The amplitude square for the cases $N=2$ and $N=3$ can be written therefore as

$$M_N^{\text{tot}}=\sum_{n=1}^{n_N^{\max}} M_N(I_N^n)\ , \tag{7}$$

where $n_2^{\max}=15$ and $n_3^{\max}=10$. It turns out that the remaining 25 helicity amplitudes, related by parity to those we explicitly compute, may be simply included by doubling the above results. Thus the total matrix element squared for the six-gluon subprocess, summed over helicities, is given by

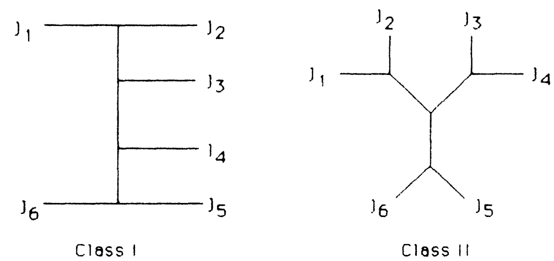


FIG. 1. The diagrams of topology class I and class II.

$$M_{6g}=2(M_2^{\text{tot}}+M_3^{\text{tot}})\ . \tag{8}$$

We focus now on the computation of the standard momentum configuration cases for $N=2$ and $N=3$, $I_2{}^1$ and $I_3{}^1$, respectively, see the top entries in Table I. For $N=2$ the simplest expression is obtained by referring $\epsilon_{1-5}$ to $k_6$, and $\epsilon_6$ to $k_3$. For $N=3$ we refer $\epsilon_{1-3}$ to $k_6$ and $\epsilon_{4-6}$ to $k_3$. There are 220 Feynman diagrams for each helicity case. Of these 90 are of the ladder topology, called I, and 15 are of the Mercedes type, called II, see Fig. 1. The remaining 115 diagrams contain four-gluon couplings. All these diagrams may be trivially incorporated by summing over all "contracted" versions of diagram types I and II. For diagrams I we may contract either one internal propagator or two noncontiguous internal propagators; for II we may contract only one internal propagator. These contractions are performed on the momentum-space structure of a given diagram keeping the color structure unchanged. The contraction operation is defined in Fig. 2. Most of these contractions give zero, and the nonzero results are extremely simple. Only helicity case $N=3$ yields nonzero contractions. The contraction technique automatically includes appropriate color factors.

We shall now discuss the evaluation of the momentum-space structure of the various diagrams. Color structure will be incorporated later. Within each

TABLE I. Independent momentum permutations: $I_2^n$ and $I_3^n$ for $++-++-$ and $+++---$ helicity cases, respectively. Column labels indicate helicity assignments for given momentum.

$I_2^n=$

| + | + | − | + | + | − |
|---|---|---|---|---|---|
| 1 | 2 | 3 | 4 | 5 | 6 |
| 1 | 2 | 4 | 3 | 5 | 6 |
| 1 | 2 | 5 | 3 | 4 | 6 |
| 1 | 3 | 2 | 4 | 5 | 6 |
| 1 | 2 | 4 | 3 | 6 | 5 |
| 1 | 2 | 3 | 4 | 6 | 5 |
| 1 | 3 | 2 | 4 | 6 | 5 |
| 1 | 2 | 3 | 5 | 6 | 4 |
| 1 | 3 | 2 | 5 | 6 | 4 |
| 1 | 4 | 2 | 5 | 6 | 3 |
| 2 | 3 | 1 | 4 | 5 | 6 |
| 2 | 3 | 1 | 4 | 6 | 5 |
| 2 | 3 | 1 | 5 | 6 | 4 |
| 2 | 4 | 1 | 5 | 6 | 3 |
| 3 | 4 | 1 | 5 | 6 | 2 |

$I_3^n=$

| + | + | + | − | − | − |
|---|---|---|---|---|---|
| 1 | 2 | 3 | 4 | 5 | 6 |
| 1 | 4 | 6 | 2 | 3 | 5 |
| 1 | 3 | 6 | 2 | 4 | 5 |
| 1 | 2 | 6 | 3 | 4 | 5 |
| 1 | 4 | 5 | 2 | 3 | 6 |
| 1 | 3 | 5 | 2 | 4 | 6 |
| 1 | 2 | 5 | 3 | 4 | 6 |
| 1 | 3 | 4 | 2 | 5 | 6 |
| 1 | 2 | 4 | 3 | 5 | 6 |
| 1 | 5 | 6 | 2 | 3 | 4 |

diagram type class I and II, there are a limited number of topologies that actually need to be computed. The number depends on the helicity case $N$. For $N=2$ only gluons 3 and 6 have negative helicity and, in addition, are used as polarization reference momenta. Thus the topologies are specified by the distinct locations of gluons 3 and 6. For $N=2$ class I has seven distinct topologies and class II has only two. These are labeled by a topology number $m_2=1-9$. Each topology must be evaluated for the various independent permutations of the remaining gluons 1–4. Thus, for example, topology $m_2=1$ corresponds to $j_5=3$ and $j_6=6$ for diagram type I in Fig. 1, and is evaluated for 12 independent orderings of 1–4 for the indices $j_1-j_4$. Topology $m_2=8$ corresponds to $j_5=3$ and $j_6=6$ for diagram type II; only three independent permutations of gluons 1–4 are possible. By including all independent permutations of 1–4 among $j_1-j_4$ for each of the nine topologies we generate the complete set of 105 Feynman diagrams for this helicity case. The full list of $j_1-j_6$ assignments and topology types, $m_2$, appears in Table II, according to diagram number $l=1-105$.

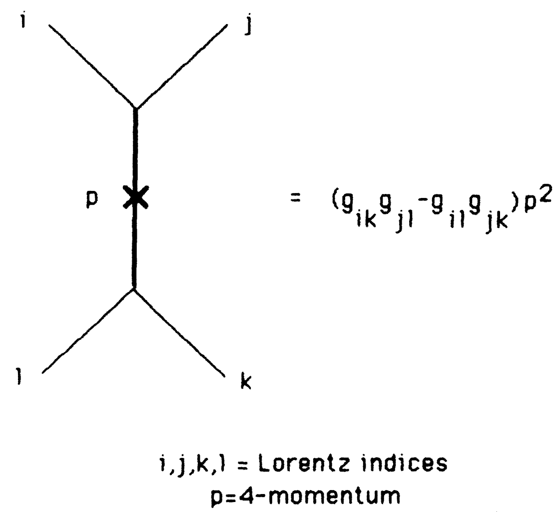


FIG. 2. The momentum-space contraction rule used to include four-gluon couplings.

TABLE II. Momentum labels $j_1-j_6$ and topology types $m_3$ and $m_2$ as a function of the diagram number $l$.

| $l$ | $j_1-j_6$ | $m_3$ | $m_2$ | $l$ | $j_1-j_6$ | $m_3$ | $m_2$ | $l$ | $j_1-j_6$ | $m_3$ | $m_2$ |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 125436 | 01 | 1 | 36 | 241635 | 10 | 3 | 71 | 614253 | 26 | 7 |
| 2 | 124536 | 01 | 1 | 37 | 126354 | 27 | 4 | 72 | 614523 | 25 | 7 |
| 3 | 152436 | 02 | 1 | 38 | 156324 | 29 | 4 | 73 | 621543 | 24 | 7 |
| 4 | 154236 | 03 | 1 | 39 | 146325 | 29 | 4 | 74 | 621453 | 24 | 7 |
| 5 | 142536 | 02 | 1 | 40 | 546312 | 28 | 4 | 75 | 625143 | 26 | 7 |
| 6 | 145236 | 03 | 1 | 41 | 246315 | 29 | 4 | 76 | 625413 | 25 | 7 |
| 7 | 251436 | 02 | 1 | 42 | 256314 | 29 | 4 | 77 | 624153 | 26 | 7 |
| 8 | 254136 | 03 | 1 | 43 | 162354 | 16 | 5 | 78 | 624513 | 25 | 7 |
| 9 | 241536 | 02 | 1 | 44 | 261354 | 16 | 5 | 79 | 651243 | 23 | 7 |
| 10 | 245136 | 03 | 1 | 45 | 165324 | 15 | 5 | 80 | 651423 | 22 | 7 |
| 11 | 541236 | 04 | 1 | 46 | 561324 | 14 | 5 | 81 | 652143 | 23 | 7 |
| 12 | 542136 | 04 | 1 | 47 | 164325 | 15 | 5 | 82 | 652413 | 22 | 7 |
| 13 | 254361 | 07 | 2 | 48 | 461325 | 14 | 5 | 83 | 654123 | 21 | 7 |
| 14 | 542361 | 08 | 2 | 49 | 265314 | 15 | 5 | 84 | 654213 | 21 | 7 |
| 15 | 245361 | 07 | 2 | 50 | 562314 | 14 | 5 | 85 | 641253 | 23 | 7 |
| 16 | 541362 | 08 | 2 | 51 | 264315 | 15 | 5 | 86 | 641523 | 22 | 7 |
| 17 | 145362 | 07 | 2 | 52 | 462315 | 14 | 5 | 87 | 642153 | 23 | 7 |
| 18 | 154362 | 07 | 2 | 53 | 564312 | 13 | 5 | 88 | 642513 | 22 | 7 |
| 19 | 142365 | 06 | 2 | 54 | 465312 | 13 | 5 | 89 | 645123 | 21 | 7 |
| 20 | 124365 | 05 | 2 | 55 | 126534 | 17 | 6 | 90 | 645213 | 21 | 7 |
| 21 | 241365 | 06 | 2 | 56 | 126435 | 17 | 6 | 91 | 125436 | 30 | 8 |
| 22 | 125364 | 05 | 2 | 57 | 156234 | 18 | 6 | 92 | 152436 | 31 | 8 |
| 23 | 251364 | 06 | 2 | 58 | 156432 | 19 | 6 | 93 | 142536 | 31 | 8 |
| 24 | 152364 | 06 | 2 | 59 | 146235 | 18 | 6 | 94 | 615423 | 32 | 9 |
| 25 | 125634 | 09 | 3 | 60 | 146532 | 19 | 6 | 95 | 625413 | 32 | 9 |
| 26 | 251634 | 10 | 3 | 61 | 256134 | 18 | 6 | 96 | 612453 | 35 | 9 |
| 27 | 152634 | 10 | 3 | 62 | 256431 | 19 | 6 | 97 | 652413 | 34 | 9 |
| 28 | 254631 | 11 | 3 | 63 | 246135 | 18 | 6 | 98 | 612543 | 35 | 9 |
| 29 | 542631 | 12 | 3 | 64 | 246531 | 19 | 6 | 99 | 642513 | 34 | 9 |
| 30 | 245631 | 11 | 3 | 65 | 546132 | 20 | 6 | 100 | 621453 | 35 | 9 |
| 31 | 541632 | 12 | 3 | 66 | 546231 | 20 | 6 | 101 | 651423 | 34 | 9 |
| 32 | 145632 | 11 | 3 | 67 | 612543 | 24 | 7 | 102 | 621543 | 35 | 9 |
| 33 | 154632 | 11 | 3 | 68 | 612453 | 24 | 7 | 103 | 641523 | 34 | 9 |
| 34 | 142635 | 10 | 3 | 69 | 615243 | 26 | 7 | 104 | 651243 | 33 | 9 |
| 35 | 124635 | 09 | 3 | 70 | 615423 | 25 | 7 | 105 | 641253 | 33 | 9 |

A similar procedure applies to helicity case $N=3$. It is most economical to use the symmetric referencing stated earlier. Topologies are then specified both by the (independent) locations of the referenced gluons 3 and 6 and by the (independent) arrangements of the remaining two positive and two negative helicities, relative thereto, among the locations $j_1-j_6$ in Fig. 1. In all we obtain 29 topologies from diagram type I and six topologies from type II, labeled by $m_3=1-35$. Again each is evaluated for a number of different orderings of the remaining $j$'s. The complete list of $j_1-j_6$ assignments and topology types, $m_3$, appears as a function of diagram number $l=1-105$ in Table II.

In fact the expressions for the various different topology types for a given $N$ are not all independent, and some are even zero. For $N=2$ there are only six independent nonzero type-I topologies, and only one for type II. For $N=3$ there are 18 independent nonzero type-I topologies, and four of type II.

We now state the results (up to a certain overall factor, to be specified later) for the amplitudes associated with each of the $l=1-105$ diagrams for cases $N=2$ and $N=3$. Their numerators are denoted $A_N(l,m_N^l,i_1,i_2,i_3,i_4,i_5,i_6)$ and their denominators $D_N(l,i_1,i_2,i_3,i_4,i_5,i_6)$ where $l$ is the diagram number, $m_N^l$ is the associated topology type read from Table II, and $i_{1-6}$ specify a certain momentum permutation, to be given in detail shortly. The $A_N$ are given in Table III for $N=2$ and in Table IV

TABLE III. Auxiliary function calls for $A_2(l,m_2^l,i_1,i_2,i_3,i_4,i_5,i_6)$ as a function of the topology type, $m_2^l$. We use shorthand notation $i_1\equiv 1, i_2\equiv 2,\ldots,i_6\equiv 6$.

| $m_2^l$ | $A_2(l,m_2^l,\ldots)$ |
|---|---|
| 1 | 0 |
| 2 | $C_2(v,1,2,3,4,5,6)$ |
| 3 | $A_2(l-15)$ for $l>27$, otherwise $A_2(l-3)$ |
| 4 | $C_4(v,1,2,3,4,5,6)$ |
| 5 | $C_5(v,1,2,3,4,5,6)$ |
| 6 | $C_6(v,1,2,3,4,5,6)$ |
| 7 | $C_7(v,1,2,3,4,5,6)$ |
| 8 | 0 |
| 9 | $C_9(v,1,2,3,4,5,6)$ |

TABLE IV. Auxiliary function calls for $A_3(l,m_3^l,i_1,i_2,i_3,i_4,i_5,i_6)$ as a function of the topology type, $m_3$. We use shorthand notation $i_1\equiv 1, i_2\equiv 2,\ldots,i_6\equiv 6$.

| $m_3^l$ | $A_3(l,m_3^l,\ldots)$ |
|---|---|
| 1 | 0 |
| 2 | 0 |
| 3 | 0 |
| 4 | 0 |
| 5 | $B_5(v,1,2,3,4,5,6)$ |
| 6 | $B_6(v,1,2,3,4,5,6)+B'_6(v,1,2,3,4,5,6)$ |
| 7 | $-B_6(\bar{v},2,1,3,5,4,6)-B'_6(v,2,1,3,4,5,6)$ |
| 8 | $-B_5(\bar{v},2,1,3,5,4,6)$ |
| 9 | $A_3(22)$ for $l=25$, $A_3(20)$ for $l=35$ |
| 10 | $A_3(23)$ for $l=26$, $A_3(24)$ for $l=27$, $A_3(19)$ for $l=34$, $A_3(21)$ for $l=36$ |
| 11 | $A_3(l-15)$ |
| 12 | $A_3(l-15)$ |
| 13 | $B_{13}(v,1,2,3,4,5,6)$ |
| 14 | $B_{14}(v,1,2,3,4,5,6)+B'_{14}(v,1,2,3,4,5,6)$ |
| 15 | $B_{15}(v,1,2,3,4,5,6)+B'_{14}(v,1,2,3,4,5,6)$ |
| 16 | $B_{16}(v,1,2,3,4,5,6)$ |
| 17 | $B_{16}(\bar{v},6,5,4,3,2,1)$ |
| 18 | $B_{15}(\bar{v},6,5,4,3,2,1)+B'_{18}(v,1,2,3,4,5,6)$ |
| 19 | $B_{14}(\bar{v},6,5,4,3,2,1)+B'_{18}(v,1,2,3,4,5,6)$ |
| 20 | $B_{13}(\bar{v},6,5,4,3,2,1)$ |
| 21 | $B_{21}(v,1,2,3,4,5,6)$ |
| 22 | $B_{22}(v,1,2,3,4,5,6)+B'_{22}(v,1,2,3,4,5,6)+B''_{22}(v,1,2,3,4,5,6)$ |
| 23 | $B_{25}(\bar{v},6,5,4,3,2,1)+B'_{22}(v,1,2,3,4,5,6)-B''_{22}(v,1,2,3,4,5,6)$ |
| 24 | $B_{24}(v,1,2,3,4,5,6)$ |
| 25 | $B_{25}(v,1,2,3,4,5,6)+B'_{22}(v,1,2,3,4,5,6)-B''_{22}(v,1,2,3,4,5,6)$ |
| 26 | $B_{26}(v,1,2,3,4,5,6)+B'_{22}(v,1,2,3,4,5,6)+B''_{22}(v,1,2,3,4,5,6)$ |
| 27 | $-B_{13}(v,5,6,3,4,2,1)$ |
| 28 | $A_3(37)$ |
| 29 | $A_3(l-3)$ for $l>40$, otherwise $B_{29}(v,1,2,3,4,5,6)+B'_{29}(v,1,2,3,4,5,6)$ |
| 30 | 0 |
| 31 | 0 |
| 32 | $B_{32}(v,1,2,3,4,5,6)$ |
| 33 | $B_{33}(v,1,2,3,4,5,6)$ |
| 34 | $B_{34}(v,1,2,3,4,5,6)$ |
| 35 | $B_{35}(v,1,2,3,4,5,6)$ |

TABLE V. (a) Subsidiary quantities $e$, $f$, and $g$ for helicity case $N=2$. (b) Subsidiary quantities $e$, $f$, and $g$ for helicity case $N=3$. Note that $f$ and $g$ are not required for $N=2$. Note also that here shorthand notation is not employed. These subsidiary quantities are evaluated directly in terms of the $s$ and $t$ quantities defined in Eq. (5).

(a)

| | | | |
|---|---|---|---|
| $e_{12}=-s_{26}t_{12}$ | $e_{13}=-s_{36}t_{13}$ | $e_{14}=-s_{46}t_{14}$ | $e_{15}=-s_{56}t_{15}$ |
| $e_{21}=s_{16}t_{12}$ | $e_{23}=-s_{36}t_{23}$ | $e_{24}=-s_{46}t_{24}$ | $e_{25}=-s_{56}t_{25}$ |
| $e_{31}=s_{13}t_{16}$ | $e_{32}=s_{23}t_{26}$ | $e_{34}=-s_{34}t_{46}$ | $e_{35}=-s_{35}t_{56}$ |
| $e_{41}=s_{16}t_{14}$ | $e_{42}=s_{26}t_{24}$ | $e_{43}=s_{36}t_{34}$ | $e_{45}=-s_{56}t_{45}$ |
| $e_{51}=s_{16}t_{15}$ | $e_{52}=s_{26}t_{25}$ | $e_{53}=s_{36}t_{35}$ | $e_{54}=s_{46}t_{45}$ |
| $e_{61}=s_{16}t_{13}$ | $e_{62}=s_{25}t_{23}$ | $e_{64}=-s_{46}t_{34}$ | $e_{65}=-s_{56}t_{35}$ |

(b)

| | | | |
|---|---|---|---|
| $e_{12}=-s_{26}t_{12}$ | $e_{13}=-s_{36}t_{13}$ | $e_{14}=-s_{46}t_{14}$ | $e_{15}=-s_{56}t_{15}$ |
| $e_{21}=s_{16}t_{12}$ | $e_{23}=-s_{36}t_{23}$ | $e_{24}=-s_{46}t_{24}$ | $e_{25}=-s_{56}t_{25}$ |
| $e_{31}=s_{16}t_{13}$ | $e_{32}=s_{26}t_{23}$ | $e_{34}=-s_{46}t_{34}$ | $e_{35}=-s_{56}t_{35}$ |
| $e_{41}=s_{14}t_{13}$ | $e_{42}=s_{24}t_{23}$ | $e_{45}=s_{45}t_{35}$ | $e_{46}=s_{46}t_{36}$ |
| $e_{51}=s_{15}t_{13}$ | $e_{52}=s_{25}t_{23}$ | $e_{54}=-s_{45}t_{34}$ | $e_{56}=s_{56}t_{36}$ |
| $e_{61}=s_{16}t_{13}$ | $e_{62}=s_{26}t_{23}$ | $e_{64}=-s_{46}t_{34}$ | $e_{65}=-s_{56}t_{35}$ |

$f_{14}=-2s_{46}t_{13}$ $f_{24}=-2s_{46}t_{23}$ $f_{15}=-2s_{56}t_{13}$ $f_{25}=-2s_{56}t_{23}$ $g=f_{15}f_{24}/2$

TABLE VI. (a) Auxiliary functions $C_m(v,i_1,i_2,i_3,i_4,i_5,i_6)$ for helicity case $N=2,++-++-$. We use shorthand notation $i_1\equiv 1,\ldots,$ in the $C$ expressions. (b) Auxiliary functions $B_m(v,i_1,i_2,i_3,i_4,i_5,i_6)$ for helicity case $N=3,+++---$. We use shorthand notation $i_1\equiv 1,\ldots,$ in the $B$, $B'$, and $B''$ expressions. If $C_m$ or $B_m$ is called as a function of $\bar{v}$ then, in the above expressions, each $s$ should be replaced by the corresponding $t$ and vice versa. The $e$'s, $f$'s, and $g$ are not altered.

(a)

$C_2=4d_{45}e_{64}{}^2t_{12}s_{65}p_{312}$ $C_4=-4d_{43}t_{21}t_{56}s_{43}{}^2p_{312}{}^2$

$C_5=4d_{42}t_{65}s_{42}e_{31}e_{41}p_{213}$ $C_6=4d_{53}e_{65}t_{12}s_{53}p_{312}p_{412}$

$C_7=4d_{16}e_{56}e_{32}e_{26}p_{456}$ $C_9=4d_{16}t_{34}s_{21}e_{56}e_{26}p_{234}$

$$p_{ijk}=e_{ij}+e_{ik}$$

(b)

$B_5=2e_{46}p_{412}{}^2p_{312}t_{12}s_{65}$

$B_6=-e_{46}[-2s_{65}(-t_{13}e_{46}+t_{34}e_{12})(-e_{21}e_{46}+p_{256}e_{42})-ge_{46}(d_{31}+d_{32})]$

$B_{13}=-2p_{456}{}^3p_{312}t_{56}s_{12}$

$B_{14}=2\delta\{s_{12}[-e_{41}{}^2e_{65}+e_{46}(e_{43}e_{61}-e_{41}p_{653})]+s_{62}e_{41}e_{13}e_{46}\}+2s_{12}t_{34}e_{56}p_{612}p_{416}p_{456}-2ge_{41}(p_{456}\rho+q_{413}y_{465}+q_{465}\sigma)$

$B_{15}=2e_{41}[p_{436}(t_{54}e_{65}s_{32}e_{13}-t_{14}s_{62}e_{56}p_{312})-g(p_{456}\rho+q_{413}y_{465}+q_{465}\sigma)]$

$B_{16}=-2p_{413}{}^3p_{312}t_{14}s_{65}$, $B_{21}=2p_{645}{}^2p_{456}p_{312}t_{56}s_{21}$

$B_{22}=-2t_{56}s_{21}[(p_{635}e_{32}+e_{63}e_{34})(p_{624}e_{45}+e_{14}e_{43})]+\tau$

$B_{24}=2p_{623}{}^2e_{32}e_{65}t_{26}s_{54}$

$B_{25}=-2e_{62}t_{56}p_{256}(p_{634}p_{312}s_{41}+e_{63}e_{65}s_{43})+\tau$

$B_{26}=-2e_{62}e_{65}(e_{65}t_{42}+e_{23}t_{46})(-e_{12}s_{53}+e_{54}s_{31})+\tau$

$\bar{B}_{29}=2p_{412}(-p_{412}\alpha\beta-t_{54}e_{14}e_{45}\beta-s_{23}e_{42}e_{63}\alpha)$

$B_{29}=\bar{B}_{29}-g[-p_{412}(q_{465}x_{1234}+q_{421}x_{3456})+p_{412}{}^2(\gamma/2)+2q_{465}q_{421}\sigma]$

$B_{32}=2t_{56}s_{43}p_{652}{}^2e_{62}p_{256}$, $B_{33}=2t_{43}s_{21}p_{634}{}^2e_{65}p_{534}$

$\bar{B}_{34}=-r_{562}(e_{43}f_{32}e_{12}+f_{34}r_{124})-r_{215}(e_{65}f_{54}e_{34}+f_{34}r_{564})$

$B_{34}=\bar{B}_{34}-f_{52}e_{65}e_{12}(-e_{43}p_{356}+e_{34}e_{45}+f_{34}\zeta)-g[(d_{56}e_{12}-d_{21}e_{65})q_{143}-d_{43}e_{65}e_{12}]$

$B_{35}=e_{62}e_{65}[f_{25}(e_{43}p_{356}-e_{34}e_{45})+f_{34}(e_{23}e_{54}-e_{24}e_{53})-f_{24}e_{34}p_{512}+f_{35}e_{43}p_{234}-2g\zeta]+g[(d_{65}e_{12}-d_{21}e_{65})q_{134}-d_{43}e_{12}e_{65}]$

$B'_6=-g(e_{46}{}^2d_{12}+e_{46}q_{421}w_{123})$, $B'_{14}=-(g/2)p_{456}q_{465}d_{12}$, $B'_{18}=(g/2)p_{512}q_{521}d_{56}$

$B'_{22}=g(e_{62}q_{632}d_{56}+e_{65}q_{645}d_{12})$, $B''_{22}=ge_{65}e_{62}w_{123}$, $B'_{29}=-gq_{421}q_{456}w_{123}$

$\alpha=t_{15}e_{45}-t_{14}e_{56}$, $\beta=s_{63}e_{21}+s_{26}e_{42}$, $\delta=t_{54}e_{31}+t_{34}e_{56}$

$\gamma=d_{43}+d_{21}+d_{56}$, $\zeta=y_{634}+y_{534}$, $\tau=-ge_{65}e_{62}(d_{21}-d_{45}-d_{46})$

$\rho=(d_{63}-d_{61}-d_{62}-d_{53}+d_{51}+d_{52})/4$, $\sigma=(d_{43}-d_{41}-d_{42}-d_{63}+d_{61}+d_{62}-d_{53}+d_{51}+d_{52})/4$

$p_{ijk}=e_{ij}+e_{ik}$, $q_{ijk}=(e_{ij}-e_{ik})/2$, $r_{ijk}=e_{ij}e_{jk}-e_{ji}e_{ik}$

$w_{ijk}=d_{ij}+d_{ik}+d_{jk}$, $y_{ijk}=(d_{ij}-d_{ik})/2$, $x_{ijkl}=d_{ik}+d_{il}+d_{jk}+d_{jl}$

TABLE VII. Nonzero entries for color matrix $R(k,l)$.

| $k$ | $l$ values for $+1$ entries | $l$ values for $-1$ entries |
|---|---|---|
| 1 | [1,12,13,25,37,43,55,67,91,98] | [7,8,14,26] |
| 2 | [2,14,15,35,56,68,96] | [9,10,12,36,37,43,91] |
| 3 | [3,7,8,10,26,27,38,45,57,69,92] | [13,15,98] |
| 4 | [4,14,15,33,58,70,94] | [10,11,12,31,38,45,92] |
| 5 | [5,8,9,10,34,36,39,47,59,71,93] | [13,15,96] |
| 6 | [6,11,12,13,31,32,60,72] | [8,14,39,47,93,94] |
| 7 | [11,18,44,73,102] | [1,3,4,16,25,27,37,55,91] |
| 8 | [16,17,37,74,91,100] | [2,5,6,11,34,35,44,56] |
| 9 | [3,4,6,7,26,27,42,49,61,75] | [17,18,93,102] |
| 10 | [8,16,17,28,62,76,93,95] | [6,11,12,29,42,49] |
| 11 | [4,5,6,9,34,36,41,51,63,77] | [17,18,92,100] |
| 12 | [10,11,12,18,29,30,64,78,92] | [4,16,41,51,95] |
| 13 | [9,20,46,79,104] | [1,2,3,21,25,27,38,57,92] |
| 14 | [19,21,38,80,92,101] | [4,5,6,9,32,33,46,58] |
| 15 | [1,2,5,25,50,81,93] | [7,19,20,26,42,61,104] |
| 16 | [19,21,42,82,97] | [5,8,9,10,28,30,50,62,93] |
| 17 | [2,5,6,11,31,32,40,53,65,83] | [19,20,91,101] |
| 18 | [9,10,12,20,29,30,66,84,91] | [2,21,40,53,97] |
| 19 | [7,22,48,85,105] | [1,2,5,23,34,35,39,59,93] |
| 20 | [23,24,39,86,93,103] | [3,4,6,7,32,33,48,60] |
| 21 | [1,2,3,35,52,87,92] | [9,22,24,36,41,63,105] |
| 22 | [23,24,41,88,99] | [3,7,8,10,28,30,52,64,92] |
| 23 | [1,3,4,33,54,89,91] | [11,22,24,31,40,65,103] |
| 24 | [7,8,22,28,40,90] | [1,12,23,29,54,66,91,99] |
| | all other $R(k,l)=0$ | |

for $N=3$. These tables assume that the diagrams are calculated in the order $l=1-105$ for fixed $i_{1-6}$; some later diagrams can then be specified trivially in terms of ones already evaluated in the sequence. Note that in these tables the shorthand notation $i_1\equiv 1,\ldots,i_6\equiv 6$ has been employed. In Tables III and IV auxiliary functions, $C_k$ for $N=2$ and $B_k$, $B_k'$, and $B_k''$ for $N=3$, appear. They are evaluated as functions of a collective variable set $v$ or $\bar{v}$ and integers $i_1-i_6$. The variable sets $v$ and $\bar{v}$ are defined by

$$v\equiv(s,t,d,e,f,g),\quad \bar{v}\equiv(t,s,d,e,f,g)\ , \tag{9}$$

where $s$, $t$, and $d$ are the subscripted spinor overlaps and dot products defined in Eq. (5) and $e$, $f$ (both subscripted), and $g$ derive from $e_{ij}=\epsilon_i\cdot p_j$, $f_{ij}=\epsilon_i\cdot\epsilon_j$, and $g=\epsilon_1\cdot\epsilon_4\epsilon_2\cdot\epsilon_5/2$, respectively. These latter quantities depend on the helicity case ($N=2$ or $N=3$) as given in (a) and (b) of Table V. It should be noted that, in going from $v$ to $\bar{v}$, the quantities $d$, $e$, $f$, and $g$ are not changed, even though they are computed in terms of $s$ and $t$. The quantities $e$, $f$, and $g$ may thus be evaluated for the two helicity cases at the same time as $s$, $t$, and $d$. All are obtained immediately in terms of the six input four-momenta vectors. (The two helicity cases must, of course, be kept separate even though we have, for simplicity, used a common notation.) The auxiliary functions, $C_k$ or $B_k$, $B_k'$, and $B_k''$, appear in (a) and (b) of Table VI, respectively. In these tables the shorthand notation $i_1,\ldots,i_6\equiv 1,\ldots,6$ is used.

Before giving our final expressions for the squared amplitudes, $M_N$, it is necessary to discuss the color structure of the various contributing diagrams. In all there are 24 independent color structures, which we shall label by the index $k=1-24$. The choice of structures is dictated by convenience. We have taken the color structures of diagrams $l=67-90$ as independent. The color factors associated with any other diagram may be decomposed in terms of these. We define a matrix $R(k,l)$ which gives the decomposition of the color factor for diagrams $l$ into our independent structures labeled by $k$. The matrix $R$ has only 0, 1, and $-1$ entries. In Table VII we specify the nonzero entries by giving the contributing $l$ values for $k=1-24$. In squaring the amplitude for a given $N$ we must compute the squares and overlaps of our 24 independent color structures. These are given by the matrix $C(k,k')$ in Table VIII.

## IV. FINAL FORMULAS

We are now in a position to give our final expressions. To optimize the numerical program based on our expressions we imagine that a given set of six four-momenta have been generated, labeled by $k_{1-6}$. We then precompute the $s$'s, $t$'s, and $d$'s of Eq. (5). Using these we also precompute the two sets of $e$'s, $f$'s, and $g$ specified in (a)

TABLE VIII. Color squaring matrix $C(k,k')$. For the standard SU($n$)-color group (defining $N=n^2-1$) we have $a=n^4N/8$, $b=3n^2N/2$, $c=2a$, $d=a+b$, $e=4a$, $f=8a$.

$$C=\begin{pmatrix}
f & e & e & c & c & 0 & e & c & c & a & a & 0 & c & a & 0 & 0 & d & b & a & 0 & 0 & b & b & b \\
e & f & c & 0 & e & c & c & e & a & 0 & c & a & a & 0 & 0 & b & b & b & c & a & 0 & 0 & d & b \\
e & c & f & e & 0 & c & c & a & 0 & 0 & d & b & e & c & c & a & a & 0 & 0 & a & b & b & 0 & b \\
c & 0 & e & f & c & e & a & 0 & 0 & b & b & b & c & e & a & 0 & c & a & a & c & d & b & 0 & 0 \\
c & e & 0 & c & f & e & a & c & d & b & 0 & 0 & 0 & a & b & b & 0 & b & e & c & c & a & a & 0 \\
0 & c & c & e & e & f & 0 & a & b & b & 0 & b & a & c & d & b & 0 & 0 & c & e & a & 0 & c & a \\
e & c & c & a & a & 0 & f & e & e & c & c & 0 & 0 & 0 & c & a & b & d & 0 & b & a & 0 & b & b \\
c & e & a & 0 & c & a & e & f & c & 0 & e & c & 0 & b & a & 0 & b & b & 0 & 0 & c & a & b & d \\
c & a & 0 & 0 & d & b & e & c & f & e & 0 & c & c & a & e & c & 0 & a & b & b & 0 & a & b & 0 \\
a & 0 & 0 & b & b & b & c & 0 & e & f & c & e & a & 0 & c & e & a & c & d & b & a & c & 0 & 0 \\
a & c & d & b & 0 & 0 & c & e & 0 & c & f & e & b & b & 0 & a & b & 0 & c & a & e & c & 0 & a \\
0 & a & b & b & 0 & b & 0 & c & c & e & e & f & d & b & a & c & 0 & 0 & a & 0 & c & e & a & c \\
c & a & e & c & 0 & a & 0 & 0 & c & a & b & d & f & e & e & c & c & 0 & b & 0 & b & b & a & 0 \\
a & 0 & c & e & a & c & 0 & b & a & 0 & b & b & e & f & c & 0 & e & c & 0 & 0 & b & d & c & a \\
0 & 0 & c & a & b & d & c & a & e & c & 0 & a & e & c & f & e & 0 & c & b & b & b & 0 & 0 & a \\
0 & b & a & 0 & b & b & a & 0 & c & e & a & c & c & 0 & e & f & c & e & b & d & 0 & 0 & a & c \\
d & b & a & c & 0 & 0 & b & b & 0 & a & b & 0 & c & e & 0 & c & f & e & a & c & 0 & a & e & c \\
b & b & 0 & a & b & 0 & d & b & a & c & 0 & 0 & 0 & c & c & e & e & f & 0 & a & a & c & c & e \\
a & c & 0 & a & e & c & 0 & 0 & b & d & c & a & b & 0 & b & b & a & 0 & f & e & e & c & c & 0 \\
0 & a & a & c & c & e & b & 0 & b & b & a & 0 & 0 & 0 & b & d & c & a & e & f & c & 0 & e & c \\
0 & 0 & b & d & c & a & a & c & 0 & a & e & c & b & b & b & 0 & 0 & a & e & c & f & e & 0 & c \\
b & 0 & b & b & a & 0 & 0 & a & a & c & c & e & b & d & 0 & 0 & a & c & c & 0 & e & f & c & e \\
b & d & 0 & 0 & a & c & b & b & b & 0 & 0 & a & a & c & 0 & a & e & c & c & e & 0 & c & f & e \\
b & b & b & 0 & 0 & a & b & d & 0 & 0 & a & c & 0 & a & a & c & c & e & 0 & c & c & e & e & f
\end{pmatrix}$$

and (b) of Table V. These are not recomputed in the following sum over permutations. In Eq. (7), for a given $N$ and given $I_N^n$ [for which we use the generic notation I, with components $I(1)$–$I(6)$], we compute

$$M_N(I)=\sum_{k=1}^{24}\sum_{k'=1}^{24} C(k,k')T_N(k,I)T_N^*(k',I)/K_N(I)\ , \tag{10}$$

where $K_N(I)$ is a normalization factor given by

$$K_N(I)=\begin{cases}2^{10}\,|\,d_{I(1)I(6)}d_{I(2)I(6)}d_{I(3)I(6)}{}^2d_{I(4)I(6)}d_{I(5)I(6)}\,|\ , & N=2\ ,\\ 2^{10}\,|\,d_{I(1)I(6)}d_{I(2)I(6)}d_{I(3)I(6)}{}^2d_{I(4)I(3)}d_{I(5)I(3)}\,|\ , & N=3\ .\end{cases} \tag{11}$$

The factor $K_N{}^{-1}$ in Eq. (10) derives from trivial overall factors that are common to all diagrams of a given $N$. $K_N$ includes the normalization factors for the polarization vectors in Eq. (6). The quantities $T_N(k,I)$ are the independent color amplitudes and receive contributions from the $l=1$–$105$ diagrams according to the color decomposition matrix $R(k,l)$. We find

$$T_N(k,I)=\sum_{l=1}^{105} R(k,l)\frac{A_N(l,m_N^l,I(j_1^l),I(j_2^l),I(j_3^l),I(j_4^l),I(j_5^l),I(j_6^l))}{D_N(l,I(j_1^l),I(j_2^l),I(j_3^l),I(j_4^l),I(j_5^l),I(j_6^l))}\ . \tag{12}$$

Recall that the numerator functions $A_N$ appear in Tables III and IV, and that the $I$ of Eq. (12) is one of the integer-valued vectors of Table I, while the $m_N^l$ and $j_i^l$ are the topology types and integer permutations, respectively, of Table II. The functions $D_N(l,i_1,i_2,i_3,i_4,i_5,i_6)$ are easily specified

$$D_N(l,1,2,3,4,5,6)=\begin{cases}d_{12}d_{56}(d_{12}+d_{13}+d_{23}), & l=1\text{–}90\ ,\\ d_{12}d_{34}d_{56}, & l=91\text{–}105\ ,\end{cases} \tag{13}$$

where we simplify notation as in Tables III and IV, denoting $i_1$–$i_6$ by 1–6.

## V. EXPLICIT EXAMPLE

As an example let us consider $N=3$ and $I=I_3{}^3$ (from Table I) in Eq. (10), together with $l=14$ in Eq. (12). Then the components $I(1)=1$, $I(2)=3$, $I(3)=6$, $I(4)=2$, $I(5)=4$, $I(6)=5$ are used in Eqs. (11) and (12). In particular, referring to Table II we evaluate $A_3(l=14,\ m_3{}^{14}=8,\ I(5),I(4),I(2),I(3),I(6),I(1))$ and similarly for $D_3$. Using Table IV and then (b) of Table VI we find

$$A_3(14,8,4,2,3,6,5,1)=-B_5(\bar{v},2,4,3,5,6,1)$$
$$=-2e_{51}p_{524}{}^2p_{324}s_{24}t_{16} \tag{14}$$

with $p_{524}=e_{52}+e_{54}$, etc., as precomputed. Finally from Eq. (13) we have

$$D_3(14,4,2,3,6,5,1)=d_{42}d_{51}(d_{42}+d_{43}+d_{23}) \;. \qquad (15)$$

## VI. CONCLUSIONS

We have given an explicit and easily programed expression for the six-gluon scattering amplitude. Of the two independent helicity amplitudes that must be directly computed, $N=2$ and $N=3$, the $N=2$ amplitude (with two negative and four positive gluon helicities) is extremely simple and can undoubtedly be further reduced by combining all terms that contribute to a definite color structure. This was indeed the case for the simpler helicity amplitude of the *ggqqll* calculation of Ref. 4. We have not pursued this possibility. In contrast, the $N=3$ amplitude appears to have a multivariable pole structure that is not obviously amenable to further reduction.

As a matter of curiosity, we have compared the running times of the three different six-gluon programs on the same computer; we will refer to them by initials K for Ref. 6, PT for Ref. 5, and KG for the present work. All three programs use the parity-symmetry short cut, cf. Eq. (8), and set up the color, permutation mappings, etc., as part of a program initialization process. On the SLACVM system the running times for computation of a single momentum configuration using the core analytic program were K=0.46 sec, PT=0.29 sec, and KG=0.18 sec. Since none of the groups has made a serious effort to maximize efficiency, it is clear that all the techniques should be regarded as comparable. Anyone of the three programs runs fast enough that phenomenological Monte Carlo calculations become possible.

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