QCD-based relativistic Hartree-Fock calculations for identical quarks

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As was first pointed out by Witten, a large number of colors (N_c) leads to a simplification in the theory of baryon masses in that the quarks may be assumed to move in a mean field which can be found self-consistently. The interquark potential in such a description can be borrowed from the meson-sector phenomenology in the absence of an accurate evaluation of it from large-N_c quantum chromodynamics (QCD). We have carried out this program with such a potential due to Richardson, used often by workers in the meson sector. This potential has the advantage of incorporating the two main features of QCD: namely, confinement and asymptotic freedom. In view of the small number of parameters involved, the results agree surprisingly well with experiment for the case of three identical quarks.

I. INTRODUCTION

Although QCD was formulated for three colors, many of its features are more readily understandable if one lets the number of colors N_c become arbitrarily large. This suggestion of 't Hooft[1] became an extremely powerful tool for understanding hadrons after the work of Witten.[2] The $1/N_c$ expansion gives, to leading order, most of the qualitative features expected of low-energy QCD, such as confinement[2] and chiral-symmetry breaking.[3] For mesons it leads to a Bethe-Salpeter equation with a $q\bar{q}$ potential, while for baryons it yields a Hartree-Fock-type equation. An alternative for baryons is to solve classical meson Lagrangians, as in the Skyrme model,[4] but this approach is currently faced with the problem of justifying the truncation of the Lagrangian[5] and incorporating important low-mass mesons into the model.[6] The relativistic Hartree-Fock (HF) method looks therefore like a very promising alternative for describing baryons[7] and it is the purpose of the present paper to explore this alternative while keeping as close to QCD as present-day techniques permit.

Starting from the action for a system of interacting quarks and gluons, we obtain, after a series of approximations, a Dirac Hamiltonian, Eq. (2.9), with a two-body static potential, which is the actual point of departure of our HF calculations. Present-day techniques do not permit summing up all the planar gluon diagrams which would yield unambiguously such a potential. As an alternative, we borrow a potential devised for meson-sector calculations, the Richardson potential.[8] It has the correct renormalization-group behavior at high q^2 and a q^{-4} behavior at small q^2, as suggested by various groups.[9] It fits the charmonium data, and additional cross checks are possible on its only free parameter Λ, which is related to the string tension and hence to other parameters such as the critical temperature T_c through lattice gauge calculations.[10]

In the spirit of remaining close to the first principles of QCD, we fixed the quark masses m_q at the current-algebra values.[11] With $m_s = 150$ MeV for the strange quark, we find very good agreement with the Ω^- energy for a reasonable value of the parameter Λ in Richardson's potential. The result is not very sensitive to uncertainties in the value of m_q, since the kinetic energy decreases as m_q increases. On the other hand, with a mass of about 10 MeV for u and d quarks, there are problems of confinement connected with the Lorentz-vector nature of the effective potential as developed in Sec. II. These can however be alleviated by splitting the confining part of the interaction into a scalar and a vector part, and this prescription allows one to deal with light-quark systems quite successfully.

In Sec. II we derive the relativistic HF equations and we describe the method that allows subtraction of the spurious center-of-mass kinetic energy. Throughout the paper we limit our considerations, for the sake of simplicity, to systems formed of three quarks of a single flavor in the same orbital wave function. Numerical results are presented in Sec. III. In particular, we study the confining properties of the various potentials.

II. GENERAL FORMALISM

The action for a system of interacting quarks and gluons can be written as

$$S = S_{\text{gluon}} + \int [\bar{q}(i\partial - m)q + j_\mu^a A^{a\mu}] d^4x \ , \tag{2.1}$$

where S_{gluon} denotes, collectively, the action of the free gluonic fields $A^{a\mu}$, the gauge-fixing terms, and the action of the ghost fields, while j_μ^a is the quark current

$$j_\mu^a = \frac{g}{2} \bar{q} \gamma_\mu \lambda^a q \ , \tag{2.2}$$

where the λ^a's are the Gell-Mann matrices. The connect-

ed Green's functions of gluons are generated by the functional

$$e^{iW(j)}=\int \exp\left[i\left[S_{\text{gluon}}+\int j^a_\mu A^{a\mu}d^4x\right]\right]d\mu[A]\,, \tag{2.3}$$

where the volume element in path space, $d\mu[A]$, includes the ghost fields. The full generating functional is given by

$$Z=\int e^{iS}d\mu[A][d\bar{q}\,dq]\,. \tag{2.4}$$

Using (2.3), one can integrate formally over the gluons and the ghosts, and write

$$Z=\int \exp\left[i\left[\int \bar{q}(i\partial\!\!\!/-m)q\,d^4x+W(j)\right]\right][d\bar{q}\,dq]$$
$$\equiv\int e^{iS_{\text{eff}}}[d\bar{q}\,dq]\,, \tag{2.5}$$

thus obtaining an effective action for the quarks:

$$S_{\text{eff}}=\int d^4x\left[\bar{q}(i\partial\!\!\!/-m)q-\frac{1}{2!}\int j^{a\mu}(x)V^{ab}_{\mu\nu}(x,y)j^{b\nu}(y)d^4x\,d^4y\right.$$
$$\left.-\frac{1}{3!}\int j^{a\mu}(x)j^{b\nu}(y)j^{c\rho}(z)V^{abc}_{\mu\nu\rho}(x,y,z)d^4x\,d^4y\,d^4z+\cdots\right], \tag{2.6}$$

where the V's are connected Green's functions. Now, Eq. (2.6) is an infinite expansion and it is absolutely essential to have a truncation scheme if we want to extract meaningful numbers. For N_c-quark systems like baryons, the $1/N_c$ expansion provides such a scheme. This can be seen by going back to the formalism of canonical quantization and considering S_{eff} as a function of the field operators q and $\bar{q}$. For an N_c-body baryon, the expectation value of the N' currents ($N'>N_c$) that are contracted with $V^{a_1\cdots a_{N'}}_{\mu_1\cdots\mu_{N'}}$ involves

$$\langle N_c q's\,|\,\underbrace{\bar{q}q\cdots\bar{q}q}_{N'\text{ times}}\,|\,N_c q's\rangle\sim\Psi^*_{N_c}(1,\ldots,N_c)\langle 0\,|\,\underbrace{\bar{q}q\cdots\bar{q}q}_{(N'-N_c)\text{ times}}\,|\,0\rangle\Psi_{N_c}(1,\ldots,N_c)\,. \tag{2.7}$$

Now, the factor $\langle 0\,|\,\bar{q}q\cdots\bar{q}q\,|\,0\rangle$ corresponds to the production of virtual $\bar{q}q$ pairs and quark loops. It is suppressed by $1/N_c$ and all terms involving more than N_c currents can be dropped in a similar manner.

In spite of this restriction, one cannot actually compute even the two-point Green's function $V^{ab}_{\mu\nu}$ to all orders (Fig. 1). Moreover, a fully relativistic treatment of the two-body problem keeping all components of $V^{ab}_{\mu\nu}$ (1,2) is prevented by the standard relative-time difficulties in the Bethe-Salpeter equation. We therefore go to the static limit in our calculation and use for V_{00} (1,2) the Richardson potential, the simplest interpolation incorporating both asymptotic freedom and confinement. Quite obviously, this is a drastic approximation. It is probably quite adequate for heavy quarks, but its validity is much more questionable for light ones. Unfortunately, it seems impossible for the time being, to assess quantitatively its degree of validity, but it should be borne in mind that it may affect the conclusions of this paper.

With such a static limit for all Green's functions in (2.6), it is easy to write down a Dirac equation for the subspace of N_c-quark states:

$$\left[\sum_{i=1}^{N_c}(\boldsymbol{\alpha}_i\cdot\mathbf{p}_i+m_i\beta_i)+\tfrac{1}{4}\sum_{i<j}\lambda(i)\cdot\lambda(j)V(\mathbf{x}_i-\mathbf{x}_j)+\cdots+3\text{-},\ldots,N_c\text{-body terms}\right]\Psi(\mathbf{x}_1,\ldots,\mathbf{x}_{N_c})$$
$$=E\Psi(\mathbf{x}_1,\ldots,\mathbf{x}_{N_c})\,. \tag{2.8}$$

Little is known about the interactions involving more than two quarks, except that they should be all of the same order of magnitude as far as the $1/N_c$ expansion is concerned. Other considerations however may reduce their importance. For three-quark systems with $N_c=3$, the diagram shown in Fig. 2(a) gives no contribution,[12] because the expectation value of $f^{abc}\lambda^a\lambda^b\lambda^c$ between color singlets vanishes. Some higher-order diagrams [Fig. 2(b)] can easily be shown[13] to vanish in the same manner. In view of the present lack of knowledge on the remaining nonvanishing three-body diagrams, we will drop them in our computational work. When dealing with more than three colors, we will do the same for all Green's functions involving more than two quarks. Our Hamiltonian will thus be reduced to

$$H=\sum_{i=1}^{N}(\boldsymbol{\alpha}_i\cdot\mathbf{p}_i+m_i\beta_i)+\tfrac{1}{4}\sum_{i<j}\lambda(i)\cdot\lambda(j)V(r_{ij})\,, \tag{2.9}$$

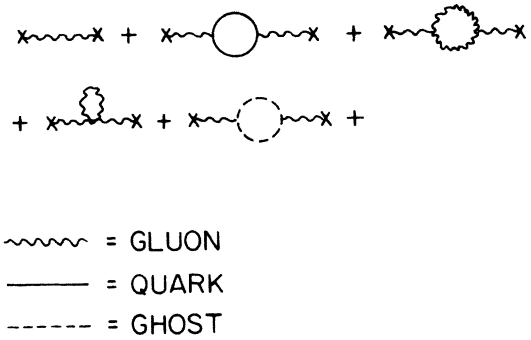

FIG. 1. Some of the terms appearing in the two-point Green's function for the gluons. As discussed in the text, the second diagram is suppressed by $1/N_c$ and is dropped.

with the Richardson potential for $\frac{1}{4}\lambda(i)\cdot\lambda(j)V(r_{ij})$.

Let us come now to the HF approximation. In order to avoid the complexities of open-shell and multiconfiguration HF calculations, we will limit ourselves in this paper to the simplest case, in which N_c quarks all have the same flavor and occupy the same state $\phi_{jm}(\mathbf{r})$. In the formalism of second quantization, this Slater determinant is written as

$$\Psi=\prod_{c=1}^{N_c} a_{fcjm}^{\dagger}\,|\,0\rangle\ , \tag{2.10}$$

where $a_{fcjm}^{\dagger}$ creates a quark of color c and flavor f in state (jm). The state ϕ_{jm} is chosen so as to minimize the average value of the energy,

$$E=\langle\Psi\,|\,H\,|\,\Psi\rangle\ . \tag{2.11}$$

In order to compute this quantity, one needs the matrix element

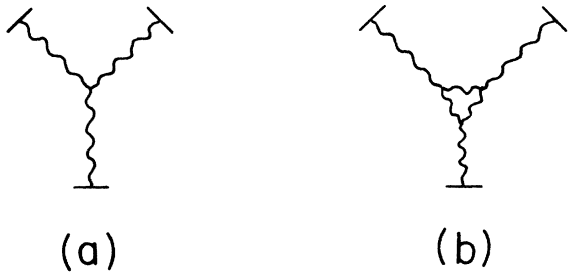

FIG. 2. Vanishing three-body diagrams.

$$\langle\text{color singlet}\,|\,\lambda(i)\cdot\lambda(j)/4\,|\,\text{color singlet}\rangle=-\frac{N_c+1}{2N_c}\ . \tag{2.12}$$

One then obtains readily

$$E=N_c\int\phi_{jm}^{\dagger}(\mathbf{r})t\phi_{jm}(\mathbf{r})d\mathbf{r}-\frac{N_c^2-1}{4}\int\phi_{jm}^{\dagger}(\mathbf{r}_1)\phi_{jm}^{\dagger}(\mathbf{r}_2)V(r_{12})\phi_{jm}(\mathbf{r}_1)\phi_{jm}(\mathbf{r}_2)d\mathbf{r}_1d\mathbf{r}_2\ , \tag{2.13}$$

where

$$t\equiv\boldsymbol{\alpha}\cdot\mathbf{p}+\beta m\ . \tag{2.14}$$

Varying this expression with respect to ϕ_{jm}, subject to the constraint

$$\int\phi_{j'm'}^{\dagger}(\mathbf{r})\phi_{jm}(\mathbf{r})d\mathbf{r}=\delta_{jj'}\delta_{mm'}\ , \tag{2.15}$$

yields the HF equation

$$\left[t-\frac{N_c^2-1}{2N_c}\int\phi_{jm}^{\dagger}(\mathbf{r}')V(\mathbf{r}-\mathbf{r}')\phi_{jm}(\mathbf{r}')d\mathbf{r}'\right]\phi_{jm}(\mathbf{r})=\epsilon\phi_{jm}(\mathbf{r})\ , \tag{2.16}$$

where ϵ is the single-particle energy. One gets a single equation, instead of a set of coupled equations, since all particles are in the same orbital ϕ_{jm}. For the same reason, there is no exchange matrix element for V and the single-particle self-consistent potential

$$w_{\text{av}}(\mathbf{r})=-\frac{N_c^2-1}{2N_c}\int\phi_{jm}^{\dagger}(\mathbf{r}')V(\mathbf{r}-\mathbf{r}')\phi_{jm}(\mathbf{r}')d\mathbf{r}' \tag{2.17}$$

is local. This self-consistent potential obviously behaves like the time component of a vector, and not as a mass term. The total energy E_{HF} is related to the single-quark energy ϵ by the usual relation

$$E_{\text{HF}}=N(\epsilon-\tfrac{1}{2}\langle\phi_{jm}\,|\,w_{\text{av}}\,|\,\phi_{jm}\rangle)\ . \tag{2.18}$$

For quarks in the lowest $(1s\frac{1}{2})$ orbital, one may write

$$\phi_{1s\frac{1}{2}m}(\mathbf{r})=\begin{bmatrix}\frac{1}{4\pi}\end{bmatrix}^{1/2}\begin{bmatrix}iG(r)\chi_m\\ \boldsymbol{\sigma}\cdot\hat{\mathbf{r}}F(r)\chi_m\end{bmatrix}\ , \tag{2.19}$$

where χ_m is a Pauli spinor, and Eq. (2.16) yields the system of coupled equations:

$$\frac{dG}{dr}-(m-w_{\text{av}}+\epsilon)F=0\ , \tag{2.20a}$$

$$\frac{dF}{dr}+\frac{2}{r}F+(\epsilon-w_{\text{av}}-m)G=0\ , \tag{2.20b}$$

where

$$w_{\text{av}}\equiv-\frac{N_c^2-1}{2N_c}\int[G^2(r')+F^2(r')]V_0(r,r')r'^2dr'\ , \tag{2.21}$$

$V_0(r,r')$ being the $l=0$ part of $V(\mathbf{r}-\mathbf{r}')$:

$$V_0(r,r')=\tfrac{1}{2}\int_0^{\pi}V(\mathbf{r}-\mathbf{r}')\sin\theta\,d\theta\ . \tag{2.22}$$

Quite obviously the single-particle energy ϵ does not depend on N_c when w_{av} is itself independent of this parameter. In the limit $N_c\to\infty$, this condition is seen to be realized if V is proportional to $1/N_c$, in accordance with the philosophy of the $1/N_c$ expansion.[2]

It may be noticed, in Eqs. (2.20), that the single-particle potential (2.21) is added to the energy term. This will be shown later to lead to too large a radius for the confined system. Indeed, confinement problems connected with potentials considered as the time component of a four-vector have been known for quite a long time.[14] Some authors have tried to overcome them by considering scalar potentials, but these are not without problems either, since an oscillatorlike scalar potential does not generate real eigenvalues of the Dirac equation.[15] A much more common prescription[16] has been to use an equal mixture of a static vector part and a scalar part. This prescription has no sound theoretical justification so far. One expects a potential which will behave like a mass term if the explicit chiral-symmetry breaking is introduced as a constraint in the action, as done, for example, by Simić,[17] but it is not

at all obvious that this potential would be approximately equal to the static vector potential. Still, in spite of its lack of justification this prescription has a certain number of appealing features: it contributes no spin-orbit splitting to the p states,[16] in agreement with phenomenology; it realizes an exact SU(2)-spin symmetry of the Dirac Hamiltonian[16,18] and allows an exact Melosh transformation.[19] It has been applied successfully to low-lying S-wave baryons[20] and to quark-antiquark systems,[21,22] Crater and Van Alstine,[22] in particular, have broken up the confining part of the Richardson potential into an equal vector and scalar part and put the terms appropriately with the energy and the mass, respectively. We have tried both approaches: (i) treating the entire potential like an energy term and (ii) splitting the confining part into half-energy-like and half-mass-like terms, and the results will be discussed in the following section.

It is well known that HF solutions violate the translation invariance of the underlying Hamiltonian (2.9), since they are formed of single-particle wave functions derived from an average potential $w_{\rm av}$ that is not translation invariant. As a consequence, the center-of-mass momentum is not well defined in HF solutions and this entails a spurious contribution from the center-of-mass kinetic energy to the total energy. Since the relative importance of this spurious effect increases as the number of particles decreases, it is important that it should be corrected for systems formed of few quarks. This can be done by extending to the relativistic HF equations the Peierls-Yoccoz[23] procedure of nuclear physics.

Let $\Psi(\mathbf{x})$ be the HF wave function in configuration space, where $\mathbf{x}$ denotes in a collective way the quark coordinates. It is possible to expand $\Psi(\mathbf{x})$ on the basis of eigenvectors of the center-of-mass momentum:

$$\Psi(\mathbf{x})=\int e^{i\mathbf{k}\cdot\mathbf{R}}\psi_{\mathbf{k}}^{\rm int}(\mathbf{x}-\mathbf{R})d\mathbf{k}\ , \tag{2.23}$$

where $\mathbf{R}$ is the center-of-mass coordinate and $\psi_{\mathbf{k}}^{\rm int}(\mathbf{x}-\mathbf{R})$ describes the internal motion of the system. Of course, this internal wave function will be different for every value of $\mathbf{k}$, since there is no factorization of the center-of-mass motion. The Peierls-Yoccoz method allows one to project out of $\Psi(\mathbf{x})$ a state $\Psi_{\mathbf{k}}^{\rm PY}$ of definite total momentum $\mathbf{k}$:

$$\begin{aligned}\Psi_{\mathbf{k}}^{\rm PY}(\mathbf{x})&=\frac{1}{(2\pi)^3}\int e^{i\mathbf{k}\cdot\mathbf{a}}\Psi(\mathbf{x}-\mathbf{a})d\mathbf{a}\\&=e^{i\mathbf{k}\cdot\mathbf{R}}\psi_{\mathbf{k}}^{\rm int}(\mathbf{x}-\mathbf{R})\ ,\end{aligned} \tag{2.24}$$

where $\Psi(\mathbf{x}-\mathbf{a})$ is the HF wave function translated in such a way that it is centered at point $\mathbf{a}$.

Once this projection is performed, it is quite easy to compute the expectation value of any operator $F(\mathbf{P})$ depending on the total momentum $\mathbf{P}$, when the system is in the state $\Psi(\mathbf{x})$:

$$\begin{aligned}\langle\Psi(\mathbf{x})\,|\,F(\mathbf{P})\,|\,\Psi(\mathbf{x})\rangle&=\int\langle\Psi_{\mathbf{k}'}^{\rm PY}(\mathbf{x})\,|\,F(\mathbf{P})\,|\,\Psi_{\mathbf{k}}^{\rm PY}\rangle d\mathbf{k}\,d\mathbf{k}'\\&=\int F(\mathbf{k})\langle\Psi_{\mathbf{k}'}^{\rm PY}\,|\,\Psi_{\mathbf{k}}^{\rm PY}\rangle d\mathbf{k}\,d\mathbf{k}'\ .\end{aligned} \tag{2.25}$$

Now, it is trivial to show that

$$\langle\Psi_{\mathbf{k}'}^{\rm PY}\,|\,\Psi_{\mathbf{k}}^{\rm PY}\rangle=\delta(\mathbf{k}-\mathbf{k}')I(\mathbf{k})\ , \tag{2.26}$$

where

$$I(\mathbf{k})\equiv\frac{1}{(2\pi)^3}\int e^{i\mathbf{k}\cdot\boldsymbol{\alpha}}N(\boldsymbol{\alpha})d\boldsymbol{\alpha} \tag{2.27}$$

is the Fourier transform of the overlap kernel

$$N(\mathbf{a}-\mathbf{a}')\equiv\langle\Psi(\mathbf{x}-\mathbf{a}')\,|\,\Psi(\mathbf{x}-\mathbf{a})\rangle\ . \tag{2.28}$$

Substituting (2.26) into (2.25), one gets

$$\langle\Psi(\mathbf{x})\,|\,F(\mathbf{p})\,|\,\Psi(\mathbf{x})\rangle=\int F(\mathbf{k})I(\mathbf{k})d\mathbf{k}\ . \tag{2.29}$$

One is now left with the task of computing $I(\mathbf{k})$. The single-quark overlap kernel for wave functions (2.19) is

$$\begin{aligned}n(\boldsymbol{\alpha})&=\langle\phi_{1s\frac{1}{2}m}(\mathbf{r}+\boldsymbol{\alpha}/2)\,|\,\phi_{1s\frac{1}{2}m}(\mathbf{r}-\boldsymbol{\alpha}/2)\rangle\\&=\frac{1}{4\pi}\int[\,G(r_+)G(r_-)\\&\qquad+F(r_+)F(r_-)\boldsymbol{\sigma}\cdot\hat{\mathbf{r}}_+\boldsymbol{\sigma}\cdot\hat{\mathbf{r}}_-]d\mathbf{r}\ ,\end{aligned} \tag{2.30}$$

where

$$\mathbf{r}_\pm\equiv\mathbf{r}\pm\boldsymbol{\alpha}/2\ . \tag{2.31}$$

Now,

$$\boldsymbol{\sigma}\cdot\hat{\mathbf{r}}_+\boldsymbol{\sigma}\cdot\hat{\mathbf{r}}_-=\frac{(r^2-\alpha^2/4)+\boldsymbol{\sigma}\cdot\mathbf{r}\times\boldsymbol{\alpha}}{r_+r_-}\ ,$$

and the term in $\boldsymbol{\sigma}\cdot\mathbf{r}\times\boldsymbol{\alpha}$ brings no contribution to (2.30) since it yields an integrand that is odd in $\mathbf{r}$. One is thus left with

$$\begin{aligned}n(\boldsymbol{\alpha})&=n(a)\\&=\tfrac{1}{2}\int_0^\infty r^2dr\int_{-1}^{1}du\left[G(r_+)G(r_-)+\frac{F(r_+)F(r_-)}{r_+r_-}\left[r^2-\frac{\alpha^2}{4}\right]\right],\end{aligned} \tag{2.32}$$

where u is the cosine of the angle between $\mathbf{r}$ and $\boldsymbol{\alpha}$.

The overlap for N_c quarks in the same orbital is simply

$$N(\alpha)=[n(\alpha)]^{N_c} \tag{2.33}$$

and

$$\begin{aligned}I(\mathbf{k})&=I(k)\\&=\frac{1}{2\pi^2k}\int_0^\infty\alpha\sin k\alpha[n(\alpha)]^{N_c}d\alpha\ .\end{aligned} \tag{2.34}$$

Taking for $F(\mathbf{P})$ the center-of-mass kinetic energy operator

$$T_{\rm c.m.}=F(\mathbf{P})=(M^2+\mathbf{P}^2)^{1/2}-M\ , \tag{2.35}$$

where M is the mass of the system, one gets finally

$$\begin{aligned}\langle\Psi\,|\,T_{\rm c.m.}\,|\,\Psi\rangle=\frac{2}{\pi}\int_0^\infty k\alpha[(M^2+k^2)^{1/2}-M]\\\times\sin(k\alpha)[n(\alpha)]^{N_c}dk\,d\alpha\ .\end{aligned} \tag{2.36}$$

TABLE I. Hartree-Fock energy E_{HF} and center-of-mass kinetic energy $\langle T_{c.m.} \rangle$ for a timelike vector average potential derived from the Richardson two-body interaction for the Ω^- with $m_s = 150$ MeV. All quantities except the average radius $\langle r^2 \rangle^{1/2}$ (fm) are in MeV. E_0 is the single-particle energy and $\Delta\epsilon$ is the nodal excitation energy when the Hartree-Fock field is kept unchanged.

Λ	E_{HF}	$\langle T_{c.m.} \rangle$	$E_{HF} - \langle T_{c.m.} \rangle$	E_0	$\Delta\epsilon$	$(\langle r^2 \rangle)^{1/2}$
400	2247	197	2050	1081	334	0.92
350	1950	155	1795	916	294	0.99
325	1831	140	1691	860	281	1.08
300	1690	122	1568	781	263	1.13

The calculation of this quantity thus involves a fourfold integral. Particular attention has to be paid to the integration over k, which was handled through Filon's method.

Although the Peierls-Yoccoz method enables one to subtract the average value of the center-of-mass kinetic energy, it does obviously not eliminate all effects connected with spurious center-of-mass motion: in particular, it does not change anything to the fact that the HF wave function is a linear combination or internal wave functions, Eq. (2.23), each of which yields a different average value of the internal energy.

III. NUMERICAL RESULTS

Taking for granted the legitimacy of describing hadrons by an effective Hamiltonian with two-body interactions, there remains the question of the validity of the HF approximation. We obviously have no exact relativistic solutions for the three-body problem to compare our HF solution with, and the best we can do is to perform such a calculation in a situation where relativistic effects are presumably small. We took as a reference Richard's calculation[24] of the Ω^- mass, using the Martin[25] potential in the hyperspherical formalism. Since the strange quark is given a mass of 518 MeV in this calculation, one might expect the situation to be nonrelativistic. Richard found a mass of 1617 MeV before including spin-dependent corrections. With the same potential and quark mass, we obtained $E_{HF} = 1821$ MeV. The average value of the center-of-mass energy was computed as described in Sec. II. Since such a procedure removes the center-of-mass spurious behavior in an approximate manner only, we did not try to reach self-consistency between the mass used in computing the correction and the corrected mass. We thus used the experimental value of the Ω^- mass in Eq. (2.36), finding $\langle T_{c.m.} \rangle = 237$ MeV. The corrected mass thus went down to 1584 MeV. Although this is reasonably close to Richard's value, it turns out to be difficult to draw any definite conclusion from this agreement, since the wave functions, somewhat to our surprise, turned out to have relatively large small components (Fig. 3), and thus not to be nonrelativistic. This raises, among other things, the question of the appropriateness of the Martin potential for our calculation, since it was devised for nonrelativistic systems.

Such a question does not arise, of course, for Richardson's potential:

TABLE II. Dependency of the Hartree-Fock energy $E_{HF}(m_s)$ on the strange-quark mass m_s. All quantities are in MeV.

Λ	E_{HF} (150)	E_{HF} (300)	E_{HF} (500)
400	2247	2320	2605
300	1690	1833	2198

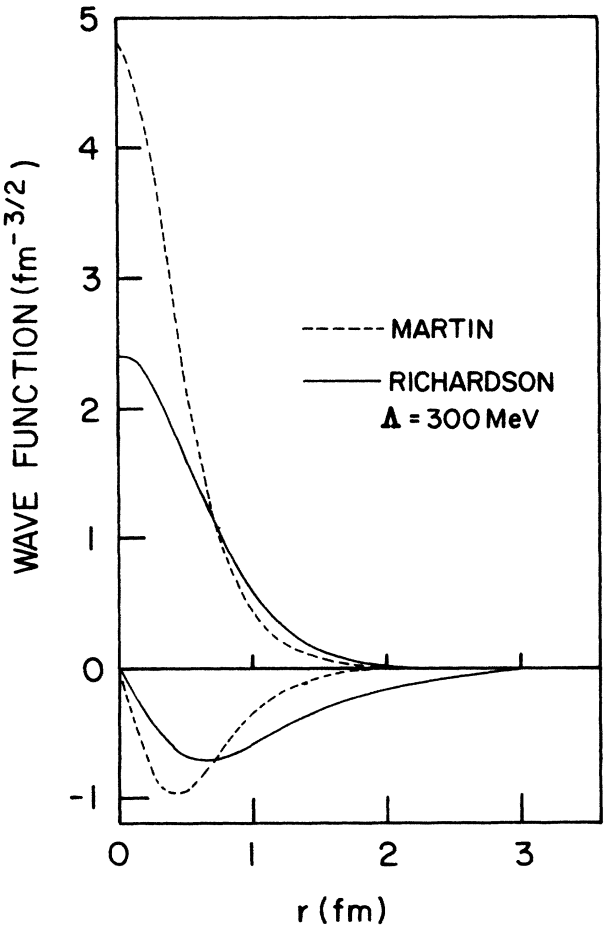

FIG. 3. Large (upper half of the figure) and small (lower half) components of the wave functions for the potentials of Richardson ($\Lambda = 300$ MeV) and Martin.

TABLE III. Hartree-Fock energy for half-vector, half-scalar confining potential for nucleonlike systems with $m_q = 10$ MeV. The notation is as in Table I, μ is the magnetic moment in nuclear magnetons.

Λ	E_{HF}	$\langle T_{c.m.} \rangle$	$E_{HF} - \langle T_{c.m.} \rangle$	$\langle r^2 \rangle^{1/2}$	ϵ_0	$\Delta\epsilon$	μ
400	1523	154	1369	0.96	647	282	2.48
350	1373	124	1249	1.03	562	245	2.48
300	1212	87	1125	1.15	481	189	2.38

$$V(r) = \frac{\lambda(1)\cdot\lambda(2)}{4} \frac{6\pi}{33-2N_f} \Lambda \left[\Lambda r - \frac{f(\Lambda r)}{\Lambda r} \right], \quad (3.1)$$

where N_f is the number of flavors, taken to be three, and

$$f(t) \equiv 1 - 4 \int_1^\infty \frac{dq}{q} \frac{e^{-qt}}{[\ln(q^2-1)]^2 + \pi^2} . \quad (3.2)$$

We have computed the Ω^- mass with this potential, which contains only one free parameter: the scale factor Λ. Since our calculation is relativistic, we need not work with an unrealistically large quark mass and we can take the current-algebra value,[11] $m_s = 150$ MeV. The results for the case where the entire potential is treated like the time part of a vector [Eqs. (2.20a) and (2.20b)] are given in Table I for several values of Λ. They are rather sensitive to the value of this parameter. In view of uncertainties in the value of the quark mass, we have checked how sensitive the HF energy is to m_s. One sees in Table II that doubling the mass from 150 to 300 MeV changes E_{HF} by less than 5% for $\Lambda = 400$ MeV and 10% for $\Lambda = 300$ MeV. This relative lack of sensitivity is due to the fact that the increase in m_s is compensated, to a large extent, by a decrease of the kinetic energy. However, as shall be seen later, a decrease of the quark mass enhances the small component of the wave function and leads to confinement problems.

The single-quark wave functions obtained with the Martin and the Richardson ($\Lambda = 300$ MeV) potentials are compared in Fig. 3 and appear to be less confined in the second case. As can be expected, the Richardson wave functions are less confined for $\Lambda = 300$ MeV than for $\Lambda = 400$ MeV (Fig. 4), and more relativistic for $m_s = 150$ MeV than for $m_s = 500$ MeV (Fig. 5).

It can be seen in Fig. 5 that as the quark mass is decreased the wave function, particularly the small component, is enhanced at large radius. The average radius of about 1 fm given in Table I is already large. If one were to calculate the radius of systems like the nucleon or the delta isobar one would have to use current-quark masses of the order of 10 MeV. With $\Lambda = 300$ MeV and $m_q = 10$ MeV the Richardson potential yields wave functions leading to an average radius of about 1.7 fm. Clearly this is unrealistic and the purely vector potential is unable to confine massless quarks. As we have indicated before, this problem is well known and we circumvent it by dividing the linear part of the Richardson potential in Eq. (3.1)

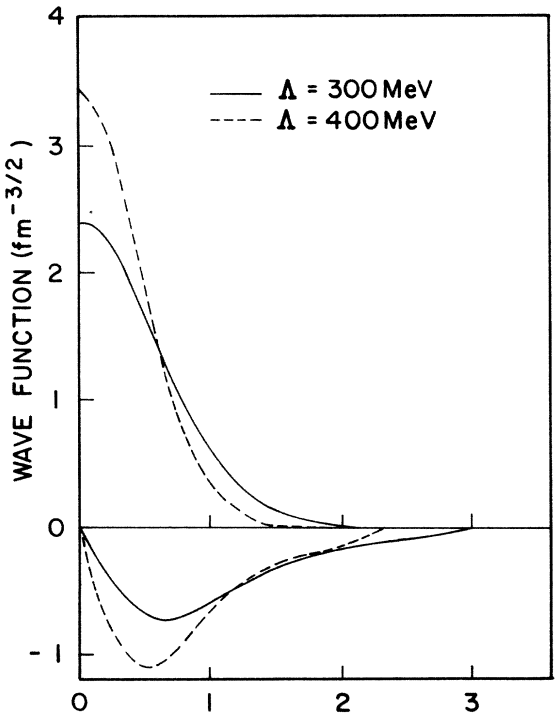

FIG. 4. Large (upper half of the figure) and small (lower half) components of the wave functions for two different values of the parameter Λ in the Richardson potential.

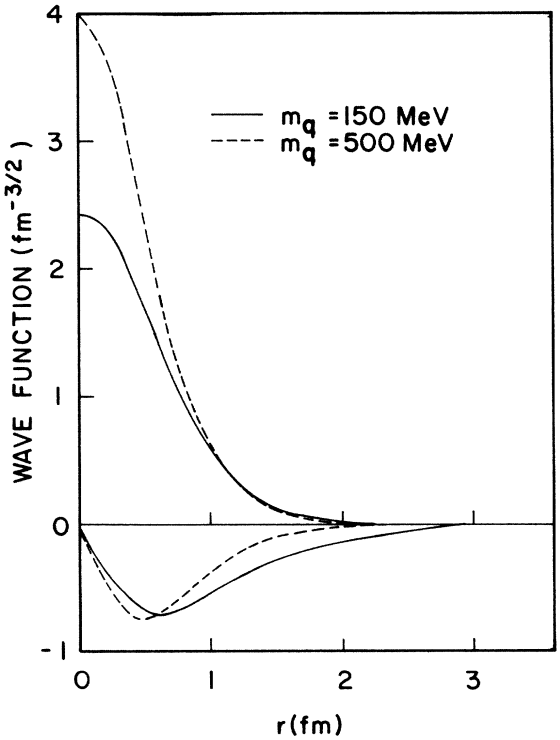

FIG. 5. Large (upper half of the figure) and small (lower half) components of the wave functions for two different values of the quark mass and the Richardson potential ($\Lambda = 300$ MeV).

TABLE IV. Same as in Table III but for quark mass $m_s = 150$ MeV appropriate for the Ω^-.

Λ	E_{HF}	$\langle T_{c.m.} \rangle$	$E_{HF} - \langle T_{c.m.} \rangle$	$\langle r^2 \rangle^{1/2}$ (fm)	ϵ_0	$\Delta\epsilon$	μ
400	1778	190	1588	0.75	652	345	−1.81
350	1598	150	1448	0.83	578	304	−2.19
300	1419	115	1304	0.93	503	264	−2.00

into two equal parts, as done by Crater and Van Alstine,[22] $S = V = 8\pi\Lambda^2 r/54$, the vector linear part in the average potential w_{av} [Eq. (2.21)] being half of what it was, while the contribution from the second term of the Richardson potential remains the same. The S term now gives a v_{av} which is found through Eqs. (2.21) and (2.22), with V replaced by S, so that Eqs. (2.20) are changed to

$$\frac{dG}{dr} - (m + v_{av} - w_{av} + \epsilon)F = 0 , \tag{3.3a}$$

$$\frac{dF}{dr} + \frac{2}{r}F + (\epsilon - w_{av} - m - v_{av})G = 0 . \tag{3.3b}$$

For $m_q = 10$ MeV and $\Lambda = 300$ MeV, the radius of the system now becomes 1.1 fm, which is more reasonable. The results for various values of Λ are given in Table III. For the Ω^- system as shown in Table IV the mass is 1588 MeV for $\Lambda = 400$ MeV in very good agreement with experiment. The magnetic moment of the Ω^-, $\mu(\Omega^-)$, is not experimentally known. However, in large-N_c expansion it is related to the $\mu(\Lambda^0)$, as shown by Karl and Paton,[26]

$$\mu(\Omega^-) = 3\mu(\Lambda^0) . \tag{3.4}$$

Since $\mu(\Lambda^0)$ is known to be -0.6138 ± 0.0047 nuclear magneton[27] the agreement between the $\mu(\Omega^-)$ in Table IV for $\Lambda = 400$ MeV and Eq. (3.4) is very good. In Table III we give the calculated magnetic moment of the nucleonlike system assuming it is a proton composed of *uud*. Whereas we neglected the difference between the *u* and *d* flavors in the Hartree-Fock calculation and assumed a shell of identical quarks, we put appropriate charges on these quarks in the magnetic calculation. In the absence of strong spin-dependent forces, this may be justified. Again, the magnetic moments in Table III are close to the experimental proton magnetic moment. The mass of the nucleonlike system is close to the mean of the nucleon and the delta isobar for $\Lambda = 300$ MeV. But we feel less sure about the results of Table III since in the case of light quarks there are (i) greater uncertainties about the legitimacy of using a static potential, (ii) greater uncertainties in the masses, (iii) mass difference between the two flavors and, and (iv) effects from the open-shell character of the wave function. We prefer the value of $\Lambda = 400$ MeV since this is in accord with lattice string tension estimates and meson sector results.

IV. CONCLUSION

In conclusion we are of the opinion that the Richardson potential with $\Lambda = 400$ MeV can explain baryon properties as well as meson properties provided the confinement part is split into equal scalar and vector parts. For a purely vector potential there are problems with confinement. It is remarkable that a large-N_c theory with practically no free parameters can achieve such good results.

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