Exact parapositroniumlike solution to two-body Dirac equations

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Recently we used a supersymmetric version of Dirac's constraint mechanics to derive coupled Dirac equations for a quark and antiquark in mutual chromodynamiclike interaction. Here we investigate the properties of our equations for two spinning particles in mutual electromagneticlike interaction. In the chiral Dirac matrix representation we explicitly obtain a family of exact 16-component solutions with closed-form energy spectrum that agrees with that of parapositronium through order α^4. We also find that various rearrangements of our Dirac equations simultaneously yield (for weak potentials) the relativistic wave equations of Todorov and Pilkuhn as well as (for weak potentials and slow motion) the semirelativistic interaction structure of Breit.

Various authors have used spin-dependent relativistic wave equations (with varying degrees of success) to describe the bound $q\bar{q}$ system.[1] But spectral and decay-width results of the light-quark systems are extremely sensitive to the form of the wave equation employed (or equivalently to the form of the resulting relativistic wave function). Since relativistic quark interactions are governed by the nonperturbative structure of QCD (which is largely unknown), there is apparently great latitude in the choice of wave equations and potentials for phenomenological descriptions of the $q\bar{q}$ system. We contend, however, that any relativistic quantum mechanics that, in principle, can describe weak forces as well as strong ones must, to be taken seriously, be capable of duplicating the well-known weak potential results of perturbative quantum field theory. In particular, acceptable spin-dependent relativistic wave equations must at least be capable of incorporating the detailed interaction structure of quantum electrodynamics. Elsewhere, we have shown how to achieve this goal for spinless particles described by quantum wave equations obtained through quantization of relativistic constraint mechanics.[2] In this note we find that when we carry out the same program for two spin-$\frac{1}{2}$ particles, we obtain a pair of compatible coupled Dirac equations that not only reproduce perturbative spectral results of QED, but also possess an exact 16-component solution in a case of physical interest—that of singlet positronium. Furthermore, the existence of this solution is a direct consequence of the compatibility of our wave equations.

The relativistic wave equations that we shall investigate result from the quantization of relativistic constraint mechanics.[3] Relativistic constraint mechanics for two spinless particles is a covariant canonical dynamics generated by coupled mass-shell conditions $\mathcal{H}_1=p_1^2+m_1^2+\Phi_1\approx 0$, $\mathcal{H}_2=p_2^2+m_2^2+\Phi_2\approx 0$ that must be compatible in the sense that $\{\mathcal{H}_1,\mathcal{H}_2\}_{\rm PB}\approx 0$. This requirement is satisfied if $\Phi_1=\Phi_2=\Phi$ ("third law") and if Φ depends on $x\equiv x_1-x_2$ only through $x_\perp$ such that $P\cdot x_\perp\equiv 0$, where $P\equiv p_1+p_2$ (covariant regulation of the "relative time").

The quantum version is governed by coupled Klein-Gordon equations $\mathcal{H}_1\psi=(p_1^2+m_1^2+\Phi)\psi=0$, $\mathcal{H}_2\psi=(p_2^2+m_2^2+\Phi)\psi=0$ which are supposed to be compatible in the sense that $[\mathcal{H}_1,\mathcal{H}_2]\psi=0$. In terms of the collective variables P, $p\equiv(\varepsilon_2/w)p_1-(\varepsilon_1/w)p_2$ (where $w\equiv(-P^2)^{1/2}$, $\varepsilon_1=\frac{1}{2}w[1+(m_1^2-m_2^2)/w^2]$, $\varepsilon_2=\frac{1}{2}w[1+(m_2^2-m_1^2)/w^2]$), the difference $(\mathcal{H}_1-\mathcal{H}_2)\psi=2P\cdot p\psi=0$ becomes a "ghost-killing" condition that removes the relative time in the c.m. rest frame, while the weighted sum

$$\mathcal{H}\psi=\left(\frac{\varepsilon_2}{w}\mathcal{H}_1+\frac{\varepsilon_1}{w}\mathcal{H}_2\right)\psi=(-\varepsilon_w^2+m_w^2+p^2+\Phi)\psi=0$$

(where $\varepsilon_w=\frac{1}{2}w[1-(m_1^2+m_2^2)/w^2]$ and $m_w=m_1m_2/w$) becomes a Klein-Gordon equation for the two-body *system.*

For two spin-$\frac{1}{2}$ particles, the quantum system is governed by the simultaneous "two-body Dirac equations"[4]

$$D_1\psi\equiv\gamma_1^5\mathcal{S}_1\psi=0,\ D_2\psi\equiv\gamma_2^5\mathcal{S}_2\psi=0\ , \tag{1}$$

which will be compatible in the sense that $[D_1,D_2]\psi=0$ if $[\mathcal{S}_1,\mathcal{S}_2]\equiv 0$.

We construct such a compatible system that has an *electromagnetic* structure (with the correct heavy-particle limits) by introducing the minimal substitutions $p_1\to p_1-A_1\equiv\pi_1$, $p_2\to p_2-A_2\equiv\pi_2$ into free Dirac equations so that $D_1\psi=(\pi_1\cdot\gamma_1+m_1)\psi=0$, $D_2\psi=(\pi_2\cdot\gamma_2+m_2)\psi=0$ are compatible:

$$[\gamma_1^5(\pi_1\cdot\gamma_1+m_1),\gamma_2^5(\pi_2\cdot\gamma_2+m_2)]\equiv 0\ , \tag{2}$$

where[2,5]

$$\begin{aligned}A_1&=[1-\tfrac{1}{2}(G+G^{-1})]p_1\\&\quad+\tfrac{1}{2}(G-G^{-1})p_2-\tfrac{1}{2}i(\partial G\cdot\gamma_2)\gamma_2\ ,\\A_2&=[1-\tfrac{1}{2}(G+G^{-1})]p_2\\&\quad+\tfrac{1}{2}(G-G^{-1})p_1+\tfrac{1}{2}i(\partial G\cdot\gamma_1)\gamma_1\end{aligned} \tag{3}$$

$[G^2=1/(1-2\mathcal{A}/w)]$ incorporate the Gordon decomposition of the electromagnetic current. "Squaring" these

Dirac equations leads to $\mathcal{H}_1\psi=0$, $\mathcal{H}_2\psi=0$, where

$$\mathcal{H}_1=\mathcal{S}_1{}^2=\pi_1{}^2-\tfrac{1}{2}\sigma_1^{\mu\nu}F_{1\mu\nu}+m_1{}^2 \ ,$$

$$\mathcal{H}_2=\mathcal{S}_2{}^2=\pi_2{}^2-\tfrac{1}{2}\sigma_2^{\mu\nu}F_{2\mu\nu}+m_2{}^2 \qquad \left(F_a^{\mu\nu}\equiv\frac{1}{i}[\pi_a^\mu,\pi_a^\nu]\right) .$$

The difference $(\mathcal{H}_1-\mathcal{H}_2)\psi=2P\cdot p\psi=0$ (just as in the spinless case) while the weighted sum

$$\mathcal{H}\psi=\left(\frac{\varepsilon_2}{w}\mathcal{H}_1+\frac{\varepsilon_1}{w}\mathcal{H}_2\right)\psi=0$$

yields a system wave equation whose "upper-upper" four-component piece reduces for weak potentials ($\mathcal{A}\ll m_1c^2$, m_2c^2) to Todorov's homogeneous quasipotential equation for spinor quantum electrodynamics,[6] when $\mathcal{A}=-\alpha/r$ [where $r=(x_\perp{}^2)^{1/2}$]:

$$\left[\mathbf{p}^2+m_w{}^2-(\varepsilon_w-\mathcal{A})^2+\frac{\mathcal{A}'}{r}\left(\frac{w+m_1}{w(\varepsilon_1+m_1)}\mathbf{L}\cdot\boldsymbol{\sigma}_1+\frac{w+m_2}{w(\varepsilon_2+m_2)}\mathbf{L}\cdot\boldsymbol{\sigma}_2\right)\right.$$
$$\left.+\frac{\nabla^2\mathcal{A}}{2}\left(\frac{w+m_1}{w(\varepsilon_1+m_1)}+\frac{w+m_2}{w(\varepsilon_2+m_2)}-\frac{1}{w}(1-\tfrac{2}{3}\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2)\right)+\tfrac{1}{6}(3\boldsymbol{\sigma}_1\cdot\hat{\mathbf{r}}\boldsymbol{\sigma}_2\cdot\hat{\mathbf{r}}-\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2)\left(-\frac{\mathcal{A}''}{w}+\frac{\mathcal{A}'}{rw}\right)\right]\phi=0 \quad (4)$$

(in the c.m. rest frame).

For weak binding energies, division of the spin-dependent interaction by $2m_1m_2/(m_1+m_2)$ gives directly the complete Fermi-Breit spin-dependent interaction for electromagnetism, while approximate solution for the binding energy leads directly to Schwinger's method for calculation of the perturbative positronium spectrum.[2,7]

But, does the full 16-component system determined by $D_1\psi=0$, $D_2\psi=0$ make sense? We are aided in answering this question by a special property of our interaction. Since Dirac γ matrices appear in $\mathcal{H}_1$, $\mathcal{H}_2$, and $\mathcal{H}$ only through $\sigma_1^{\mu\nu}$ and $\sigma_2^{\mu\nu}$, the "squared" Dirac and system wave equations are automatically block diagonal in the "chiral" γ-matrix representation, related to the usual "Dirac" representation by

$$\gamma_{\rm ch}^0=-\gamma_D^5,\ \boldsymbol{\gamma}_{\rm ch}=\boldsymbol{\gamma}_D,\ \gamma_{\rm ch}^5=\gamma_D^0$$
$$\Longrightarrow\boldsymbol{\sigma}_{\rm ch}=\begin{pmatrix}\boldsymbol{\sigma}&0\\0&\boldsymbol{\sigma}\end{pmatrix},\ \sigma_{\rm ch}^{0j}=\begin{pmatrix}i\sigma^j&0\\0&-i\sigma^j\end{pmatrix}=i\gamma_{\rm ch}^5\sigma^j .$$

For

$$\psi_{\rm ch}=\frac{1}{G}\begin{pmatrix}\phi_{++}\\\phi_{+-}\\\phi_{-+}\\\phi_{--}\end{pmatrix} \quad (5)$$

with subspaces labeled by the eigenvalues of γ_1^5 and γ_2^5, $\mathcal{H}_{\rm ch}\psi_{\rm ch}=0$ implies

$$\{\mathbf{p}^2+m_w{}^2-(\varepsilon_w-\mathcal{A})^2+(\nabla\ln G\times\mathbf{p})\cdot(\boldsymbol{\sigma}_1+\boldsymbol{\sigma}_2)$$
$$\mp i[(\varepsilon_2-\mathcal{A})\boldsymbol{\sigma}_1-(\varepsilon_1-\mathcal{A})\boldsymbol{\sigma}_2]\cdot\nabla\ln G+\tfrac{1}{2}(1+\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2)\nabla^2\ln G+(\nabla\ln G)^2\}\phi=0 \quad (6)$$

for $\phi=\phi_{++}$, ϕ_{--} (which for weak potentials $\mathcal{A}$ becomes an equation recently discovered by Pilkuhn[8]) and

$$\{\mathbf{p}^2+m_w{}^2-(\varepsilon_w-\mathcal{A})^2+(\nabla\ln G\times\mathbf{p})\cdot(\boldsymbol{\sigma}_1+\boldsymbol{\sigma}_2)\mp i[(\varepsilon_2-\mathcal{A})\boldsymbol{\sigma}_1+(\varepsilon_1-\mathcal{A})\boldsymbol{\sigma}_2]\cdot\nabla\ln G$$
$$+\tfrac{1}{6}(3+\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2)[\nabla^2\ln G+2(\nabla\ln G)^2]-\tfrac{1}{3}(3\boldsymbol{\sigma}_1\cdot\hat{\mathbf{r}}\boldsymbol{\sigma}_2\hat{\mathbf{r}}-\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2)[\ln''G-\ln'G/r+2(\nabla\ln G)^2]\}\phi=0 \quad (7)$$

for $\phi=\phi_{+-}$, ϕ_{-+}—a new equation that collapses for an equal-mass singlet wave function ϕ_s to

$$[\mathbf{p}^2+m_w^2-(\varepsilon_w-\mathcal{A})^2]\phi_s=0 . \quad (8)$$

For a single free spin-$\frac{1}{2}$ particle, since $\mathcal{S}^2=\mathcal{H}=p^2+m^2$ where $\mathcal{S}=\gamma_5D=\gamma_5(p\cdot\gamma+m)$, we see that $\psi\equiv\mathcal{S}\Psi$ *automatically* solves $\mathcal{S}\psi=0$ (and hence $D\psi=0$) if $\mathcal{H}\Psi=0$; i.e., $\mathcal{S}\psi=\mathcal{S}^2\Psi=\mathcal{H}\Psi=0$. Then, since $[\mathcal{H},\gamma^5]\equiv0$, $\psi=\mathcal{S}\gamma^5\Psi=\gamma^5D\gamma^5\Psi=\tilde{D}\Psi=(-p\cdot\gamma+m)\Psi$ *automatically* solves $D\psi=0$ if $\mathcal{H}\Psi=0$. Appropriate choices for Ψ (solving the Klein-Gordon equation) lead to ψ's that are the u and v solutions of the free Dirac equation.

For two spin-$\frac{1}{2}$ particles governed by our "two-body Dirac equations" compatibility leads to a similar construction. Since $\mathcal{S}_1{}^2=\mathcal{H}_1$ and $\mathcal{S}_2{}^2=\mathcal{H}_2$, where $\mathcal{S}_1=\gamma_1^5D_1$ and $\mathcal{S}_2=\gamma_2^5D_2$, we see that $\psi\equiv\mathcal{S}_1\mathcal{S}_2\Psi$ *automatically* solves $\mathcal{S}_1\psi=0$ and $\mathcal{S}_2\psi=0$ (and hence both $D_1\psi=0$ and $D_2\psi=0$) if $\mathcal{H}_1\Psi=0$ and $\mathcal{H}_2\Psi=0$; i.e., $\mathcal{S}_1\psi=\mathcal{S}_1{}^2\mathcal{S}_2\Psi=\mathcal{S}_2\mathcal{H}_1\Psi=0$ and $\mathcal{S}_2\psi=\mathcal{S}_2\mathcal{S}_1\mathcal{S}_2\Psi=\mathcal{S}_1\mathcal{H}_2\Psi=0$. Note that such a solution exists only because the two Dirac equations are compatible in the sense that $[\mathcal{S}_1,\mathcal{S}_2]\equiv0$. Then, since $\mathcal{H}_1$ and $\mathcal{H}_2$ commute with γ_1^5 and γ_2^5 (and since $[\gamma_1^5,D_2]=0$),

$$\psi=\gamma_1^5D_1\gamma_1^5\gamma_2^5D_2\gamma_2^5\Psi=\tilde{D}_1\tilde{D}_2\Psi$$
$$=(-\pi_1\cdot\gamma_1+m_1)(-\pi_2\cdot\gamma_2+m_2)\Psi \quad (9)$$

automatically solves both $D_1\psi=0$ and $D_2\psi=0$ if both $\mathcal{H}_1\Psi=0$ and $\mathcal{H}_2\Psi=0$.

Rearrangement of $\mathcal{H}_1$ and $\mathcal{H}_2$ into the difference and weighted sum implies that both $D_1\psi=0$ and $D_2\psi=0$ if both $P\cdot p\Psi=0$ and $\mathcal{H}\Psi=0$. Since we can solve two of the $\mathcal{H}$ equations for an equal-mass singlet wave function in the chiral representation, $\psi=\tilde{D}_1\tilde{D}_2\Psi$ will give us the corresponding solution to the full coupled 16-component Dirac system.

That is, since

$$\Psi_{\rm ch}=\frac{1}{G}\begin{pmatrix}0\\ \phi_s\\ \phi_s\\ 0\end{pmatrix}$$

solves $\mathcal{H}_{\rm ch}\Psi_{\rm ch}=0$ when $[\mathbf{p}^2+m_w{}^2-(\varepsilon_w-\mathcal{A})^2]\phi_s=0$, both $D_1\psi=0$ and $D_2\psi=0$ are solved by

$$\begin{aligned}\psi_{\rm ch}&=(\tilde{D}_1\tilde{D}_2)_{\rm ch}\Psi_{\rm ch}\\ &=(-\pi_1\cdot\gamma_1+m_1)_{\rm ch}(-\pi_2\cdot\gamma_2+m_2)_{\rm ch}\frac{1}{G}\begin{pmatrix}0\\ \phi_s\\ \phi_s\\ 0\end{pmatrix}\\ &=2m^2\begin{pmatrix}\dfrac{-1}{m}\left[\left(\dfrac{w}{2}-\mathcal{A}\right)+\dfrac{(\sigma_1-\sigma_2)}{2}\cdot\mathbf{p}\right]\phi_s\\ \dfrac{1}{G}\phi_s\\ \dfrac{1}{G}\phi_s\\ -\dfrac{1}{m}\left[\left(\dfrac{w}{2}-\mathcal{A}\right)-\dfrac{(\sigma_1-\sigma_2)}{2}\cdot\mathbf{p}\right]\phi_s\end{pmatrix}.\end{aligned}\tag{10}$$

Under the unitary transformation that connects the chiral γ-matrix representation to the Dirac representation,

$$\begin{aligned}\psi_{\rm ch}\to\psi_D&=\frac{1}{\sqrt{2}}(1+\gamma_1^0\gamma_1^5)\frac{1}{\sqrt{2}}(1+\gamma_2^0\gamma_2^5)\psi_{\rm ch}\\ &=-2m\begin{pmatrix}\left[\left(\dfrac{w}{2}-\mathcal{A}\right)+\dfrac{m}{G}\right]\phi_s\\ \left(\dfrac{\sigma_1-\sigma_2}{2}\right)\cdot\mathbf{p}\phi_s\\ \left(\dfrac{\sigma_1-\sigma_2}{2}\right)\cdot\mathbf{p}\phi_s\\ \left[\left(\dfrac{w}{2}-\mathcal{A}\right)-\dfrac{m}{G}\right]\phi_s\end{pmatrix}\\ &=-2m\chi\begin{pmatrix}\phi_s\\ \dfrac{1}{\chi}\sigma_1\cdot\mathbf{p}\phi_s\\ -\dfrac{1}{\chi}\sigma_2\cdot\mathbf{p}\phi_s\\ -\dfrac{1}{\chi^2}\sigma_1\cdot\mathbf{p}\cdot\sigma_2\cdot\mathbf{p}\phi_s\end{pmatrix},\end{aligned}\tag{11}$$

where $\chi\equiv(w/2-\mathcal{A})+m/G$. The final form of the solution shows how Ψ_D "evolves" out of the direct product of u solutions to two free Dirac equations (for which $G=1$) as the potential $\mathcal{A}$ is turned on.

To find electromagneticlike bound-state solutions of the two-body Dirac system, we combine this form with bound-state solutions of $[\mathbf{p}^2+m_w{}^2-(\varepsilon_w-\mathcal{A})^2]\phi_s=0$ for $\mathcal{A}=-\alpha/r$ which we seek by first factoring off the angular dependence, then transforming the radial equation (using the dimensionless relativistic Coulomb variable $x=\varepsilon_w\alpha r$) into the time-independent Schrödinger-like equation:

$$\left[-\frac{d^2}{dx^2}-\frac{2}{x}\frac{d}{dx}+\frac{\lambda(\lambda+1)}{x^2}-\frac{2}{x}\right]f_l(x)=\frac{\varepsilon_w{}^2-m_w{}^2}{\varepsilon_w{}^2\alpha^2}f_l(x)\tag{12}$$

with "shifted angular momentum" [since $\lambda(\lambda+1)=l(l+1)-\alpha^2$]

$$\lambda=-\tfrac{1}{2}+[(l+\tfrac{1}{2})^2-\alpha^2]^{1/2}\equiv l-\delta_l\ .$$

Then, the radial part takes the form

$$f_l(r)=(\varepsilon_w\alpha r)^\lambda\exp[-(m_w{}^2-\varepsilon_w{}^2)^{1/2}r]\times F\left[\left(\lambda+1-\frac{\varepsilon_w\alpha}{(m_w{}^2-\varepsilon_w{}^2)^{1/2}}\right),2\lambda+2;2(m_w{}^2-\varepsilon_w{}^2)^{1/2}r\right],\tag{13}$$

where $F[a,b;\rho]$ is the confluent hypergeometric function solution of

$$\left[\rho\frac{d^2}{d\rho^2}+(b-\rho)\frac{d}{d\rho}-a\right]F[a,b;\rho]=0\ .\tag{14}$$

But if $f_l(r)$ is to remain finite as $r\to\infty$, the negative of the first argument of F must be a non-negative integer:

$$-(\lambda+1)+\frac{\varepsilon_w\alpha}{(m_w{}^2-\varepsilon_w{}^2)^{1/2}}=n_r\equiv n-(l+1)\ ,$$

which implies that

$$\varepsilon_w{}^2-m_w{}^2=-\frac{\varepsilon_w{}^2\alpha^2}{(n-\delta_l)^2}$$

("relativistic Balmer Formula"), which in turn gives a "Sommerfeld formula" for the total energy:

$$w=m\sqrt{2}\left[1+\left(1+\frac{\alpha^2}{(n-\delta_l)^2}\right)^{-1/2}\right]^{1/2}\tag{15}$$

in agreement with the field-theoretic parapositronium spectrum [to $O(\alpha^4)$]:

$$w=2m-\frac{m\alpha^2}{4n^2}-\frac{m\alpha^4}{2n^3(2l+1)}+\frac{11}{64}\frac{m\alpha^4}{n^4}+O(\alpha^6)\ .\tag{16}$$

Thus, we obtain a family of solutions ψ_{nlm} with parapositronium spectrum [to $O(\alpha^4)$] of the form

$$\psi_{nlm}\propto e^{-iwt}\chi\begin{pmatrix}\phi_{snlm}\\ \dfrac{1}{\chi}\sigma_1\cdot\mathbf{p}\phi_{snlm}\\ -\dfrac{1}{\chi}\sigma_2\cdot\mathbf{p}\phi_{snlm}\\ -\dfrac{1}{\chi^2}\sigma_1\cdot\mathbf{p}\sigma_2\cdot\mathbf{p}\phi_{snlm}\end{pmatrix},\tag{17}$$

where

$$\phi_{snlm}=(\varepsilon_w\alpha r)^\lambda\exp[-(m_w{}^2-\varepsilon_w{}^2)^{1/2}r]F[(-n+l+1),2\lambda+2;2(m_w{}^2-\varepsilon_w{}^2)^{1/2}r]Y_l^m(\theta,\phi)|S=0\rangle\ ,\tag{18}$$

which are eigenstates of $\mathbf{J}^2$ and J_z with eigenvalues $j(j+1)=l(l+1)$ and $m_J=m$.

Our ability to solve the second-order wave equations (7) and (8) was made possible by the Gordon form of the effective constituent potentials (3) that we introduced through minimal substitution. This form ultimately permitted us to decouple and solve the $\mathcal{H}_1$ and $\mathcal{H}_2$ equations in the chiral Dirac matrix representation. Such second-order equations might have been arrived at directly from another starting point.[8] But, what distinguishes the constraint approach from other brands of relativistic quantum mechanics (and what is most unfamiliar and novel about it) is its use of multiple but compatible simultaneous wave equations—one for each constituent. This feature forces it to treat the particles symmetrically, thereby preserving infinite-mass limits to one-body Dirac equations and including treatment of spin-dependent relativistic recoil automatically. In this note, we see that compatibility allows us to construct simultaneous solutions to the two constituent Dirac equations from solutions to the second-order equation. Compatibility ensures that this property of a single-particle or free-particle system is preserved under interaction and consequently produces a solution that degenerates asymptotically into a free-particle direct-product structure far from the center of force.

Another virtue of such a description is that its two independent Dirac equations can be rearranged in many combinations to give equivalent equations, pieces of which (for weak potential) reduce to well-known (e.g., the Fermi-Breit, Todorov, Pilkuhn equations) or new [e.g. (7)] equations for the electromagnetic system of two spin-$\frac{1}{2}$ particles. Since all of these rearrangements have (17)-(18) as an exact solution, our solution has a significance and potential utility beyond that possessed by a solution to any system governed by a single relativistic wave equation. In fact, the existence of all of these rearrangements in our procedure illustrates how apparently different relativistic wave equations can possess equivalent spectra. Furthermore, since the constraint quantum mechanics can itself be viewed as a rearrangement, the "quantum-mechanical transform" of the Bethe-Salpeter equation (according to the work of Sazdjian[9]), our solution may be an approximate but fully relativistic quantum-mechanical transform of the Bethe-Salpeter solution. During its construction, we saw that our solution was made possible by the collapse of the ϕ_{+-} second-order wave equation (in the chiral representation) to Eq. (8).[10] On the other hand, as viewed in the Dirac γ-matrix representation, the existence of our solution is a consequence of a nonperturbative cancellation between Darwin and spin-spin terms.[11] This fact underlines the important role played by Darwin terms in relativistic wave equations at short distance and argues against their neglect in the short-distance structure of QCD and resulting $q\bar{q}$ bound-state calculation.

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[1]H. J. Schnitzer, Phys. Rev. Lett. **35**, 1540 (1975); D. Gromes, Nucl. Phys. **B131**, 80 (1977); E. Eichten and F. Feinberg, Phys. Rev. Lett. **43**, 1205 (1979); D. P. Stanley and D. Robson, Phys. Rev. D **21**, 31(E) (1980); S. Gupta, S. F. Radford, and W. W. Repko, *ibid.* **26**, 3305 (1982); D. B. Lichtenberg, W. Namgung, and J. G. Wills, Z. Phys. C **19**, 19 (1983); P. Moxhay and J. Rosner, Phys. Rev. D **28**, 1132 (1983); R. W. Childers, Phys. Lett. **112B**, 485 (1983); M. Arafah, R. Bhandhri, and B. Ram, Lett. Nuovo Cimento **38**, 305 (1983); M. Bander, D. Silverman, B. Klima, and U. Maor, Phys. Rev. D **29**, 2038 (1984); H. W. Crater and P. Van Alstine, Phys. Rev. Lett. **53**, 1527 (1984); S. Godfrey and N. Isgur, Phys. Rev. D **32**, 189 (1985).

[2]H. W. Crater and P. Van Alstine, Phys. Rev. D **30**, 2585 (1984).

[3]P. A. M. Dirac, *Lectures on Quantum Mechanics* (Belfer Graduate School of Science, Yeshiva University, New York, 1964); I. T. Todorov Dubna Joint Institute for Nuclear Research Report No. E2-10175, 1976 (unpublished); M. Kalb and P. Van Alstine, Yale Reports No. C00-3075-146, 1976 (unpublished) and No. C00-3075-156, 1976 (unpublished); P. Van Alstine, Ph.D. Dissertation, Yale University, 1976 (unpublished); A. Komar, Phys. Rev. D **18**, 1887 (1978); M. King and F. Rohrlich, Phys. Rev. Lett. **44**, 62 (1980); L. P. Horwitz and F. Rohrlich, Phys. Rev. D **24**, 1528 (1981).

[4]P. Van Alstine and H. W. Crater, J. Math. Phys. **23**, 1697 (1982); H. W. Crater and P. Van Alstine, Ann. Phys. (N.Y.) **148**, 57 (1983), see also work in preparation. H. Sazdjian, Phys. Rev. D **33**, 3401 (1986). Sazdjian also derives two-body Dirac equations but from a different starting point without the use of supersymmetry.

[5]P. Van Alstine and H. W. Crater, Phys. Rev. D **33**, 1037 (1986).

[6]B. L. Aneva, J. I. Karchov, and V. A. Rizov, Bulg. J. Phys. **2**, 409 (1975); V. A. Rizov, I. T. Todorov, and B. L. Aneva, Nucl. Phys. **B98**, 447 (1975).

[7]J. Schwinger, *Particles, Sources and Fields* (Addison-Wesley, Reading, MA, 1973), Vol. 2, pp. 346–350.

[8]H. Pilkuhn, J. Phys. B **17**, 4061 (1984).

[9]H. Sazdjian, Phys. Lett. **156B**, 381 (1985).

[10]The same equation appears "accidentally" in descriptions of spinless particles if recoil (spacelike parts of A) are left out [as noted in Ref. 2 and I. T. Todorov, Phys. Rev. D **3**, 2351 (1971)].

[11]See H. W. Crater and P. Van Alstine in Refs. 1 and 4.