Vacuum fluctuations outside cosmic strings

T. M. Helliwell
Department of Physics, Harvey Mudd College, Claremont, California 91711

D. A. Konkowski
School of Mathematical Sciences, Queen Mary College, University of London, Mile End Road, London E1 4NS, United Kingdom
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The vacuum stress-energy tensor for a conformal scalar field in the exterior spacetime of a straight cosmic string is calculated. The tensor is traceless, falls off as the fourth power of the distance from the string, and is proportional to the linear mass density of the string $c^2\mu/G$ in the physically reasonable case $\mu \ll 1$. The linear energy density arising from the vacuum fluctuations is small compared with the linear mass-energy density of the string itself. A quasiregular singularity which appears in the exterior spacetime of a cosmic string with $\frac{1}{4} < \mu < \frac{1}{2}$ is shown to be unstable as a consequence of divergent vacuum fluctuations.

There has been considerable interest during the past few years in cosmic strings and their physical effects.[1] A straight cosmic string has been modeled recently by Gott[2] and by Hiscock,[3] with interior and exterior spacetimes which are exact solutions of Einstein's equations. We use the topological properties of the flat but conical exterior spacetime to derive the physical effect of the vacuum fluctuations of a conformal scalar field outside such a string.

Vacuum-polarization effects due to electromagnetic or scalar fields arise in a flat spacetime when the topology is unusual or when boundaries are present.[4] The classic example is the Casimir effect, in which the vacuum expectation value of the electromagnetic stress-energy tensor between two parallel conducting plates depends upon the plate separation, thereby producing an attractive force between the plates. The nonzero stress-energy tensor owes its existence in this case to the presence of boundaries in Minkowski spacetime.

There can also be vacuum polarization in a flat spacetime even if the manifold is complete, without boundaries: A complete manifold with an unusual topology is all that is required. Since the external spacetime of straight cosmic strings is flat but topologically different from Minkowski space, one can use the powerful tools which have been developed to study quantum field theory in similar situations.

A straight cosmic string is described by an interior spacetime metric which models the string itself, together with an exterior metric which models the Universe outside the string. The interior metric takes the string to be a cylindrically symmetric fluid with a longitudinal pressure equal to the negative of its energy density $p_z = -\rho$: The string is very taut. This interior metric is[2]

$$ds^2 = -dt^2 + dr^2 + r_0^2 \sin^2(r/r_0) d\phi^2 + dz^2 \ , \tag{1}$$

where $-\infty < t < \infty$, $0 < r < r_0\theta_M$, $0 \le \phi < 2\pi$, and $-\infty < z < \infty$, and where both r_0 and θ_M are constants.

The exterior metric is

$$ds^2 = -dt^2 + dr^2 + (1-4\mu)^2 r^2 d\phi^2 + dz^2 \ , \tag{2}$$

where the constant $\mu = \frac{1}{4}(1-\cos\theta_M)$ is proportional to the mass per unit length of the string $c^2\mu/G$. The metric applies for $0 \le \mu < \frac{1}{4}$ [for $r_b \le r < \infty$, where the boundary radius is $r_b = r_0(1-4\mu)^{-1}\sin\theta_M$], and also for $\frac{1}{4} < \mu < \frac{1}{2}$ (for $r_b \ge r \ge 0$, where the boundary radius is $r_b = r_0 \sin\theta_M$). If $\mu = \frac{1}{4}$ the exterior geometry is cylindrical; we shall not be concerned with this case.

As Gott describes,[2] the string and its exterior spacetime are conveniently viewed in terms of $t =$const, $z =$const embedding diagrams. If $\mu < \frac{1}{4}$, the interior solution is represented by a spherical cap which consists of less than one hemisphere. It is matched at $r = r_b$ to the exterior solution represented by a cone of deficit angle $D = 8\pi\mu$. If $\mu = \frac{1}{4}$, the interior solution is a hemisphere and the outer solution is a cylinder, i.e., a cone with deficit angle $D = 2\pi$. If $\frac{1}{4} < \mu < \frac{1}{2}$, the spherical cap has grown to be more than one hemisphere, and the exterior conical solution sits on the cap like a dunce hat. The coordinate r now decreases as one moves away from the string and reaches zero at the apex of the dunce hat. The line $r = 0$ is a quasiregular singularity.[5,6] It can be thought of as a singular string with zero radius, infinite density, and mass per unit length $\mu_2 = \frac{1}{2} - \mu$. One can avoid this singularity by rounding off the apex of the hat with a small spherical cap with $r_0' < r_0$. The spacetime then consists of two strings $\mu_1 + \mu_2 = \frac{1}{2}$ with empty space separating them. Finally, if $\mu = \frac{1}{2}$ the interior space is closed upon itself and cannot be matched to an exterior solution.

In all cases the exterior spacetime is flat and so can be described locally by a piece of Minkowski spacetime. However, it differs globally: Topologically, it is a cone if $0 < \mu < \frac{1}{2}(\mu \ne \frac{1}{4})$. This difference leads one to expect the presence of vacuum fluctuations induced by the spacetime's topology. In fact, such an effect exists; we have calculated, in particular, the vacuum expectation value of the stress-energy tensor for a conformally coupled scalar field in the conical ($\mu \ne \frac{1}{4}$) exterior spacetime of a cosmic string.

The calculation of $\langle \hat{T}_{\mu\nu} \rangle$ for this case is remarkably similar to a calculation made previously by Deutsch and Candelas[7] for a wedge consisting of two infinite conducting planes intersecting one another at angle α. The primary difference for the string problem is the periodic boundary condition one must impose along the seam where the cone is glued together after the deficit angle is cut out.

The scalar-field Feynman Green's function $G(x,x')$ in the flat exterior spacetime obeys the inhomogeneous wave equation

$$\Box G(x,x') = -g^{-1/2}\delta(x,x') . \quad (3)$$

The D'Alembertian may be written as the usual cylindrical-coordinate operator

$$\Box = -\frac{\partial^2}{\partial t^2} + \frac{1}{r}\frac{\partial}{\partial r}\left[r\frac{\partial}{\partial r}\right] + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2} + \frac{\partial^2}{\partial z^2} , \quad (4)$$

where the exterior metric (2) has been rendered Minkowskian with the substitution $\theta = |1-4\mu|\phi$, so that $0 \le \theta < |1-4\mu| 2\pi$. The vacuum expectation value of the stress-energy tensor may then be found from $G(x,x')$ using[4,7]

$$\langle \hat{T}_{\mu\nu} \rangle = -i \lim_{x' \to x} (\tfrac{2}{3}\nabla_{\mu\nu'} - \tfrac{1}{3}\nabla_{\mu\nu} - \tfrac{1}{6}g_{\mu\nu}\nabla_{\alpha}{}^{\alpha'})G(x,x') . \quad (5)$$

The eigenfunctions of $\Box$ are

$$u(x;\omega,k,p,n) = (p/\alpha)^{1/2} J_{|2n\pi/\alpha|}(pr)\, e^{i(kz-\omega t)} e^{i2n\pi\theta/\alpha} , \quad (6)$$

which are periodic in θ, with period $\alpha = 2\pi|1-4\mu|$. Here J is a Bessel function of the first kind, n is an integer (positive, negative, or zero), and the corresponding eigenvalues are $\omega^2 - k^2 - p^2$, where ω and k are arbitrary real numbers and p is real and positive. The eigenfunctions satisfy the completeness condition

$$\int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \int_{-\infty}^{\infty} \frac{dk}{2\pi} \int_0^{\infty} dp \sum_{n=-\infty}^{\infty} u(x;\omega,k,p,n) u^*(x';\omega,k,p,n) = \delta(x,x') . \quad (7)$$

Substitution of this expression into Schwinger's representation of the Green's function,

$$G(x,x') = i\int_0^{\infty} ds\, e^{is\Box}\delta(x,x') , \quad (8)$$

yields

$$G(x,x') = \frac{i}{\alpha}\int_{-\infty}^{\infty} \frac{d\omega}{2\pi}\int_{-\infty}^{\infty}\frac{dk}{2\pi}\int_0^{\infty} dp\,p \int_0^{\infty} ds\, e^{is(\omega^2-k^2-p^2)} e^{-i\omega(t-t')} e^{ik(z-z')} \times \sum_{n=-\infty}^{\infty} J_{|2n\pi/\alpha|}(pr) J_{|2n\pi/\alpha|}(pr') e^{i(2n\pi/\alpha)(\theta-\theta')} . \quad (9)$$

This expression is the starting point for calculation of the various derivatives of $G(x,x')$ required for $\langle \hat{T}_{\mu\nu}\rangle$. After the functions on the right have been differentiated, the integrals and sum can be evaluated. We demonstrate the method by calculating $G(x,x')$ itself, using the approach of Deutsch and Candelas.[7] Let $\omega \to i\omega$ and $s \to -is$, rotate the contours of the ω and s integrals back to the real axes, and take the limits $(t',r',z') \to (t,r,z)$. The result is

$$\lim_{(t',r',z')\to(t,r,z)} G(x,x') \equiv G(\theta,\theta') = \frac{i}{4\pi^2\alpha}\int_0^{\infty} dp\,p \int_0^{\infty} ds\, e^{-sp^2} \int_{-\infty}^{\infty} d\omega \int_{-\infty}^{\infty} dk\, e^{-s(\omega^2+k^2)} \times \sum_n J^2_{|2n\pi/\alpha|}(pr) e^{i(2n\pi/\alpha)(\theta-\theta')} . \quad (10)$$

The double integral over ω and k may be evaluated using polar coordinates in a space of $2-\varepsilon$ dimensions; then

$$\int_0^{\infty} ds \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} d\omega\, dk\, e^{-s(\omega^2+k^2+p^2)} = \frac{\pi}{p^{\varepsilon}}\Gamma\left[\frac{\varepsilon}{2}\right]\Gamma\left[1-\frac{\varepsilon}{2}\right] . \quad (11)$$

The integral over p is[8]

$$\int_0^{\infty} dp\, p^{1-\varepsilon} J^2_{|2n\pi/\alpha|}(pr) = \frac{2^{1-\varepsilon}\Gamma(-1+\varepsilon)\Gamma(|2n\pi/\alpha|+1-\varepsilon/2)}{r^{2-\varepsilon}\Gamma^2(\varepsilon/2)\Gamma(|2n\pi/\alpha|+\varepsilon/2)} , \quad (12)$$

and so, taking $\varepsilon \to 0$,

$$G(\theta,\theta') = -\frac{i}{\alpha^2 r^2}\sum_{n=1}^{\infty} n\cos\frac{2n\pi}{\alpha}(\theta-\theta') . \quad (13)$$

The sum is evaluated by differentiation of the geometric series

$$\sum_{n=1}^{\infty} e^{inz} = (e^{-iz}-1)^{-1} , \quad (14)$$

resulting finally in

$$G(\theta,\theta') = \frac{i}{4\alpha^2 r^2}\csc^2\frac{\pi(\theta-\theta')}{\alpha} . \quad (15)$$

The corresponding Minkowski-space Green's function is

$$\lim_{(t',r',z')\to(t,r,z)} G_0(x,x') \equiv G_0(\theta,\theta') = \frac{i}{4\alpha^2 r^2}\csc^2\left[\frac{\theta-\theta'}{2}\right] , \quad (16)$$

which follows from Eq. (15) by taking $\alpha \to 2\pi$.

The derivatives of $G(x,x')$ required for $\langle \hat{T}_{\mu\nu}\rangle$ are derived in a similar way. Derivatives with respect to r and r' can be found with the aid of the formulas[8]

$$\int_0^\infty dp\, p^{1-\varepsilon}J_\nu(pr)J_\nu(pr')=\frac{(rr')^\nu\Gamma(\nu+1-\varepsilon/2)}{2^{-1+\varepsilon}(r+r')^{2\nu+2-\varepsilon}\Gamma(\nu+1)\Gamma(\varepsilon/2)}F\left[\nu+1-\frac{\varepsilon}{2},\nu+\frac{1}{2},2\nu+1;\frac{4rr'}{(r+r')^2}\right] \tag{17}$$

and

$$dF(\alpha,\beta,\gamma,z)/dz=(\alpha\beta/\gamma)F(\alpha+1,\beta+1,\gamma+1,z)\ . \tag{18}$$

All of the derivatives may be related to derivatives of $G(\theta,\theta')$ with respect to θ. For example,

$$\lim_{x'\to x}\frac{\partial^2 G(x,x')}{\partial t^2}=-\lim_{x'\to x}\frac{\partial^2 G(x,x')}{\partial t\,\partial t'}=\lim_{x'\to x}\frac{\partial^2 G(x,x')}{\partial z\,\partial z'}=\frac{1}{3r^2}\lim_{\theta\to\theta'}(1+\partial^2/\partial\theta^2)G(\theta,\theta')\ , \tag{19}$$

$$\lim_{x'\to x}\frac{\partial^2 G(x,x')}{\partial\theta\,\partial\theta'}=-\lim_{\theta\to\theta'}\frac{\partial^2}{\partial\theta^2}G(\theta,\theta')\ , \tag{20}$$

$$\lim_{x'\to x}\frac{\partial^2 G(x,x')}{\partial r\,\partial r'}=\frac{1}{3r^2}\lim_{\theta\to\theta'}\left[4+\frac{\partial^2}{\partial\theta^2}\right]G(\theta,\theta')\ . \tag{21}$$

The expectation value of the renormalized stress tensor then becomes

$$\langle\hat T_\mu{}^\nu\rangle_{\rm ren}=-\frac{i}{3r^2}\lim_{\theta'\to\theta}\frac{\partial^2}{\partial\theta^2}[G(\theta,\theta')-G_0(\theta,\theta')]\times{\rm diag}(1,1-3,1)\ , \tag{22}$$

where $\langle\hat T_\mu{}^\nu\rangle$ is renormalized by subtracting off its value in ordinary Minkowski space. We then obtain the result

$$\langle\hat T_\mu{}^\nu\rangle_{\rm ren}=\frac{1}{1440\pi^2r^4}\left[\left(\frac{2\pi}{\alpha}\right)^4-1\right]{\rm diag}(1,1,-3,1)=\frac{1}{1440\pi^2r^4}[(1-4\mu)^{-4}-1]\,{\rm diag}(1,1,-3,1)\ , \tag{23}$$

which is traceless, falls off as the fourth power of the distance from the string, and is proportional to the mass density of the string in the limit of small mass densities.

For the physically reasonable case $\mu<\frac{1}{4}$, where the string is surrounded by an infinite flat but conical external space, the scalar-field vacuum energy per unit length along the string outside the boundary r_b is

$$\varepsilon=\int_{r_b}^\infty\langle T_0{}^0\rangle 2\pi(1-4\mu)r\,dr=\frac{(1-4\mu)\hbar c}{1440\pi r_b{}^2}[(1-4\mu)^{-4}-1]\simeq\frac{\mu\hbar c}{90\pi r_b{}^2}\quad(\text{if }\mu\ll1)\ , \tag{24}$$

where $\hbar c$ has been inserted to give the dimensions. The ratio of this linear energy density to the linear mass-energy density $c^2\mu/G$ of the string itself is therefore (if $\mu\ll1$)

$$\varepsilon_{\rm vac}/\varepsilon_{\rm string}\simeq1/90\pi R_b{}^2\ , \tag{25}$$

where R_b is the boundary radius in units of Planck length. The boundary radius is roughly the Compton wavelength of the typical boson mass when the string was formed: If $m\sim10^{16}$ GeV, then $R_b\sim10^4$, so the external vacuum energy is very small.

For the case $\frac{1}{4}<\mu<\frac{1}{2}$, the coordinate r begins at r_b and decreases as one moves away from the string. A quasiregular singularity, i.e., a singularity at which the curvature tensor is at least bounded in a parallel-propagated orthonormal frame, is encountered at $r=0$. However, our result for $\langle\hat T_\mu{}^\nu\rangle_{\rm vac}$ diverges as $r\to0$, which suggests that the quasiregular singularity is unstable, converting to a curvature singularity as a consequence of the unbounded vacuum energy density. It is easy to show that the divergence of $\langle\hat T_\mu{}^\nu\rangle$ is not a coordinate effect: For example, the quadratic stress-energy scalar

$$\langle\hat T_{\mu\nu}\hat T^{\mu\nu}\rangle=[(1-4\mu)^{-4}-1]^2/172\,800\pi^4r^8 \tag{26}$$

also diverges. It is also possible that rather than induce a curvature singularity in this case, the vacuum energy might serve in effect to round off the cone apex at $r=0$, inducing a second string of finite density. In either case the quasiregular singularity would be unstable, a feature it would share with other quasiregular singularities which have been studied.[9-14]

[1]For a recent review, see A. Vilenkin, Phys. Rep. **121**, 263 (1985).

[2]J. R. Gott III, Astrophys. J. **288**, 422 (1985).

[3]William A. Hiscock, Phys. Rev. D **31**, 3288 (1985).

[4]N. D. Birrell and P. C. W. Davies, *Quantum Fields in Curved Space* (Cambridge Univ. Press, Cambridge, England, 1982).

[5]G. F. R. Ellis and B. G. Schmidt, Gen. Relativ. Gravit. **8**, 915 (1977).

[6]F. J. Tipler, C. J. S. Clark, and G. F. R. Ellis, in *General Relativity and Gravitation,* edited by A. Held (Plenum, New York, 1980).

[7]D. Deutsch and P. Candelas, Phys. Rev. D **20**, 3063 (1979).

[8]I. S. Gradshteyn and I. W. Ryzhik, *Table of Integrals, Series, and Products* (Academic, New York, 1965).

[9]C. W. Misner and A. H. Taub, Zh. Eksp. Teor. Fiz. **55**, 233 (1968) [Sov. Phys. JETP **28**, 122 (1969)].

[10]V. Moncrief and J. Isenberg, Commun. Math Phys. **89**, 387 (1983).

[11]D. A. Konkowski and L. C. Shepley, Gen. Relativ. Gravit. **14**, 61 (1982).

[12]William A. Hiscock and D. A. Konkowski, Phys. Rev. D **26**, 1225 (1982).

[13]D. A. Konkowski, T. M. Helliwell, and L. C. Shepley, Phys. Rev. D **31**, 1178 (1985).

[14]D. A. Konkowski and T. M. Helliwell, Phys. Rev. D **31**, 1195 (1985).