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Low-energy meson action from QCD: Extended Skyrme model

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In the limit of extremely low energies QCD describes essentially the interactions between the approximately massless pseudoscalars. They are the bound states of the "chiral" quarks. Equivalently they are collective Goldstone modes of dynamically broken chiral symmetry, and interact in a nonlinear way with the "constituent" quarks. These two pictures are related by going into the "constituent gauge," which we define in this paper as a QCD analog of the unitary gauge in theories with Higgs scalars. We develop a framework for extracting the low-energy dynamics of pions directly from QCD in the limit of a large number of colors, and under some additional assumptions we calculate the pure pion theory truncated to four derivatives. The model obtained is a somewhat extended Skyrme model, and contains the anomaly term (Wess-Zumino term) as well as the nontopological terms, the coefficients of which depend on a classical scalar-meson-field background. Stability of the soliton is discussed. We show that in the limit in which symmetry breaking is turned off, the coefficients in front of the pseudoscalar interactions vanish. Under a plausible assumption about the behavior of a scalar-meson background, we interpret this as a natural realization of the space cutoff entering the topological soliton bag model.

I. INTRODUCTION

One of the outstanding problems of particle physics today is to understand the properties of hadrons, starting from what is supposed to be a microscopic theory of hadrons—quantum chromodynamics (QCD). The apparent success of phenomenological chiral Lagrangians in describing many properties of the low-energy interactions between hadrons makes this problem somewhat more concrete—one might try to understand how the phenomenological Lagrangian appears in a certain limit from the microscopic theory of quarks and gluons, QCD. This is clearly a difficult task, yet some simplification could occur if one is willing to consider QCD in the limit of a large number of colors.[1,2] It is well known[2] that many of the properties of low-energy hadron interactions can be qualitatively explained in the context of the large-N limit. It is also known[1,2] that QCD in this limit should be completely equivalent to some local, pure meson theory. It is a weakly coupled meson theory because the (quartic) meson coupling constant is $1/N$. It has been conjectured by Witten[2] that baryons should appear as solitons in this theory. But only recently, after the old Skyrme model[3] was reexamined,[4,5] did it become more clear in which sense baryons are in fact solitons.[5–7] It seems, therefore, that the large-N limit is a good starting point for establishing the connection between QCD and the effective chiral Lagrangians, known to describe so well the low-energy hadronic world.[8]

In this paper, we will argue that in the limit of large N and extremely low energies, QCD reduces to a pure pseudoscalar theory which, under some additional assumptions stated and elaborated in Sec. II, can be calculated. We have calculated this theory truncated to four derivatives.[9] Here we give a detailed exposition of the motivation, derivation, and the calculation of the low-energy effective action for pseudoscalars. In Sec. III we discuss the basic strategy for our calculation. In Sec. IV the effective potential is derived whose minimization would in principle lead to determination of the ground state. While the expression for the potential is not explicit enough to enable one to demonstrate spontaneous breaking of the chiral symmetry, assuming that it is broken we obtain an interesting relation for our order parameter $\langle \bar{q}q \rangle$ and the pion mass (m_π), linking them to the three important low-energy scales F_π, m_Q (the constituent-quark mass), and Λ (the "chiral-symmetry-breaking scale"). In Secs. V and VI, and in the Appendix we present in detail our calculation. In Sec. VII we discuss the implication of our results to the question of stability of the soliton. We show that the coefficients in front of the pseudoscalar interactions vanish whenever the scalar-meson background field, which figures importantly in our framework, vanishes. Under a plausible assumption about the behavior of $\sigma_{\rm cl}(x)$, we interpret this as a possible, natural realization of the space cutoff which enters *ad hoc* in the (topological) soliton bag models.[10,11]

II. QCD CONNECTION

In this section we will introduce and elaborate the basic physical ideas and assumptions within which it is possible to extract the dynamics of pseudoscalar mesons from QCD in the limit of a large number of colors and extremely low energies. Apart from certain technical assumptions which will be stated in the appropriate context later, our assumptions are the following. (i) QCD(N) confines at arbitrarily large N. (ii) The theory is characterized by a dynamically generated scale Λ (proportional but not necessarily equal to $\Lambda_{\rm QCD}$) below which chiral $U(N_F)\times U(N_F)$ is spontaneously broken down to diagonal $U(N_F)$. (iii) The breaking is characterized by a nonvanishing vacuum expectation value of some quark bilinear transforming as $\mathbf{N}_F^* \times \mathbf{N}_F$ under $U(N_F)\times U(N_F)$ and is realized by minimization of some local potential, a chiral-invariant function of quark bilinears.[12] (iv) At large N and energies below the chiral-symmetry-breaking scale, QCD is equivalent to some local effective pure meson theory.[2] The lightest mesons are pseudoscalars while all other mesons are much heavier (as is the case in reality, particularly for $N_F=2$). Those decouple in the limit of very low energies in which case it makes sense to talk about a pure pseudoscalar effective theory. We will make the last "decoupling" assumption more precise later.

Consider now the following composite operator:

$$\phi_j^i(x)=\bar{q}\,{}_R^i q_{Lj}(x),\quad i,j=1,2,\ldots,N_F\ . \tag{2.1}$$

The vacuum expectation value of this operator is a convenient order parameter. At high energies[13] $\langle \phi_j^i \rangle=0$ and global $U(N_F)\times U(N_F)$ is unbroken. At low energies, below the chiral-symmetry-breaking scale, the ground state of the theory will be characterized by the operator ϕ_j^i being frozen to its large expectation value. The lowest-energy states above the ground state are Goldstone bosons of broken-chiral-symmetry pseudoscalar mesons. Fluctuations of $\phi_j^i(x)$ around its vacuum expectation value have the correct quantum numbers to be identified as scalar and pseudoscalar mesons. One can think of the pseudoscalars as being collective massless excitations corresponding to the broken axial generators of $U(N_F)\times U(N_F)$. It is therefore natural to define what we mean by scalar and pseudoscalar mesons by enforcing the identification

$$\phi(x)=V^\dagger \sigma V^\dagger(x)\ , \tag{2.2}$$

where $\sigma(x)$ is an $N_F\times N_F$ Hermitian matrix of scalar mesons. The collective pseudoscalar field matrix is $\mathbf{U}=V\cdot V=\exp[i(2/F_\pi)\pi^B Q^B]$ and transforms in a nonlinear realization[14] of $SU_L(N_F)\times SU_R(N_F)$ in a standard way—under an $SU_L\times SU_R$ transformation realized by pair of unitary matrices (L,R), $\mathbf{U}$ transforms as $\mathbf{U}\to L\mathbf{U}R^\dagger$. Consider now the partition function for QCD with the measure appropriately extended to include integration over the collective fields defined above:

$$\int [d\mathbf{G}_\mu]d\bar{\mathbf{q}}\,d\mathbf{q}[d\mathbf{U}\,d\sigma]\delta(\bar{\mathbf{q}}_L\mathbf{q}_R - V\sigma V)\delta(\bar{\mathbf{q}}_R\mathbf{q}_L - V^\dagger\sigma V^\dagger) \times\exp\left[i\left[\mathbf{S}_{\mathrm{QCD}}-m\int\bar{\mathbf{q}}\mathbf{q}\right]\right]\ . \tag{2.3}$$

We have included above the explicit mass term for the quarks (current-quark mass). For simplicity as we are just interested in the leading effects of the chiral-symmetry breaking, we take a common current mass for all the quarks. Let us imagine now separating the measure into a short- and long-wavelength part with respect to some physical scale Λ (the chiral-symmetry-breaking scale). Above this scale one has a weakly coupled theory of quarks and gluons, moving with high relative momenta. There are no bound states. A good description, of the physics in this region is given by perturbation in the color coupling. Below the scale Λ the color forces are becoming rapidly strong, the chiral symmetry breaks down and the bound states are formed. At large N the bound states are mesons, which interact weakly and are certainly much better candidates for the physical states than strongly interacting quarks and gluons. Clearly, quantizing around the perturbative QCD vacuum is not appropriate anymore—QCD as usually written should be rewritten such that quantization of the small vibrations around the right ground state is made possible. The first step in this direction was to rewrite the action as in (2.3). Let us now integrate out gluons. Define $G[J]$ to be the resulting effective action: i.e.,

$$\exp(iG[J])=\int[d\mathbf{G}_\mu]\exp\left[i\int d^4x\left(-\tfrac{1}{2}\,\mathrm{Tr}\mathbf{G}_{\mu\nu}\mathbf{G}^{\mu\nu} - J_\mu^a G^{a\mu}\right)\right]\ ,$$

where $J_\mu^a=\bar{\mathbf{q}}\gamma_\mu \mathbf{Q}^a\mathbf{q}$ is the color current. $G[J]$ is a functional of the color current and has therefore a full local chiral invariance. It contains all kinds of higher-dimension composite operators as well as the effective potential which, following Ref. 12, we assume to be some chiral invariant of quark bilinears $\phi_j^i(x)$ defined above. If we now expand the composite operators around their vacuum values and quantize the small vibrations around the vacuum values, we get an effective theory of mesons interacting with massive quarks. Needless to say, calculating $G[J]$ is a very hard task which we will not attempt in this paper. Instead, the question we would like to ask is whether we can somehow extract the low-energy dynamics of the lightest mesons, pseudoscalars, without really explicitly calculating the effective action due to gluons $G[J]$. The central observation which makes this indeed plausible is that $G[J]$ is invariant under local chiral transformations on the quark fields and therefore although itself not necessarily local, it has a full local chiral invariance. To illustrate our idea, forget for the moment about the quark kinetic term and the measure and note that at the level of the gluon action this invariance is essentially trivial, coming from the fact that gluons couple to the flavor-singlet color current. However, once the gluons are integrated out the resulting action $G[J]$ is highly nontrivial (nonlocal and nonrenormalizable) and is made out of all kinds of flavor-nonsinglet composite quark operators which are in a rough correspondence with the low-energy meson degrees of freedom contained in $G[J]$. What was a trivial local chiral invariance before gluons have been integrated out, becomes a highly nontrivial constraint on the form of $G[J]$—the local chiral invariance [with $U(N_F)\times U(N_F)$ as the gauge group] has to be realized at the composite level. To see what this means, and to make our discussion more concrete, let us consider now $G[J]$ in the limit of extremely low energies and large number of colors. In terms of the low-energy degrees of freedom, $G[J]$ is expected to contain scalar, vector, and axial-vector mesons interacting with massless pseudoscalars and massive quarks. We will now make the important assumption that, at least in this limit, the mesons are associated in $G[J]$ with an appropriate set of *local* composite quark operators. For example, it is natural within our framework to represent the scalar mesons and pions simply as the scalar and pseudoscalar part of the composite "chiral" quark operator, $\phi^i_j = \bar{q}^i_R q_{Lj}(x)$, respectively:

$$\sigma_{ij}(x) = \bar{q}_i q_j(x) \ \text{ and } \ \pi_{ij}(x) = \bar{q}_i \gamma^5 q_j(x) . \tag{2.4a}$$

$G[J]$ can therefore be written, we assume, as some very general, Lorentz-singlet, nonlocal action of composite mesons and massive quarks. There is no reason why such a complicated action would not contain all possible Lorentz-singlet terms, subject to the constraint of a full local chiral invariance. In particular, any derivative term of the composite meson fields appearing in $G[J]$ will have to be a covariant derivative—the appearance of the gauge fields associated with local chiral $U(N_F) \times U(N_F)$ at the composite level seems, therefore, highly plausible.[15]

Consider now the pseudoscalars in $G[J]$. According to (2.4a), pions are, roughly speaking, composites of two "chiral" quarks. Below the chiral-symmetry-breaking scale those quarks become massive and the pion appears as a massless bound state of the massive chiral quarks. Alternatively, but equivalently as far as the physics is concerned, the pion is a collective Goldstone mode associated with breakdown of the chiral symmetry and, using the local chiral invariance of $G[J]$, can be completely eliminated as independent degree of freedom by the appropriate redefinition of the quark fields. What we have in mind here is similar and generalize somewhat the unitary gauge trick in theories with Higgs scalars—using the gauge freedom, one removes manifestly the Goldstone bosons by a particular gauge redefinition of the Higgs scalars, fermion and gauge fields. To see, more clearly, how this comes out, note that a local chiral redefinition of the quark fields will necessarily induce local chiral redefinitions (i.e., gauge transformations) on the composite meson fields in $G[J]$. Let $V(x) = \exp[i\pi(x)/F_\pi]$ be the field matrix of the collective Goldstone mode defined in (2.2) and (2.3). We may pass now from the "chiral" quarks, (q), to the "constituent" quarks (q^U) via the following chiral redefinition of the quark fields:

$$\mathbf{q}_L = V^\dagger(x)\mathbf{q}^U_L, \ \ \mathbf{q}_R = V(x)\mathbf{q}^U_R \tag{2.5}$$

which exponentiates the Goldstone bosons in $G[J]$. Because $G[J] = G[J^U]$ is form invariant under this transformation, it is still the same functional, with the difference being that composite operators associated in $G[J^U]$ with mesons, are now made out of the constituent-quark fields q^U instead of the chiral quark fields q. The important thing is that there is no explicit dependence on $V(x)$ in $G[J^U]$; that is, the Goldstone mesons are removed from $G[J^U]$. In this "gauge" (the "constituent gauge"), roughly speaking, the mesons are composed of the constituent quarks q^U. The pion is, however, a massless Goldstone mode, and one should not think of it as the composite of two constituent quarks. Indeed, combining our identification of the collective pion (2.2) and (2.3) with (2.5) we find

$$\pi^U_{ij}(x) = \bar{q}^U_i \gamma^5 q^U_j = 0 . \tag{2.4b}$$

Clearly, in this "gauge" it is manifest that $G[J^U]$ does not contain the operators capable of creating light pseudoscalar from the vacuum. What happens one can interpret as, that by passing to the description in terms of the constituent quarks, one removes a zero-mass two-quark bound state from the spectrum and replace it with the zero-mass collective pion state. Collective pions, being removed from $G[J^U]$, will couple through those parts of the QCD action which do not have a local chiral invariance—the quark kinetic term, explicit current-quark mass term—and, because they are introduced through a finite chiral rotation of the quarks, will also couple to the quark measure $[J[U]$ term in Eq. (2.6) below].

The partition function in this "gauge" reads

$$\begin{aligned} Z_{\rm QCD} = \int [d\mathbf{U}\, d\sigma] d\bar{\mathbf{q}}^U d\mathbf{q}^U \delta(\bar{\mathbf{q}}^U_L \mathbf{q}^U_R - \sigma)\delta(\bar{\mathbf{q}}^U_R \mathbf{q}^U_L - \sigma) \\ \times \exp\left[i \int d^4x \bar{\mathbf{q}}^U [\not{D}(\mathbf{U}) - m\mathbf{U}] q^U + \ln J[\mathbf{U}] \right] \\ \times \exp(iG[J^U]) , \end{aligned} \tag{2.6}$$

where $\not{D}(\mathbf{U}) = i\gamma^\mu(\partial_\mu + V_\mu + \gamma^5 A_\mu)$ and V_μ and A_μ are collective flavor gauge fields made out of pions:

$$V_\mu = \tfrac{1}{2}(V^\dagger \partial_\mu V + V \partial_\mu V^\dagger), \ \ A_\mu = \tfrac{1}{2}(V^\dagger \partial_\mu V - V \partial_\mu V^\dagger) . \tag{2.7}$$

$G[J^U]$ now contains only operators capable of creating the heavy mesons from the vacuum. Those however are not excited in the limit of very low energies and being interested for the lowest-energy excitations above the ground state we keep those operators frozen to their vacuum values. This amounts to keeping only the zero-momentum term, i.e., potential, and we will drop out a complicated and unknown piece of $G[J^U]$ containing chiral invariants made out of the heavy currents and derivatives of the scalar-meson-type quark bilinears. From the point of view of calculating the effective, pure pseudoscalar theory this approximation means that we neglect contributions to the coefficients of this theory, due to exchanges of the heavy mesons. Those contributions are suppressed by inverse powers of the heavy-meson masses, and as long as the momenta involved are much smaller than the exchange mass, are not important.

The long-wavelength part of the QCD partition functional becomes

$$Z^L_{\rm QCD} \approx \int [d\mathbf{U}\, d\sigma] d\bar{\mathbf{q}}^U d\mathbf{q}^U d\mathbf{S}\, d\mathbf{P} \exp\left[i \left[\int d^4x \{ \bar{\mathbf{q}}^U [\not{D}(U) - (m\mathbf{U} + \mathbf{S} + i\gamma^5 \mathbf{P})] q^U + 2\,\mathrm{Tr}\mathbf{S}\sigma - V_{\rm gluon}(\sigma) \} - i \ln J[\mathbf{U}] \right] \right] . \tag{2.8}$$

We have used above the pair $(\mathbf{S},\mathbf{P})$ of auxiliary fields to exponentiate the δ-function constraints in (2.6). Integrating out quarks, the long-wavelength part of the partition function becomes

$$Z^L_{\rm QCD} \simeq \int [d\mathbf{U}\, d\sigma] \exp[iW_{\rm eff}(\mathbf{U},\sigma)] ,$$

where

$$\exp(iW_{\rm eff})\equiv\int d\mathbf{S}\,d\mathbf{P}\exp\left[N_c\mathrm{Tr}\ln[\not{D}(\mathbf{U})-(m\mathbf{U}+\mathbf{S}+i\gamma^5\mathbf{P})]+\ln J[\mathbf{U}]+i\int d^4x[2\,\mathrm{Tr}\mathbf{S}\sigma-V_{\rm gluon}(\sigma)]\right]. \tag{2.9}$$

Consider now the integral over the auxiliary fields **P** and **S**. At large N it is dominated by a stationary phase. Using the equations of motion we find

$$\mathbf{P}_{\rm cl}=0 \text{ and } \mathbf{S}_{\rm cl}\equiv\Sigma(\sigma)=\frac{\partial V_{\rm gluon}}{\partial\sigma}+\cdots. \tag{2.10}$$

While $\mathbf{P}_{\rm cl}=0$ can be simply understood by noticing that $\mathbf{P}_{\rm cl}\neq0$ is forbidden by parity conservation,[16] to find $\mathbf{S}_{\rm cl}$ we differentiate $\exp[iW_{\rm eff}(\mathbf{U},\sigma)]$ above with respect to σ and we get

$$\frac{\delta W_{\rm eff}}{\delta\sigma(x)}e^{iW_{\rm eff}}=\int d\mathbf{S}\,d\mathbf{P}\left[\mathbf{S}(x)-\frac{\partial V_{\rm gluon}}{\partial\sigma(x)}\right]\exp\{N_c\mathrm{Tr}[\not{D}(\mathbf{U})-(m\mathbf{U}+\mathbf{S}+i\gamma^5\mathbf{P})]+\cdots\},$$

from which follows

$$\mathbf{S}_{\rm cl}(x;\sigma)=\frac{\partial V_{\rm gluon}}{\partial\sigma(x)}+\frac{\delta W_{\rm eff}}{\delta\sigma(x)}. \tag{2.11}$$

When $\sigma(x)$ satisfies the classical equations of motion the last term drops out which proves the second equation in (2.10). We will, therefore, approximately evaluate the integral over **P** and **S**, at large N, by replacing $\mathbf{P}=\mathbf{P}_{\rm cl}$ and $\mathbf{S}=\mathbf{S}_{\rm cl}$. Within this approximation the partition function describing the low-energy dynamics of scalar and pseudoscalar mesons is obtained and reads

$$Z^L_{\rm QCD}\approx\int[d\mathbf{U}\,d\sigma]\exp\left[N_c\mathrm{Tr}\ln[\not{D}(\mathbf{U})-(m\mathbf{U}+\Sigma)]+\ln J[\mathbf{U}]+i\int d^4x[2\,\mathrm{Tr}\sigma\Sigma(\sigma)-V_{\rm gluon}(\sigma)]\right] \tag{2.12}$$

or equivalently, before the quarks are integrated out,

$$\int d\bar{\mathbf{q}}^U d\mathbf{q}^U[d\mathbf{U}\,d\sigma]\exp\left[i\int d^4x\{\bar{\mathbf{q}}^U[\not{D}(\mathbf{U})-(m\mathbf{U}+\Sigma)]\mathbf{q}^U+2\,\mathrm{Tr}\sigma\Sigma(\sigma)-V_{\rm gluon}(\sigma)\}+\ln J[\mathbf{U}]\right]. \tag{2.13}$$

With $\Sigma(\sigma)$ defined in (2.10), this is our first result. It is derived to be valid in the limit of large number of colors and for energies below some physical cutoff scale Λ (the chiral-symmetry-breaking scale) which is of the order of magnitude or smaller than the typical mass of the heavy mesons (0.8 GeV). It makes manifest where the pseudoscalars are localized and shows that the pure pseudoscalar low-energy effective theory is to leading order in the decoupling of heavy mesons completely calculable essentially in terms of two scales, Λ and $\langle\Sigma\rangle$. The model derived is in fact a somewhat generalized form of the σ model,[17] with the tree-level potential

$$V_0(\sigma)=V_{\rm gluon}(\sigma)-2\,\mathrm{Tr}\sigma\frac{\partial V_{\rm gluon}}{\partial\sigma} \tag{2.14}$$

and dynamically generated quark mass $\langle\Sigma\rangle$ which is, through defining relation (2.10), related to the gluon induced potential $V_{\rm gluon}(\sigma)$.

III. LOW-ENERGY EFFECTIVE ACTION FOR GOLDSTONE BOSONS IN QCD

After integrating over quark degrees of freedom with momenta lower than our physical cutoff scale Λ, one in principle ends up with pure pseudoscalar effective theory with the coefficients explicitly dependent on the physical cutoff, signaling the fact that the theory is not valid beyond this scale. To be precise, all the coefficients of the effective theory we are going to calculate will depend on the "chiral-symmetry-breaking scale" Λ and dynamical quark mass $\langle\Sigma\rangle$. This means that we will be able to relate those two scales with the experimentally measurable parameters F_π and m_π. To see how this comes out, and the full low-energy Lagrangian, let us start with the effective action after the quarks are integrated out, (2.12):

$$iW_{\rm eff}(\mathbf{U},\sigma)=N_c\mathrm{Tr}\ln\{\not{D}(\mathbf{U})-[m\mathbf{U}+\Sigma(\sigma)]\}+\ln J[\mathbf{U}]+i\int d^4x[2\,\mathrm{Tr}\sigma\Sigma(\sigma)-V_{\rm gluon}(\sigma)]. \tag{3.1}$$

The central point is to calculate the first two terms above. The pseudoscalars' action should come out from this calculation naturally as an expansion in the number of derivatives. The expansion will include derivatives of an arbitrary order but in this paper we will calculate this action truncated to four derivatives. We use the proper-time method. It is easy to show that this regularization respects the vector gauge invariance.[18] The proper-time integration has to be truncated at some maximal momentum Λ which we interpret to be the chiral-symmetry-breaking scale.

IV. THE LOW-ENERGY EFFECTIVE POTENTIAL

Clearly (at least within our framework) the crucial information about the low-energy limit of QCD at large N is contained in $V_{\rm gluon}(\sigma)$. Yet in order to extract this information one has to determine first the ground state. For example, to calculate a dynamical quark mass one has to evaluate the first derivative of $V_{\rm gluon}$ at the ground state. $V_{\rm gluon}$ itself is not the potential to be minimized in order to find the ground state. We have seen, however, how the tree-level potential $V_0(\sigma)$ is determined given V_{gluon} [see (2.14)]. A complete low-energy effective potential is obtained after contribution induced by quark-meson interaction is added to $V_0(\sigma)$. Calculation of this contribution is trivial—one just has to evaluate the fermion determinant [first term in (3.1)] for the case of constant meson fields. For the effective potential we find then[9]

$$V_{\text{eff}}(\sigma,\mathbf{U})\simeq\frac{N_c\Lambda^4}{32\pi^2}\times\text{Tr}\int_1^\infty\frac{ds}{s^3}\exp\left[-\frac{s}{\Lambda^2}(m\mathbf{U}+\boldsymbol{\Sigma})^\dagger(m\mathbf{U}+\boldsymbol{\Sigma})\right]-2\,\text{Tr}\sigma\boldsymbol{\Sigma}(\sigma)+V_{\text{gluon}}(\sigma)\,. \tag{4.1}$$

To find the ground state we minimize V_{eff} with respect to $\langle U\rangle$ and $\langle\sigma\rangle$. We find $\langle U\rangle=1$ and

$$\frac{\partial V_{\text{eff}}}{\partial\langle\sigma\rangle}\sim\frac{\partial^2V_{\text{gluon}}}{\partial\langle\sigma\rangle^2}\left[\langle\bar{q}q\rangle+\frac{N_c\Lambda^3}{4\pi^2}\left[\frac{m_Q}{\Lambda}\right]\times\int_1^\infty\frac{ds}{s^2}e^{-s(m_Q/\Lambda)^2}\right]=0\,,$$

where $m_Q=m+\langle\partial V_{\text{gluon}}/\partial\sigma\rangle$ is the constituent-quark mass and $\langle\bar{q}q\rangle=\langle u_Lu_R\rangle=\langle d_Ld_R\rangle=\cdots$. Assuming $\langle V''_{\text{gluon}}\rangle\neq0$ we obtain the following interesting relation:

$$-\langle\bar{q}q\rangle\simeq\frac{N_c\Lambda^3}{4\pi^2}\left[\frac{m_Q}{\Lambda}\right]\int_1^\infty\frac{ds}{s^2}e^{-s(m_Q/\Lambda)^2}\,. \tag{4.2}$$

Indeed expanding the effective potential in powers of m/Λ and keeping only the leading $O(m/\Lambda)$ term, we obtain

$$V_{\text{eff}}(\sigma,\mathbf{U})\simeq V_{\text{eff}}(\sigma,\mathbb{1})+\frac{N_c\Lambda^3}{4\pi^2}\left[\frac{\langle\Sigma\rangle}{\Lambda}\right]\int_1^\infty\frac{ds}{s^2}e^{-s(\langle\Sigma\rangle/\Lambda)^2}\text{Tr}\frac{m}{2}(\mathbf{U}^\dagger+\mathbf{U})+\cdots+O\left[\frac{m^2}{\Lambda^2}\right]\,. \tag{4.3}$$

Expanding to quadratic order in the pion field we deduce

$$m_\pi^{\,2}\simeq\frac{m}{F_\pi^{\,2}}\frac{N_c\Lambda^3}{\pi^2}\left[\frac{m_Q}{\Lambda}\right]\int_1^\infty\frac{ds}{s^2}e^{-s(m_Q/\Lambda)^2}\,. \tag{4.4}$$

This is just a familiar current-algebra result $m_\pi^{\,2}F_\pi^{\,2}=2m\langle\bar{u}_Lu_R+\bar{d}_Ld_R\rangle$ providing (4.2) is satisfied.

Without some additional knowledge about V_{gluon} we are not able to demonstrate that the chiral symmetry is broken, yet assuming that this is the case we have deduced relations (4.2) and (4.4), linking into a simple consistency condition four important low-energy parameters m_π, F_π, m_Q, and Λ.

V. DERIVATIVE ACTION

Consider the proper-time representation for the first term of the effective meson action (3.1)

$$\Gamma(\mathbf{U},\boldsymbol{\Sigma})=\frac{N_c}{2}\text{Tr}\{\not{D}(\mathbf{U})-[m\mathbf{U}+\boldsymbol{\Sigma}(\sigma)]\}^2\simeq-\frac{N_c}{2}\int_{\epsilon=\Lambda^{-2}}^\infty\frac{ds}{s}\text{Tr}\exp\{-s[\not{D}^2-2\langle\Sigma\rangle\not{D}+\langle\Sigma\rangle^2]\}+\cdots\,. \tag{5.1}$$

Here, we have expanded $\Sigma(\sigma)$ around its large expectation value $\langle\Sigma\rangle\gg m$ being the current-quark mass, and have neglected heavy-scalar-meson fluctuations around it, which is consistent with the basic approximation of our framework—the assumption of decoupling of heavy mesons in the limit of extremely low energies. Λ is a proper-time cutoff which reminds us that the effective meson action we are evaluating is obtained by integrating over long-wavelength components of the quark fields, i.e., with momenta less than some maximal momentum Λ. Let us introduce for calculational convenience a fifth coordinate[19] τ, defined by

$$\mathbf{U}=\exp[i\tau\xi(x)],\quad 0\le\tau\le1,\quad \xi=\frac{2}{F_\pi}\pi^B(x)Q^B\,. \tag{5.2}$$

Then it is not difficult to show that

$$\Gamma(\tau)=\Gamma(0)-2N_c\int_0^1d\tau\int_\epsilon^\infty\frac{ds}{s}e^{-s\langle\Sigma\rangle^2}\left[s\frac{d}{ds}\text{Tr}\xi\gamma^5\exp[-s(\not{D}^2-2\langle\Sigma\rangle\not{D})]-\langle\Sigma\rangle\text{Tr}\xi\gamma^5\not{D}\exp[-s(\not{D}^2-2\langle\Sigma\rangle\not{D})]\right]\,.$$

Integrating by parts one obtains

$$\begin{aligned}\Gamma(\tau)=&\text{const}+2N_c\int_0^1d\tau\,e^{-\epsilon\langle\Sigma\rangle^2}\text{Tr}\xi\gamma^5\exp[-\epsilon(\not{D}^2-2\langle\Sigma\rangle\not{D})]\\&-2N_c\langle\Sigma\rangle^2\int_0^1d\tau\int_\epsilon^\infty ds\,e^{-s\langle\Sigma\rangle^2}\text{Tr}\xi\gamma^5\exp[-s(\not{D}^2-2\langle\Sigma\rangle\not{D})]\\&+2N_c\langle\Sigma\rangle^2\int_0^1d\tau\int_\epsilon^\infty ds\,e^{-s\langle\Sigma\rangle^2}\text{Tr}\xi\gamma^5\frac{\not{D}}{\langle\Sigma\rangle}\exp[-s(\not{D}^2-2\langle\Sigma\rangle\not{D})]\,.\end{aligned} \tag{5.3}$$

Tr($\cdots$) above includes space-time, Lorentz, and internal space indices. Consider now the contribution of the first term above, and let us expand the part of the exponent inside the trace in powers of $\langle\Sigma\rangle D\!\!\!/ /\Lambda^2$. We obtain

$$
\begin{aligned}
I_1 &= 2N_c \int_0^1 d\tau\, e^{-\epsilon\langle\Sigma\rangle^2} \mathrm{Tr}\,\xi\gamma^5 e^{-\epsilon(D\!\!\!/^2 - 2\langle\Sigma\rangle D\!\!\!/)} \\
&= 2N_c \int_0^1 d\tau\, e^{-\epsilon\langle\Sigma\rangle^2} \sum_{m=0} (2\langle\Sigma\rangle\epsilon^{1/2})^{2m} \\
&\qquad \times \frac{(-\epsilon)^m}{(2m)!}\frac{d^m}{d\epsilon^m}\mathrm{Tr}\,\xi\gamma^5 e^{-\epsilon D\!\!\!/^2} .
\end{aligned}
\tag{5.4}
$$

In obtaining (5.4) above we have used the heat equation, i.e., the obvious identity

$$
D\!\!\!/^{2m} e^{-\epsilon D\!\!\!/^2} = (-1)^m \frac{d^m}{d\epsilon^m} e^{-\epsilon D\!\!\!/^2} ,
$$

and the fact that the trace over an odd number of γ_μ's vanishes. The important thing achieved by deriving the last expression is that the calculation is reduced to calculation of the massless heat kernel $\langle x \mid e^{-\epsilon D\!\!\!/^2} \mid y\rangle$ in the coincidence limit $(x=y)$. It is well known[20,21] that this can be done as an expansion in $\epsilon D\!\!\!/^2$, i.e., in the number of derivatives

$$
\langle x \mid e^{-\epsilon D\!\!\!/^2} \mid x\rangle = \frac{1}{16\pi^2\epsilon^2}[1 + \epsilon \mathbf{a}_1(x,x) + \epsilon^2 \mathbf{a}_2(x,x) + \cdots] .
\tag{5.5}
$$

The coefficients $\mathbf{a}_i(x,x)$ are functionals of the pseudoscalar field matrix $\mathbf{U}$ and it is clear on dimensional ground that $\mathbf{a}_1(\mathbf{U})$ is a two-derivative term and will lead to the kinetic term for pseudoscalars, while $\mathbf{a}_2(\mathbf{U})$ contains full information about four-derivative terms. It is well known that expansion of this type is valid for slowly varying fields, i.e., momenta much lower than our physical cutoff scale $\Lambda = \epsilon^{-1/2}$. Replacing (5.5) in (5.4) we obtain

$$
I_1 = 2N_c \int_0^1 d\tau\, e^{-\epsilon\langle\Sigma\rangle^2} \sum_{m=0} \left[\frac{2\langle\Sigma\rangle}{\Lambda}\right]^{2m} \frac{m!}{(2m)!}\frac{1}{16\pi^2}\left[\frac{1}{\epsilon}\right] \mathrm{Tr}\,\xi\gamma^5 \mathbf{a}_1(\mathbf{U}) + 2N_c e^{-\epsilon\langle\Sigma\rangle^2}\frac{1}{16\pi^2}\int_0^1 d\tau\, \mathrm{Tr}\,\gamma^5\xi\, \mathbf{a}_2(\mathbf{U}) + O(\partial^6) .
$$

The summation in the first term can be done explicitly and we obtain

$$
I_1 = 2N_c \left[e^{-\langle\Sigma\rangle^2/\Lambda^2} + \sqrt{\pi}\frac{\langle\Sigma\rangle}{\Lambda}\mathrm{erf}\frac{\langle\Sigma\rangle}{\Lambda}\right]\frac{\Lambda^2}{16\pi^2}\int_0^1 d\tau\, \mathrm{Tr}\,\gamma^5\xi\, \mathbf{a}_1 + \frac{N_c}{8\pi^2} e^{-\langle\Sigma\rangle^2/\Lambda^2}\int_0^1 d\tau\, \mathrm{Tr}\,\xi\gamma^5 \mathbf{a}_2 + O(\partial^6) .
\tag{5.6}
$$

This expression completes for the moment our calculation of the derivative expansion of the first term in the expression for the massive Dirac operator, reducing it to the calculation of the coefficients $\mathbf{a}_i(x,x)$ figuring in the expansion of massless Dirac kernel (5.5). This then illustrates the way we are going to calculate the effective action Γ coming from integrating over massive quarks below the chiral-symmetry-breaking scale—we reduce it to the problem of calculating the coefficients $\mathbf{a}_i(x,x)$ of a massless Dirac determinant.

Consider now the remaining two terms in (5.3):

$$
I_2 + I_3 = -2N_c\langle\Sigma\rangle^2 \int_0^1 d\tau \int_\epsilon^\infty ds\, e^{-s\langle\Sigma\rangle^2}\mathrm{Tr}\,\xi\gamma^5\left[1 - \frac{D\!\!\!/}{\langle\Sigma\rangle}\right]\exp[-s(D\!\!\!/^2 2\langle\Sigma\rangle D\!\!\!/)] .
\tag{5.7}
$$

Let us first try to extract the four-derivative term. In order to do this let us perform a formal integration over s. We obtain

$$
I_2 + I_3 = -2N_c\langle\Sigma\rangle^2 \int_0^1 d\tau\, \mathrm{Tr}\,\xi\gamma^5\left[1 - \frac{D\!\!\!/}{\langle\Sigma\rangle}\right]\frac{e^{-\epsilon\langle\Sigma\rangle^2}}{D\!\!\!/^2 - 2\langle\Sigma\rangle D\!\!\!/ + \langle\Sigma\rangle^2}\exp[-\epsilon(D\!\!\!/^2 - 2\langle\Sigma\rangle D\!\!\!/)] .
\tag{5.8}
$$

Assuming now that $\langle\Sigma\rangle$ is sufficiently large we can expand the denominator in powers of $D\!\!\!/^2/\langle\Sigma\rangle^2$ and get

$$
I_2 = I_3 = -2N_c \int_0^1 d\tau\, \mathrm{Tr}\,\xi\gamma^5 e^{-\epsilon\langle\Sigma\rangle^2}\left[1 - \frac{D\!\!\!/}{\langle\Sigma\rangle}\right]\sum_{m=0}\frac{(-1)^m}{\langle\Sigma\rangle^{2m}}(D\!\!\!/^2 - 2\langle\Sigma\rangle D\!\!\!/)^m \exp[-\epsilon(D\!\!\!/^2 - 2\langle\Sigma\rangle D\!\!\!/)] .
$$

Expanding $\exp(2\epsilon\langle\Sigma\rangle D\!\!\!/)$ and using the heat equation, i.e., the identity $(d/d\epsilon)e^{-\epsilon D\!\!\!/^2} = -D\!\!\!/^2 e^{-\epsilon}$, we obtain

$$
\begin{aligned}
I_2 + I_3 &= -2N_c \int_0^1 d\tau\, \mathrm{Tr}\,\xi\gamma^5 e^{-\epsilon\langle\Sigma\rangle^2}\left[1 - \frac{D\!\!\!/}{\langle\Sigma\rangle}\right]\sum_{m,n=0}\langle\Sigma\rangle^{-2m}\frac{d^m}{d\epsilon^m}\frac{(2\langle\Sigma\rangle\epsilon)^n}{n!}D\!\!\!/^n e^{-\epsilon D\!\!\!/^2} \\
&= -2N_c \int_0^1 d\tau\, \mathrm{Tr}\,\xi\gamma^5 e^{-\epsilon\langle\Sigma\rangle^2}\sum_{m,n=0}\frac{(-1)^n}{\langle\Sigma\rangle^{2m}}\frac{d^m}{d\epsilon^m}\left[\frac{(2\langle\Sigma\rangle\epsilon)^{2n}}{(2n)!}\left[1 + 2\frac{\epsilon}{2n+1}\frac{d}{d\epsilon}\right]\frac{d^n}{d\epsilon^n}e^{-\epsilon D\!\!\!/^2}\right] .
\end{aligned}
$$

If we now replace $\langle x \mid e^{-\epsilon D\!\!\!/^2} \mid x\rangle$ above by its small-ϵ expansion (5.5) it is easy to see that only the $m=n=0$ term above will not annihilate its ϵ-independent, four-derivative piece. The whole contribution of the sum above to the four-derivative term is, therefore, simply

$$
(I_2 + I_3)_{\partial^4} \simeq -2N_c \int_0^1 d\tau\, \mathrm{Tr}\,\xi\gamma^5 e^{-\epsilon\langle\Sigma\rangle^2}\frac{1}{16\pi^2}\mathbf{a}_2(\mathbf{U}) .
\tag{5.9}
$$

A. Two-derivative term

Now, we would like to calculate the contribution of (5.7) to the kinetic term for pseudoscalars. This can not be done exactly, unlike the case in calculating (5.4), where one did not need to perform the integration over s. However, because of the presence of the exponential damping factor $e^{-s\langle\Sigma\rangle^2}$ in the integrand of (5.7) we assume that a proper-time integral (5.7) is going to be dominated by the small-s region extending from $\epsilon=\Lambda^{-2}$ to roughly $\langle\Sigma\rangle^{-2}$. We do not expect $\langle\Sigma\rangle$ (the constituent-quark mass) to be too far from our ultraviolet cutoff Λ and therefore within this region $s\not{D}^2$ can be considered small and we can approximate the kernel in the integrand by its small-s form (5.5). With this in mind, the contribution of (5.7) to the kinetic term can be approximately calculated:

$$I_2+I_3\simeq -2N_c\langle\Sigma\rangle^2\int_0^1 d\tau\int_\epsilon^{\sim\langle\Sigma\rangle^{-2}} ds\, e^{-s\langle\Sigma\rangle^2}\sum_{m=0}\frac{(2\langle\Sigma\rangle\sqrt{s})^{2m}}{(2m)!}s^m(-1)^m\left[\frac{d}{ds}\right]^m \underline{\mathrm{Tr}}\xi\gamma^5 e^{-s\not{D}^2}$$
$$+2N_c\langle\Sigma\rangle\int_0^1 d\tau\int_\epsilon^{\sim\langle\Sigma\rangle^{-2}} ds\, e^{-s\langle\Sigma\rangle^2}\sum_{m=0}\frac{(2\langle\Sigma\rangle\sqrt{s})^{2m+1}}{(2m+1)!}(-1)^{m+1}\frac{d^{m+1}}{ds^{m+1}}\mathrm{Tr}\xi\gamma^5 e^{-s\not{D}^2}\,.$$

Using now (5.5) we obtain the following contribution to the two-derivative term:

$$(I_2+I_3)_{\partial^2}\approx -2N_c\langle\Sigma\rangle^2\int_0^1 d\tau\frac{1}{16\pi^2}\mathrm{Tr}\xi\gamma^5\mathbf{a}_1\int_\epsilon^{\sim\langle\Sigma\rangle^{-2}}\frac{ds}{s}e^{-s\langle\Sigma\rangle^2}\sum_{m=0}\frac{(2\langle\Sigma\rangle\sqrt{s})^{2m}m!}{(2m)!}\left[1-2\frac{m+1}{2m+1}\right].$$

Fortunately, the summations above can be performed explicitly with the result

$$(I_2+I_3)_{\partial^2}\simeq N_c\int_\epsilon^{\sim\langle\Sigma\rangle^{-2}}\frac{ds}{s^2}\sqrt{\pi}\langle\Sigma\rangle\sqrt{s}\ \mathrm{erf}(\langle\Sigma\rangle\sqrt{s})\frac{1}{16\pi^2}\mathrm{Tr}\xi\gamma^5\mathbf{a}_1(\mathbf{U})\,. \tag{5.10}$$

Collecting together (5.6) and (5.10) we obtain the whole contribution to the kinetic term of the effective action $\Gamma(U,\langle\Sigma\rangle)$ defined in (5.1), i.e., (5.3):

$$\Gamma_2\simeq 2N_c\left[e^{-\langle\Sigma\rangle^2/\Lambda^2}+\sqrt{\pi}\frac{\langle\Sigma\rangle}{\Lambda}\mathrm{erf}\left[\frac{\langle\Sigma\rangle}{\Lambda}\right]+\frac{1}{2\Lambda^2}\int_{\Lambda^{-2}}^{\sim\langle\Sigma\rangle^{-2}}\frac{ds}{s^{3/2}}\sqrt{\pi}\langle\Sigma\rangle\mathrm{erf}(\langle\Sigma\rangle\sqrt{s})\right]\frac{\Lambda^2}{16\pi^2}\int_0^1 d\tau\,\mathrm{Tr}\xi\gamma^5\mathbf{a}_1\,. \tag{5.11}$$

B. Four-derivative term

As far as the calculation of a contribution from (5.7) to the four-derivative term is concerned we have assumed in evaluating (5.8) that $\langle\Sigma\rangle^2$ is large enough compared to the gradients of the pseudoscalar fields so that an expansion in powers of $\not{D}^2/\langle\Sigma\rangle^2$ is justified. This however is not the full story and we would also like to know how the overall coefficient in front of the four-derivative term behaves in the limit $\langle\Sigma\rangle\to 0$. We calculate this as well to the leading order. One has to expand the denominator in (5.8) now assuming $\langle\Sigma\rangle^2$ small. We obtain

$$I_2+I_3\simeq -2N_c\langle\Sigma\rangle^2\int_0^1 d\tau\,\mathrm{Tr}\xi\gamma^5 e^{-\epsilon\langle\Sigma\rangle^2}\left[1-\frac{\not{D}}{\langle\Sigma\rangle}\right]\frac{1}{\not{D}^2}\left[1+\left[\frac{2\langle\Sigma\rangle}{\not{D}}-\frac{\langle\Sigma\rangle^2}{\not{D}^2}\right]+\cdots\right]e^{-s(\not{D}^2-2\langle\Sigma\rangle\not{D})}\,. \tag{5.12}$$

Expanding the exponential and keeping only the leading contributions as $\langle\Sigma\rangle^2\to 0$ we find

$$I_2+I_3\approx 2N_c\langle\Sigma\rangle^2\int_0^1 d\tau\,\mathrm{Tr}\xi\gamma^5 e^{-\epsilon\Sigma^2}\left[2\epsilon+\frac{1}{\not{D}^2}\right]e^{-\epsilon\not{D}^2}+O(\epsilon^2\langle\Sigma\rangle^4)\,.$$

The leading contribution to the four-derivative terms is then obtained simply as

$$(I_2+I_3)_{\partial^4}|_{\langle\Sigma\rangle\sim 0}\simeq 2N_c\langle\Sigma\rangle^2\epsilon\int_0^1 d\tau\frac{1}{16\pi^2}\mathrm{Tr}\xi\gamma^5\mathbf{a}_2+O(\epsilon^2\langle\Sigma\rangle^4)\,. \tag{5.13}$$

To summarize, expressions (5.9) and (5.13) represent the contribution of (5.7) to the four-derivative term of the low-energy pseudoscalar theory evaluated in two different limits. For $\langle\Sigma\rangle\to 0$, however, the relevant contribution is (5.13) and we see that unlike what seems to be suggested by (5.9), the contribution above vanishes as $\epsilon\langle\Sigma\rangle^2$, as one turns off the symmetry breaking.

Collecting together (5.6), (5.9), and (5.13) we obtain a complete expression for the four-derivative part of the effective action (3.6):

$$\Gamma_4\simeq\mathcal{K}(\Sigma^2)\frac{N_c}{8\pi^2}\int_0^1 d\tau\,\mathrm{Tr}\xi\gamma^5\mathbf{a}_2\,, \tag{5.14}$$

where $\mathcal{K}(\Sigma^2)\simeq 1$ for $\langle\Sigma\rangle^2\sim 0$, while $\mathcal{K}(\Sigma^2)\simeq 0$ for $\langle\Sigma\rangle\neq 0$ and sufficiently large.

VI. THE MEASURE CONTRIBUTION, PION DECAY CONSTANT, AND THE LOW-ENERGY EFFECTIVE ACTION

In the preceding section we have calculated [up to the still undetermined coefficients $\mathbf{a}_i(\mathbf{U})$] the first term of the formal expression (3.1) for the low-energy effective meson action. Results of this calculation are expressions (5.11) and (5.14). The remaining thing to be calculated is the second term in (3.1), $\ln J[\mathbf{U}]$. This is the contribution coming from pseudoscalars coupled through the quark measure. Contribution of this is simply the negative of the $\langle\Sigma\rangle=0$ limit of the previously calculated expression, i.e.,

$$-\Gamma\,|_{m,\Sigma=0}=\ln J[\mathbf{U}]=-2N_c\frac{\Lambda^2}{16\pi^2}\int_0^1 d\tau\,\mathrm{Tr}\xi\gamma^5\mathbf{a}_1-\frac{N_c}{8\pi^2}\int_0^1 d\tau\,\mathrm{Tr}\xi\gamma^5\mathbf{a}_2\ . \tag{6.1}$$

To understand this, note that it is natural to define the quark measure to start with independent of the possible dynamically generated constituent-quark mass. Dynamical mass generation is the phenomenon occurring below a certain low-energy scale, while the quark measure is defined at all scales, being the essential part in the definition of the path integral. It is important to note, however, that the "chiral-quark" measure $d\bar{\mathbf{q}}d\mathbf{q}$ and the "constituent-quark" measure $d\bar{\mathbf{q}}^{U}d\mathbf{q}^{U}$ [see Eqs. (2.5) and (2.6) of Sec. II] have completely different transformation properties under the transformations of the global chiral $\mathrm{SU}(N_F)\times\mathrm{SU}(N_F)$. In fact, the constituent-quark measure is chiral invariant. The chiral transformation by which we have introduced pseudoscalars [Eq. (2.5)] in Sec. II is a redefinition of the chiral-quark fields and induces a nontrivial interaction, $\ln J[\mathbf{U}]$ of the low-energy part of the quark measure with pions. The quark measure relevant for calculating $\ln J[\mathbf{U}]$ is therefore defined with respect to a massless Dirac operator coupled to some "external" vector and axial-vector fields. The phenomenological requirement of having a hadronic vector current anomaly-free makes it convenient to use the Schwinger proper-time method for regularization. This is consistent with what has been done in the preceding section, where the proper-time definition of the fermion determinant is used. It is well known[18] that this regularization scheme gives automatically the vector gauge-invariant definition of the fermion determinant and the measure. Consider now a finite chiral redefinition of the quark fields similar to one [see Eq. (2.5)] by which we have localized pseudoscalars couple only through the parts of the QCD action not having the full local chiral invariance:

$$\mathbf{q}_L=V^{\dagger}(x,\tau)\mathbf{q}_L^{\tau},\quad \mathbf{q}_R=V(x,\tau)\mathbf{q}_R^{\tau}\ , \tag{6.2}$$

where $V(x,\tau)=\exp[i\gamma^5\tau\xi(x)]$. This is a finite chiral rotation, but can be viewed as a succession of infinitesimal transformations, each of which gives an anomalous change in the measure[22]

$$\delta\tau\ln J(\tau)=-2N_c\mathrm{Tr}\delta\tau\xi\gamma^5 e^{-\epsilon\not{D}^2(\tau)}\ .$$

Clearly then, summing over the infinite set of infinitesimal transformations, as τ goes from $\tau=0$ to $\tau=1$, one obtains

$$\ln J[\mathbf{U}]=-2N_c\int_0^1 d\tau\,\mathrm{Tr}\xi\gamma^5 e^{-\epsilon\not{D}^2}\ . \tag{6.3}$$

$\epsilon=\Lambda^{-2}$ is the proper-time cutoff indicating that we are considering contributions from the long-wavelength part of the measure interacting with pseudoscalars, as has been explained in the preceding sections. Formula (6.1) follows if one compares now (6.3) with the expression for $\Gamma(\mathbf{U},\boldsymbol{\Sigma})$, (5.3), and results (5.6), (5.11), and (5.14). With (6.1) and the results of the preceding section, (5.11) and (5.14), we have reduced the calculation of the low-energy pseudoscalar action (3.1), to the calculation of the coefficients $\mathbf{a}_i(x,x)$ appearing in the small-ϵ expansion of the massless heat kernel $\langle x\,|\,e^{-\epsilon\not{D}^2}|\,x\rangle$ [see Eq. (5.5)]. In the Appendix we present a general method and derive the relevant recursive formulas for evaluating the coefficients $\mathbf{a}_i(x,y)$, which we have used in our calculation. While the formulas derived could be of some pedagogical interest, we should stress that it is possible to calculate the coefficient $\mathbf{a}_2(x,x)$ in another way, using the general theorem by Gilkey[23] as has been done by Balachandran *et al.*[24] and more recently by Nepomechie.[25]

We now illustrate how the kinetic term for pseudoscalars appear, while details about the four-derivative terms are worked out in the Appendix and will be used shortly, when we present a complete expression for the low-energy effective theory for pseudoscalars. A complete expression for the two-derivative term is obtained by combining the contributions from (6.1) and (5.11). We get

$$2N_c\left[e^{-\langle\Sigma\rangle^2/\Lambda^2}+\sqrt{\pi}\frac{\langle\Sigma\rangle}{\Lambda}\mathrm{erf}\left[\frac{\langle\Sigma\rangle}{\Lambda}\right]-1+\frac{\sqrt{\pi}}{2}\frac{\langle\Sigma\rangle}{\Lambda^2}\int_{\Lambda^{-2}}^{\sim\langle\Sigma\rangle^{-2}}\frac{ds}{s^{3/2}}\mathrm{erf}(\langle\Sigma\rangle\sqrt{s}\,)\right]\frac{\Lambda^2}{16\pi^2}\int_0^1 d\tau\,\mathrm{Tr}\xi\gamma^5\mathbf{a}_1\ , \tag{6.4}$$

where for $\mathbf{a}_1(\mathbf{U})$ we find [see Eq. (A6)]

$$\mathbf{a}_1(x,x)=2A(\tau)\cdot A(\tau)+\gamma^5[D^V\cdot A(\tau)]+\tfrac{1}{2}\gamma^\mu\gamma^\nu V_{\mu\nu}(\tau)$$

with $A_\mu(\tau)$ and $V_\mu(\tau)$ defined in (A8). It follows that

$$\int_0^1 d\tau\,\mathrm{Tr}\xi\gamma^5\mathbf{a}_1=4\int_0^1 d\tau\int d^4x\,\mathrm{tr}\xi(D^V\cdot A)\ . \tag{6.5}$$

Integrating by parts and noticing that $A_\mu(\tau+\delta\tau)\simeq A_\mu(\tau)+\delta\tau D_\mu^V\xi$ we obtain

$$-4\int d^4x\,\mathrm{tr}\int_0^{A_\mu}\delta A\cdot A=-2\int d^4x\,\mathrm{tr}A\cdot A\ .$$

Using the expression (2.7) for A_μ and going back to Minkowski space, one obtains

$$\int_0^1 d\tau\,\mathrm{Tr}\xi\gamma^5\mathbf{a}_1=i\tfrac{1}{2}\int d^4x\,\mathrm{tr}\partial^\mu\mathbf{U}^\dagger\partial_\mu\mathbf{U}\ . \tag{6.6}$$

We learn, therefore, that (6.4) represents a properly normalized kinetic term for pseudoscalars providing that the following relation is satisfied:

$$F_\pi{}^2=\frac{N_c\Lambda^2}{4\pi^2}\left[e^{-(m_Q/\Lambda)^2}+\sqrt{\pi}\left[\frac{m_Q}{\Lambda}\right]\mathrm{erf}\left[\frac{m_Q}{\Lambda}\right]-1+\int_{\Lambda^{-2}}^{\sim m_Q{}^{-2}}\frac{ds}{2s^{3/2}}\sqrt{\pi}\frac{m_Q}{\Lambda^2}\mathrm{erf}(m_Q\sqrt{s})\right]. \tag{6.7}$$

This relation above is similar to the previously obtained relations (4.2) and (4.4) in that it links into a simple consistency condition some important low-energy parameters: m_Q, Λ, and F_π. One should note that (6.7) and (4.2) give, after eliminating Λ, a new (dynamical) relation between F_π, $\langle\bar{q}q\rangle$, and m_Q.

We are now in a position to present the final result of our calculation of the low-energy effective action for pseudoscalars (3.1), to which, as we have argued in Sec. II, QCD is likely to reduce in the limit of large number of colors and extremely low energies. Collecting together (4.1), (5.11), (5.14), (6.1), and using (6.4) as well as the results about the four-derivative terms presented in the Appendix, we obtain[9]

$$\begin{aligned}W_{\mathrm{eff}}(\mathbf{U},\Sigma)\simeq\int d^4x\Bigg[&\frac{F_\pi{}^2}{4}\mathrm{tr}\partial^\mu\mathbf{U}^\dagger\partial_\mu\mathbf{U}-i\frac{N_c}{48\pi^2}[1-\mathscr{K}(\Sigma^2)]\int_0^1 dx^5\epsilon^{\mu\nu\rho\sigma}\mathrm{tr}\mathbf{U}^\dagger\partial_5\mathbf{U}\mathbf{U}^\dagger\partial_\mu\mathbf{U}\mathbf{U}^\dagger\partial_\nu\mathbf{U}\mathbf{U}^\dagger\partial_\rho\mathbf{U}\mathbf{U}^\dagger\partial_\sigma\mathbf{U}\\&+\frac{N_c}{192\pi^2}[1-\mathscr{K}(\Sigma^2)]\mathrm{tr}[2\partial^2\mathbf{U}^\dagger\partial^2\mathbf{U}+\tfrac{1}{2}[\mathbf{U}^\dagger\partial_\mu\mathbf{U},\mathbf{U}^\dagger\partial_\nu\mathbf{U}]^2-(\partial^\mu\mathbf{U}^\dagger\partial_\mu\mathbf{U})^2]\\&+V_{\mathrm{eff}}(\sigma,\mathbb{1})+\frac{N_c\Lambda^3}{4\pi^2}\left[\frac{\langle\Sigma\rangle}{\Lambda}\right]\int_1^\infty\frac{ds}{s^2}e^{-s(\langle\Sigma\rangle/\Lambda)^2}\mathrm{Tr}\frac{m}{2}(\mathbf{U}^\dagger+\mathbf{U})+O\left[\frac{m^2}{\Lambda^2}\right]+\cdots\Bigg],\end{aligned} \tag{6.8}$$

where $\mathscr{K}(\Sigma^2)=1$ for $\langle\Sigma\rangle\simeq0$ (unbroken phase) while it approaches $\mathscr{K}(\Sigma^2)\simeq0$ rapidly as $\langle\Sigma\rangle$ increases above a certain scale and in particular is zero for $\langle\Sigma\rangle\simeq m_Q$, i.e., in the chiral-symmetry-broken phase ("normal vacuum"). The last two terms represent the effective potential [see (4.1) and (4.3)] expanded to leading order around the chiral limit ($m\neq0$). The second term in (6.8) is the calculated Wess-Zumino term,[5,26,27] with $\mathbf{U}=\mathbf{U}(x;x^5)$ interpolating between $\mathbf{U}=\exp[i(2/F_\pi)\pi(x)]$ at $x_5=1$ and $\mathbf{U}=\mathbb{1}$ at $x_5=0$.

VII. DISCUSSION: SKYRME OR THE BAG?

Let us first note that throughout our derivation so far, the assumption of the dynamical breakdown of the chiral symmetry, i.e., the existence of the $\langle\Sigma\rangle\neq0$ ground state, has been closely aligned with the appearance of the pseudoscalars. That is, of course, as it should be the pseudoscalars being just the pseudo-Goldstone bosons associated with the breakdown of the chiral symmetry. Our calculation illustrates that it is the dynamical symmetry breakdown which leads to the appearance of the kinetic term and the nontopological four-derivative terms for pseudoscalars. Notice that if one artificially turns off the symmetry breaking, i.e., let $\langle\Sigma\rangle\to0$, the dynamical terms of the low-energy effective action (6.8) will also be turned off, the coefficients in front of them vanishing as $\langle\Sigma\rangle\to0$. Note at this point that the same conclusion applies to the coefficients in front of all higher derivative terms as well. This can be easily seen from the fact that the effective pseudoscalar action is the difference between the massive determinant and the low-energy measure contribution [Sec. VI, Eq. (6.1)] and it vanishes in the limit $m_Q\to0$.

Consider now the four-derivative terms in (6.8). It is well known[3,6,28] that within a pure pseudoscalar low-energy model, such as the Skyrme model, those terms are crucial for the eventual stabilization of the soliton (i.e., nucleon) against shrinking to zero size.[29] It is easy to see that among the four-derivative terms calculated here,[30–33] the second (the Skyrme term) and third contribute to the stability of the soliton. However, the first term is of opposite sign and tends to destabilize the soliton. It does not seem possible to claim that the quartic terms will be manifestly positive, and we conclude therefore, that the soliton is not stabilized by the quartic, pure pseudoscalar terms.[9,34] This result, although hard to predict without calculating the four-derivative terms, is not completely surprising. After all, the effective pure pseudoscalar theory is derived to be valid at extremely low energies ($E\ll\Lambda$). This is not the relevant scale for the nucleon—Skyrmion.[35] Yet, in the context of the Skyrme model, one had hoped that a pure pseudoscalar part of the low-energy action coming from QCD, might contain a stable soliton. Our result should be interpreted as an indication that this hope is not justified (at least in the case of the theory obtained by truncating the derivative expansion to the four derivatives).[36] It might not be a bad idea to investigate in some detail the phenomenology of the model obtained by ignoring the $\partial^2U^\dagger\partial^2U$ term (that which spoils stability) in (6.8). In addition to the Skyrme term we are then left with term of different tensor structure,[37] $(\partial U^\dagger\partial U)^2$, the coefficient of which we predict[9] to be $N_c/192\pi^2$. In any case, to arrive at a more realistic meson theory, and at the same time the stable soliton, it seems that one should extend somewhat the pure pseudoscalar (Skyrme) framework. In fact, even within our (extreme) low-energy framework there is a hint of the more realistic structure appearing. To see this, note that scalar-meson field enters naturally into our formulation, and the coefficients in front of the nontopological terms of our effective pseudoscalar theory depend in a nontrivial way on the scalar-meson background σ_{cl}, i.e., $\Sigma(\sigma)$. While in the vacuum sector the energy is minimized by $\Sigma(\sigma)\simeq\langle\Sigma\rangle$, the presence of the soliton is likely to be characterized by the position dependent $\sigma_{cl}(x)$ which interpolates between $\sigma_{cl}\simeq 0$ at short distances and $\langle\sigma\rangle\neq 0$ [i.e., $\Sigma(\sigma)\simeq m_Q$] at large distances. Assuming this we see that the coefficients in front of the kinetic term (6.7) and quartic derivative terms (6.8) will rapidly vanish below a certain critical radius which characterizes the rapid falloff of $\sigma_{cl}(x)$ from its large distance ("normal vacuum") value $\langle\sigma\rangle\neq 0$ to $\langle\sigma\rangle\simeq 0$. What appears therefore is a natural realization of the space cutoff, essentially a step function, which enters in an *ad hoc* way the topological soliton bag model.[10] The topological soliton will be characterized by a core made out of the scalar-meson and chiral fields (the bag) inside of which the contribution of pseudoscalars rapidly vanishes, leaving the bubble of unbroken vacuum ($\langle\sigma\rangle\simeq 0$) with massless (chiral) quarks and gluons. The topological soliton will in this case be stable against shrinking to zero size. The baryon number will be equal to unity and composed of the fraction carried by the quarks inside the bag and the topological charge carried by the chiral soliton.[38] Note at this point that the pure pseudoscalar anomaly term [the second term in (6.8), i.e., the Wess-Zumino-Witten term] as well, enters the effective action (6.8) multiplied by a factor of $[1-\mathscr{K}(\Sigma^2)]$, a point which was not clear (to the author) when Ref. 9 was written. This result supports the result first obtained within a particular model calculation by Niemi.[39]

Consider finally the two consistency relations (4.4) and (6.7) obtained through our derivation.[40] Remarkably, those two relations have a consistent solution reasonably close to the realistic values of the parameters involved. We find, for example, $m_\pi\simeq 130$ MeV ($m=8$ MeV), $F_\pi\simeq 96$ MeV, $\Lambda\simeq 667$ MeV, and $m_Q\simeq 200$ MeV (two of these are predictions).

VIII. CONCLUDING REMARKS

In this paper we have argued that in the limit of large N and extremely low energies QCD should reduce to the pure pseudoscalar theory which under some additional assumptions, stated in Sec. II, we have calculated truncated to four derivatives. The basic ideas, making the model derived here a very plausible one for the extreme low-energy limit of QCD at large N, are explained in Sec. II. There we find it very useful to make a clear distinction between the "chiral" quarks and the "constituent" quarks. The pion is either the bound state of two chiral quarks or is (equivalently) a collective Goldstone mode interacting nonlinearly through the derivative coupling with massive constituent quarks. We define what we call the "constituent-quark gauge" as a QCD analog of the unitary gauge in the spontaneously broken theories with Higgs bosons. In this gauge the pions are made manifest as collective Goldstone modes of dynamically broken chiral symmetry and are localized to couple only through the parts of the QCD action that do not have a full local chiral invariance. Further "decoupling" and large-N arguments then singles out a particular low-energy model of the massive constituent quarks interacting with collective pions and scalar mesons [Eqs. (2.12), (2.13), and (2.14)] which we use as the starting point for our calculation of the low-energy effective pure pion theory in Secs. III, IV, V, and VI. The basic results of this calculation are expressions (4.1), (4.2), (4.4), (6.7), and (6.8). Two of these relations, (4.2) and (6.7), can be used to eliminate our physical cutoff Λ, and lead to a new (dynamical) relation between F_π, $\langle\bar{q}q\rangle$, and m_Q. Our calculation clearly illustrates that it is the dynamical symmetry breakdown which leads to the appearance of the kinetic term and all the other interaction terms for pseudoscalars. We comment about the possibility that if summed to all orders in derivatives, the pure pseudoscalar theory might have a stable soliton.[36] In this case one might have the justification to neglect one of the four-derivative terms in (6.8) (that which spoils stability). It should be interesting to investigate the phenomenology of the Skyrme model extended in this way. However, it seems very likely that one will have to extend the pure pseudoscalar framework in order to reproduce realistically all the aspects of phenomenology.[35] We show that under a plausible assumption about the behavior of the scalar-meson background $\sigma_{cl}(x)$, our results lead to a natural realization of the space cutoff entering in an *ad hoc* way in the previously proposed topological soliton bag models.[10]

Note added. Recently, the following papers appeared, in which, within a somewhat more phenomenological framework and using a different method of calculation, similar but not identical results were obtained: A. Dhar, R. Shankar, and S. R. Wadia, Phys. Rev. D 31, 3256 (1985); L. H. Chan, Phys. Rev. Lett. 55, 21 (1985). See also Ref. 9. Also, since the submission of this paper, the following relevant work appeared, I. Aitchison *et al.*, Phys. Lett. **165B**, 162 (1985), in which the contribution of the sixth-order terms to the static energy was investigated.

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APPENDIX

Consider the massless "heat kernel"

$$H(x,y;s)=\langle x\,|\exp(-s\not{D}^2)\,|\,y\rangle$$

satisfying the "heat equation"

$$\partial_s H=-\not{D}^2 H,\quad H(x,y;0)=\delta(x-y)\ . \tag{A1}$$

Let us make the ansatz

$$H(x,y;s)=H_0(x,y;s)[a_0(x,y)+s\,\mathbf{a}_1(x,y)+\cdots] \tag{A2}$$

and

$$H_0(x,y;s)=\frac{1}{16\pi^2 s^2}\exp\left[-\frac{(x-y)^2}{4s}\right],$$

where $H_0(x,y)$ is the solution of the "heat equation" for the case of the massless Dirac operator. Replacing the ansatz (A2) in the heat equation (A1), one obtains the following system of recursion relations:

$$(x-y)^\mu(\partial_\mu+V_\mu-i\sigma_{\mu\nu}\gamma^5A^\nu)\mathbf{a}_l+l\mathbf{a}_l=-\not{D}^2\mathbf{a}_{l-1}, \quad \text{(A3a)}$$

$$(x-y)^\mu(\partial_\mu+V_\mu-i\sigma_{\mu\nu}\gamma^5A^\nu)\mathbf{a}_0=0, \quad \mathbf{a}_0(x,x)=\mathbb{1}, \quad \text{(A3b)}$$

where

$$\not{D}=i\gamma^\mu(\partial_\mu+V_\mu+\gamma^5A_\mu).$$

The equation for $\mathbf{a}_0(x,y)$ can be integrated, and the solution can be represented in terms of Dyson's iterative series:

$$\mathbf{a}_0(x,y)=\mathbb{1}+\sum_{n=1}^{\infty}(-1)^n\int_y^x\int^{x_1}\cdots\int^{x_{n-1}}dx^{a_n}\cdots dx^{a_1}P_{a_n}P_{a_{n-1}}\cdots P_{a_1},$$
$$P_\mu=V_\mu-i\sigma_{\mu\nu}\gamma^5A^\nu. \quad \text{(A4)}$$

From (A3), after some easy algebra, we obtain the following useful formula:

$$\mathbf{a}_l(x,x)=-\frac{1}{l}(\not{D}^2\mathbf{a}_{l-1})_{x=y}$$
$$=\frac{1}{l(l+1)}(\not{D}^4\mathbf{a}_{l-2})_{x=y}$$
$$+\frac{1}{l(l+1)}[\gamma^\mu\gamma^\nu V_{\mu\nu}+2\gamma^5(D^V\cdot A)+4A\cdot A]_{\mathbf{a}_{l-1}}. \quad \text{(A5)}$$

This expression, together with the expression for $\mathbf{a}_0(x,y)$ in (A4) can be used to calculate directly the first two coefficients of the small-s expansion (A2) in the coincidence limit ($x=y$).

After some algebra we obtain

$$\mathbf{a}_1(x,x)=-(\not{D}^2\mathbf{a}_0)_{x=y}=2A\cdot A+\gamma^5(D^V\cdot A)+\tfrac{1}{2}\gamma^\mu\gamma^\nu V_{\mu\nu}. \quad \text{(A6)}$$

Using (A6) in (A5) we obtain a simple looking expression for $\mathbf{a}_2(x,x)$ which can be used for a direct calculation of this coefficient:

$$\mathbf{a}_2(x,x)=\tfrac{1}{6}(\not{D}^4\mathbf{a}_0)_{x=y}$$
$$+\tfrac{1}{12}[\gamma^\mu\gamma^\nu V_{\mu\nu}+2\gamma^5(D\cdot A)+4A\cdot A]^2. \quad \text{(A7)}$$

In fact, we are interested to calculate $\mathrm{Tr}\gamma^5\xi\mathbf{a}_2(x,x)$ rather than $\mathbf{a}_2(x,x)$ itself. Also, some simplification comes from the fact that vector (V) and axial-vector (A) fields appearing in our framework are made out of collective pseudoscalar fields, according to the formula

$$V_\mu=\tfrac{1}{2}(V^\dagger\partial_\mu V+V\partial_\mu V^\dagger),\quad A_\mu=\tfrac{1}{2}(V^\dagger\partial_\mu V-V\partial_\mu V^\dagger), \quad \text{(A8)}$$

where $V=V(\tau)=\exp[i\tau\xi(x)]$. Therefore V_μ and A_μ are pure gauge:

$$V_{\mu\nu}=A_{\mu\nu}=0. \quad \text{(A9)}$$

Calculation of $(\not{D}^4\mathbf{a}_0)_{x=y}$ can be done with use of (A3b). This is a straightforward but somewhat lengthy calculation. We present here only the result which agrees with one obtained some time ago by Balachandran *et al.*[24] and more recently by Nepomechie:[25]

$$\mathrm{Tr}\xi\gamma^5\mathbf{a}_2(x,x)=-i\tfrac{16}{3}\,\mathrm{tr}\epsilon^{\mu\nu\rho\sigma}\xi A_\mu A_\nu A_\rho A_\sigma$$
$$+\tfrac{1}{6}\,\mathrm{tr}\xi[-4\{D\cdot A,A^2\}$$
$$+16\{D_\mu A_\nu,A^\mu A^\nu\}+4D^2(DA)$$
$$+24A^\mu(D\cdot A)A_\mu]. \quad \text{(A10)}$$

The imaginary part of the action is therefore

$$I_{\mathrm{WZ}}=i\frac{N_c}{48\pi^2}\int_0^1 d\tau\int d^4x\,\epsilon^{\mu\nu\rho\sigma}\mathrm{tr}2i\frac{\pi}{F_\pi}$$
$$\times\mathbf{U}^\dagger\partial_\mu\mathbf{U}\mathbf{U}^\dagger\partial_\nu\mathbf{U}\mathbf{U}^\dagger\partial_\rho\mathbf{U}\mathbf{U}^\dagger\partial_\sigma\mathbf{U}.$$

Using $2i(\pi/F_\pi)\equiv U^\dagger\partial_5U$ and $x^5\equiv\tau$ we obtain the Wess-Zumino anomaly term:

$$I_{\mathrm{WZ}}=i\frac{N_c}{48\pi^2}\int d^5x\,\epsilon^{\mu\nu\rho\sigma}$$
$$\times\mathrm{tr}\mathbf{U}^\dagger\partial_5\mathbf{U}\mathbf{U}^\dagger\partial_\mu\mathbf{U}\mathbf{U}^\dagger\partial_\nu\mathbf{U}\mathbf{U}^\dagger\partial_\rho\mathbf{U}\mathbf{U}^\dagger\partial_\sigma\mathbf{U}. \quad \text{(A11)}$$

Consider now the contribution of the real part of $\mathbf{a}_2(x,x)$ to the four-derivative action:

$$S_4\simeq\frac{N_c}{48\pi^2}\int d^4x\int_0^1 d\tau\,\mathrm{tr}\xi[-4\{D\cdot A,A^2\}$$
$$+16\{D_\mu A_\nu,A^\mu A^\nu\}$$
$$+4DD(D\cdot A)$$
$$+24A^\mu(D\cdot A)A_\mu]. \quad \text{(A12)}$$

The integrand is in fact a total derivative as can be explicitly checked:

$$\frac{d}{d\tau}[4[A_\mu,A_\nu]^2-4A_\mu A_\nu A^\mu A^\nu$$
$$-4(D_\mu A_\nu)(D^\mu A^\nu)+6(D\cdot A)^2].$$

Equation (A12) can therefore be written as a four-dimensional integral:

$$S_4=\frac{N_c}{48\pi^2}\mathrm{tr}\int d^4x[4[A_\mu,A_\nu]^2-4A_\mu A_\nu A^\mu A^\nu-4(D_\mu A_\nu)(D^\mu A^\nu)+6(D\cdot A)^2] \ . \quad \text{(A13)}$$

Using the property $A_{\mu\nu}=V_{\mu\nu}=0$, i.e., $D_\mu A_\nu=D_\nu A_\mu$, (A13) can be further simplified and we obtain

$$S_4=-\frac{N_c}{48\pi^2}\int d^4x\ \mathrm{tr}[2(D\cdot A)^2+2[A_\mu,A_\nu]^2+4(A\cdot A)^2] \ . \quad \text{(A14)}$$

Using (A8), as well as the obvious identity $\partial^2\mathbf{U}^\dagger\mathbf{U}+\mathbf{U}^\dagger\partial^2\mathbf{U}=-2\gamma^\mu\mathbf{U}^\dagger\partial_\mu\mathbf{U}$ we obtain

$$S_4=-\frac{N_c}{192\pi^2}\int d^4x[2(\partial^2\mathbf{U}^\dagger\partial^2\mathbf{U})+\tfrac{1}{2}[\mathbf{U}^\dagger\partial_\mu\mathbf{U},\mathbf{U}^\dagger\partial_\nu\mathbf{U}]^2-(\partial_\mu\mathbf{U}^\dagger\partial^\mu\mathbf{U})^2] \ . \quad \text{(A15)}$$

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