Simplicial pseudorandom lattice study of a three-dimensional Abelian gauge model, the regular lattice as an extremum of the action

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We introduce a simplicial pseudorandom version of lattice gauge theory. In this formulation it is possible to interpolate continuously between a regular simplicial lattice and a pseudorandom lattice. Using this method we study a simple three-dimensional Abelian lattice gauge theory. Calculating average plaquette expectation values, we find an extremum of the action for our regular simplicial lattice. Such a behavior was found in analytical studies in one and two dimensions.

I. INTRODUCTION

The lattice gauge theory on a random simplicial lattice was introduced in a series of papers by Christ, Friedberg, and Lee.[1] Monte Carlo calculations of lattice gauge theories within this framework were reported by Ren.[2]

There are several reasons to prefer the lattice gauge theory defined on a random lattice.[3,4] We mentioned here only three of these reasons.

(i) The inherent rotational invariance of the Euclidean version of the random lattice.

(ii) Because of the higher degree of connectedness, the random lattice with the same number of points is closer to the continuum than the usual hypercubic lattice. In the hypercubic lattice in $d=4$ dimensions there are $N_{1/0}=8$ links and $N_{2/0}=24$ plaquettes per lattice point. In the random lattice the corresponding numbers are $N_{1/0}\approx 37.8$ and $N_{2/0}\approx 197$. This property seems, in the absence of phase transitions, to translate into a smoother behavior of the measured expectation values, as for instance found by Ren[2] in the case of the SU(2) lattice gauge theory.

(iii) The random lattice gauge theory is closely related to Regge's approach to quantum gravity.[5,6] Applications of the random lattice gauge theory to quantum gravity might lead to further progress in this field.[7]

The linking procedure for a large random lattice turns out to be quite complicated. So far, all numerical calculations reported[2] were done with very few actually constructed and linked random lattices. However, theoretical considerations[1,4] always use the assumption that all averages run over a large number of different random lattices. In fact, it was shown[1] that the average over all possible orientations of a Wilson loop in one large random lattice is equivalent to the quenched average calculated from an ensemble of random lattices. The size of actual random lattices constructed so far and used in calculations is, however, not sufficient to exploit this property of the quenched average. Also it seems easier to use an ensemble of not so large random, or pseudorandom lattices, than one really big random lattice.

These problems are our main motivation to introduce a pseudorandom lattice. The pseudorandom lattice should have the advantages of the random lattice when applied to lattice gauge theories but should be easy to construct and link. We derive our pseudorandom lattice from a regular simplicial lattice by randomly shifting the lattice sites, however, without relinking the resulting lattice. This can be done if the shifted lattice points remain within the so-called invariant cell.

Lattice gauge theory on simplicial lattices was studied by Drouffe and co-workers[8] in a series of papers. We stress, however, that in dimensions higher than $d=2$ the regular simplicial lattice[8] is not a simplicial lattice in our sense. We consider simplicial lattices whose elements in d dimensions are only points, links, two-simplices or triangles, three-simplices or tetrahedra, ... up to d-simplices. The simplicial lattices of Drouffe and co-workers[8] contain, for instance, plaquettes in the form of triangles as well as squares.

We arrive at our simplicial lattice by cutting the elementary cells of a regular lattice into simplices. Here we will consider only the case of $d=3$ dimensions.

It was shown by Cohen[9] that in $d=1$ dimensions the regular lattice is an extremum of the action. The corresponding property in $d=2$ dimensions was found by Pertermann.[10] Here we demonstrate numerically the same behavior in $d=3$ dimensions using our formulation of the pseudorandom lattice gauge theory, which allows a continuous transition from the initial regular simplicial lattice to the pseudorandom lattice.

In Sec. II we will define our pseudorandom lattice. The gauge theory on this lattice is defined in Sec. III, and in Sec. IV we apply this formulation to an Abelian gauge theory in $d=3$ dimensions. We find that for this model the regular simplicial lattice is an extremum of the action.

II. THE DEFINITION OF A SIMPLICIAL PSEUDORANDOM LATTICE

In order to describe the construction of a simplicial pseudorandom lattice we restrict ourselves to three dimensions. The lattice is called pseudorandom since it differs significantly from the random lattice constructed by

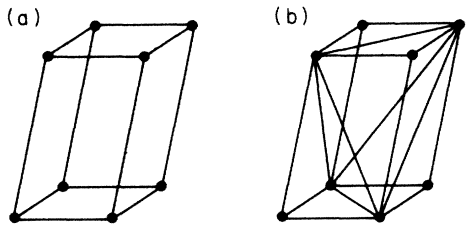

FIG. 1. The elementary cell of a three-dimensional Bravais lattice (a) in the usual sense and (b) decomposed in tetrahedra (three-simplices) by diagonal links.

Christ, Friedberg, and Lee.[1] Our construction starts from a regular simplicial lattice. Each point of the regular lattice is displaced randomly within a given invariant cell. The invariant cells are constructed in such a way that the linking structure of the initial regular lattice can be kept after randomizing the points. In this way it becomes easy to construct many different pseudorandom lattices without excessive computations for linking the lattice. Also, since the amount of randomizing the points can be controlled by a single parameter V varying between zero and one, it is easy to study the effects connected with the transition from a regular simplicial lattice to a pseudorandom lattice.

We start with lattice sites distributed on a regular Bravais lattice. The simplest case to consider is a cubic lattice, but without much additional effort we can also start from tetragonal, rhombic, rhombohedral, monoclinic, or triclinic lattices. In a usual lattice of this kind—we discuss for simplicity the cubic case—the basic lattice elements are points (sites), links, squares (plaquettes), and cubes. In our simplicial Bravais lattice the elements should be instead points, links, triangles, and tetrahedra. It is easy to go over from the usual lattice to the simplicial lattice by cutting each cube into five tetrahedra; see Fig. 1. The resultant lattice consists of two types of tetrahedra, two types of triangles, and three different kinds of links. The multiplicity of these elements increases further in case of the other noncubic lattices. The cuts in the common faces of adjacent cubes are always the same. Figure 2 gives a possible arrangement of a three-dimensional $4\times2\times2$ lattice. It is obvious that more than one possibility exists for linking the regular simplicial lattice.

Considering the rhombic configuration (which includes the cubic and tetragonal cases) we choose a Cartesian coordinate system with the axes parallel to the lattice axes. The coordinate spacings Δx, Δy, and Δz are identical to the lattice spacings a_x, a_y, and a_x along the three axes (see Fig. 3).

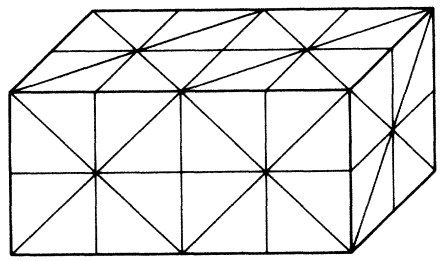

FIG. 2. Example of a regular simplicial lattice derived from a $4\times2\times2$ cubic lattice. Each elementary cube is cut as shown in Fig. 1(b).

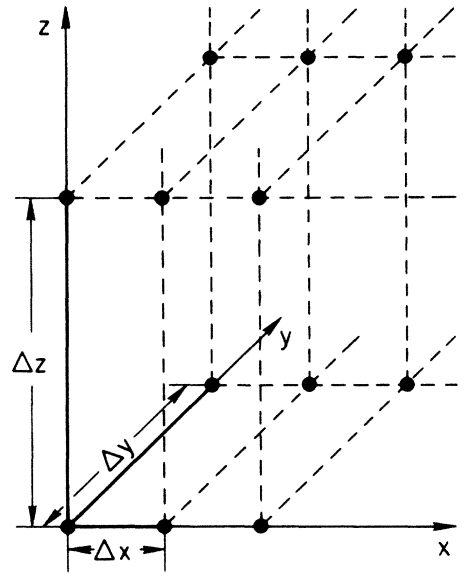

FIG. 3. The rhombic lattice and the corresponding lattice spacings.

For a triclinic lattice (which includes all other cases) it would be possible to use a skew coordinate system parallel to the lattice axes. But we prefer to describe the lattice in a Cartesian frame of reference. There are the coordinate spacings Δx, Δy, and Δz and six deviations Δxy, Δxz, Δyx, Δyz, Δzx, and Δzy. The deviation Δxy, for instance, describes the additional increment to the x coordinate if one advances by Δy in the y direction (see Fig. 4). All the important lattice quantities can be collected in a matrix

$$(g_{ij})=\begin{bmatrix}\Delta x & \Delta xy & \Delta xz\\ \Delta yx & \Delta y & \Delta yz\\ \Delta zx & \Delta zy & \Delta z\end{bmatrix} . \tag{1}$$

In $d=3$ dimensions one needs only six quantities to describe a geometric object. Therefore, we might fix three of the deviations, for instance,

$$\Delta zx=\Delta zy=\Delta yx=0 . \tag{2}$$

In this case, the xy plane and the plane of the lattice axes ξ_1 and ξ_2 (see Fig. 4) coincide. Furthermore, the x and ξ_1 axes are parallel. In this way we describe the triclinic lattice by the matrix

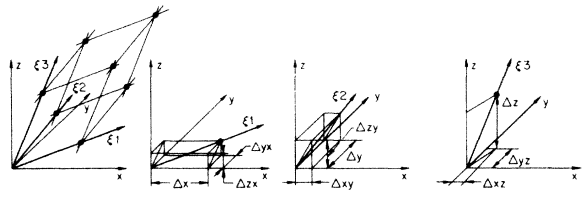

FIG. 4. The coordinate intervals Δx, Δy, and Δz and deviations Δxy, Δxz, Δyx, Δyz, Δzx, and Δzy of a triclinic Bravais lattice with axes ξ_1, ξ_2, and ξ_3.

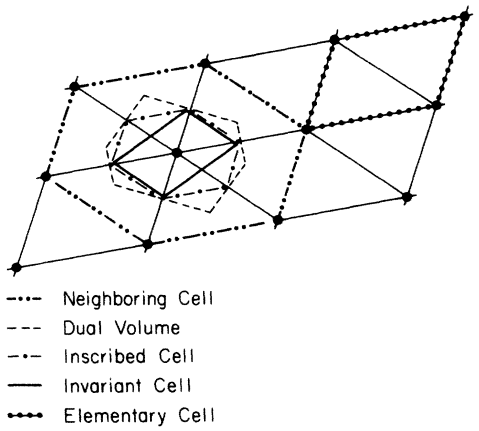

FIG. 5. A piece of a two-dimensional Bravais lattice with all important cell configurations.

$$(g_{ij})=\begin{pmatrix}\Delta x & \Delta xy & \Delta xz\\ 0 & \Delta y & \Delta yz\\ 0 & 0 & \Delta z\end{pmatrix} \tag{3}$$

and the rhombic lattice by

$$(g_{ij})=\begin{pmatrix}\Delta x & 0 & 0\\ 0 & \Delta y & 0\\ 0 & 0 & \Delta z\end{pmatrix}. \tag{4}$$

After having defined the regular lattice we turn to our method to randomize the lattice points. Allowing arbitrary movements of the points and retaining the link structure of the regular simplicial lattice, we would expect crossing of links. This would lead to the destruction of the simplicial nature of the lattice. Consequently, the random fluctuations of the points have to be small enough.

A link crossing is impossible if the site positions vary only within the so-called *invariant cell.* We briefly discuss what this is: In the simplicial Bravais lattice we have the usual *elementary cells* decomposed into *simplices.* Considering all the joint neighbors of a lattice site, including the links between them, we obtain the *neighboring cell.* Following Ref. 1 we get the *dual volume* of a site by a Voronoi construction. We define an *inscribed cell* within this dual volume. The vertices of the inscribed cell are the intersections of the links emerging from the lattice site with the faces of the dual volume. We find the inscribed cell to be similar to the corresponding neighboring cell with all links of half-length.

If the positions of the lattice sites vary within the inscribed cell of a given regular lattice, a link crossing is impossible. But the inscribed cell depends also on the nonunique linking structure. As discussed above, for a given Bravais lattice, there are several possibilities for a simplicial decomposition using diagonal links. We define the *invariant cell* of a lattice site as independent of the structure of the diagonal links. The vertices of this cell are the intersection points of the axial links emerging from the lattice site with the faces of the dual volume. In Fig. 5 we show all the cells discussed here for a two-dimensional Bravais lattice. The generalization to $d=3$ dimensions is straightforward but difficult to visualize.

We emphasize that the inscribed cell contains the invariant cell. The latter is a characteristic of the original Bravais lattice and independent of the special choice of the diagonal links for the simplicial decomposition.

We summarize the procedure.

(i) Construct a Bravais lattice.

(ii) Choose a simplicial decomposition and fix the linking structure.

(iii) Ensure the conservation of the simplicial structure independent on the special choice of the decomposition by allowing fluctuations of the lattice sites only within the invariant cell of the regular lattice.

In Fig. 6 we give an example for a pseudorandom lat-

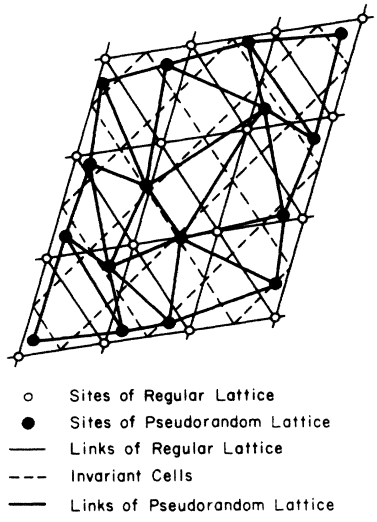

FIG. 6. A two-dimensional pseudorandom configuration over the background Bravais lattice producing the invariant cells.

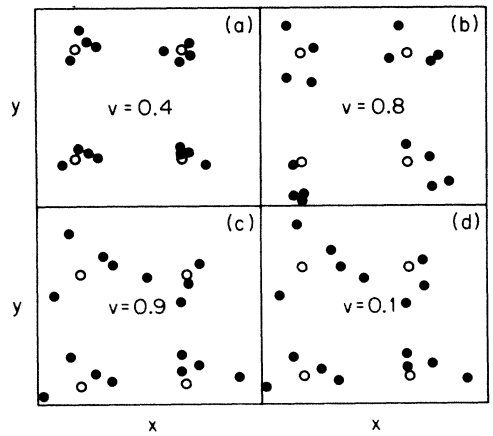

FIG. 7. The pseudorandom lattice for different values of the parameter V: (a) $V=0.4$; (b) $V=0.8$; (c) $V=0.9$; and (d) $V=1.0$. We plot the projection of part of the pseudorandom lattice on the x-y plane. The positions of the original points in a regular cubic lattice are also shown (open points).

tice constructed in this way, again in $d=2$ dimensions.

The site positions of the pseudorandom lattice are characterized by their coordinates in the regular lattice plus a random fluctuation within the invariant cell. It is convenient to generate the fluctuations in diagonal coordinates η_1, η_2, and η_3 varying within

$$-\frac{d_i}{2} \leq \eta_2 \leq \frac{d_i}{2}, \quad i=1,2,3 , \tag{5}$$

where the d_i are the corresponding half-diagonals:

$$\begin{aligned} d_1 &= \tfrac{1}{2}[(\Delta x+\Delta xy+\Delta xz)^2+(\Delta y+\Delta yx+\Delta yz)^2 \\ &\quad +(\Delta z+\Delta zx+\Delta zy)^2]^{1/2} , \\ d_2 &= \tfrac{1}{2}[(\Delta x-\Delta xy+\Delta xz)^2+(\Delta y-\Delta yx-\Delta yz)^2 \\ &\quad +(\Delta z+\Delta zx-\Delta zy)^2]^{1/2} , \\ d_3 &= \tfrac{1}{2}[(\Delta x-\Delta xy-\Delta xz)^2+(\Delta y-\Delta yx+\Delta yz)^2 \\ &\quad +(\Delta z-\Delta zx+\Delta zy)^2]^{1/2} . \end{aligned} \tag{6}$$

Using the cosines between the Cartesian and diagonal coordinates, we find the corresponding variations in the Cartesian coordinates of the sites

$$\begin{bmatrix} \delta x \\ \delta y \\ \delta z \end{bmatrix} = V \begin{bmatrix} \frac{\Delta x+\Delta xy+\Delta xz}{2d_1} & \frac{\Delta x-\Delta xy+\Delta xz}{2d_2} & -\frac{\Delta x-\Delta xy+\Delta xz}{2d_3} \\ \frac{\Delta y+\Delta yx+\Delta yz}{2d_1} & -\frac{\Delta y-\Delta yx-\Delta yz}{2d_2} & \frac{\Delta y-\Delta yx+\Delta yz}{2d_3} \\ \frac{\Delta z+\Delta zx+\Delta zz}{2d_1} & \frac{\Delta z+\Delta zx-\Delta zy}{2d_2} & \frac{\Delta z-\Delta zx+\Delta zy}{2d_3} \end{bmatrix} \begin{bmatrix} \eta_1 \\ \eta_2 \\ \eta_3 \end{bmatrix} , \tag{7}$$

where V is a parameter, introduced in order to interpolate continuously between the regular simplicial lattice and the pseudorandom lattice. For $V=0$, the variations δ_i vanish, we get the regular lattice, for $V=1$ we get the pseudorandom lattice with maximum randomization of the points.

The differences between our version of a pseudorandom lattice and the random lattice introduced by Christ, Friedberg, and Lee[1] do not concern the simplex decomposition alone but also the dual lattice. So, negative dual lengths of plaquettes (triangles) and negative dual areas of links are possible and occur indeed occasionally corresponding to an overlapping of dual cells. Such negative dual quantities appear because the definition of lattice site clusters[1] is not valid in our version. Nevertheless, the dual of the pseudorandom lattice is a Voronoi construction. Therefore, the relations between the lattice quantities discussed in Ref. 1 are valid. Let Ω be the given volume of the space-time. We have

$$L^{\mu\nu} \equiv \sum_{(ij)} l_{ij}^{\mu} l_{ij}^{\nu} \lambda_{ij} = \Omega \delta^{\mu\nu} , \tag{8}$$

where

$$\lambda_{ij} = \frac{\sigma_{ij}}{l_{ij}} ,$$

λ_{ij}, l_{ij}, and σ_{ij} being the coupling, length, and the dual area of the link joining the sites i and j. The l_{ij}^{μ} are the components of the corresponding link vector. The sum runs over all the links (ij) of the lattice. Concerning the plaquettes we have

$$T^{\mu\nu\rho\sigma} \equiv \sum_{p} \omega_p \Delta_p^{\mu\nu} \Delta_p^{\rho\sigma} = (\delta^{\mu\rho}\delta^{\nu\sigma} - \delta^{\mu\sigma}\delta^{\nu\rho})\Omega ,$$

where

$$\Delta_p^{\mu\nu} = \tfrac{1}{6}[l_{ij}^{\mu} l_{jk}^{\nu} + l_{jk}^{\mu} l_{ki}^{\nu} + l_{ki}^{\mu} l_{ij}^{\nu} - (\mu \leftrightarrow \nu)]$$

and

$$\omega_p = \frac{l_p}{\Delta_p} . \tag{9}$$

Here p denotes the plaquette (triangle) with vertices i,j,k and links (ij), (jk), (ki). The ω_p, Δ_p, and l_p are the coupling, area, and dual length of the plaquette. It follows from Eqs. (8) and (9) that

$$\sum_{(ij)} \tfrac{1}{3} \sigma_{ij} l_{ij} = \Omega , \tag{10}$$

$$\sum_{p} \tfrac{1}{3} l_p \Delta_p = \Omega . \tag{11}$$

These formulas have a simple meaning. In three space-time dimensions using the dual areas and the half-lengths of the links we get pyramids. The sum over all these pyramids [Eq. (10)] has to be Ω. In the same way it is possible to construct tetrahedra consisting of the areas and the half-dual-lengths of plaquettes. Again, the sum [Eq. (11)] over these tetrahedra gives Ω.

For a pseudorandom lattice represented by 64 sites in a cubic volume $\Omega=64.0$ (linear size $AL=4$) we found this value within the numerical accuracy of the computer used.

III. THE LATTICE GAUGE THEORY ON THE SIMPLICIAL PSEUDORANDOM LATTICE

We define the gauge theory on our simplicial pseudorandom lattice following Christ, Friedberg, and Lee.[1] The theory in $d=3$ dimensions is defined by the partition function

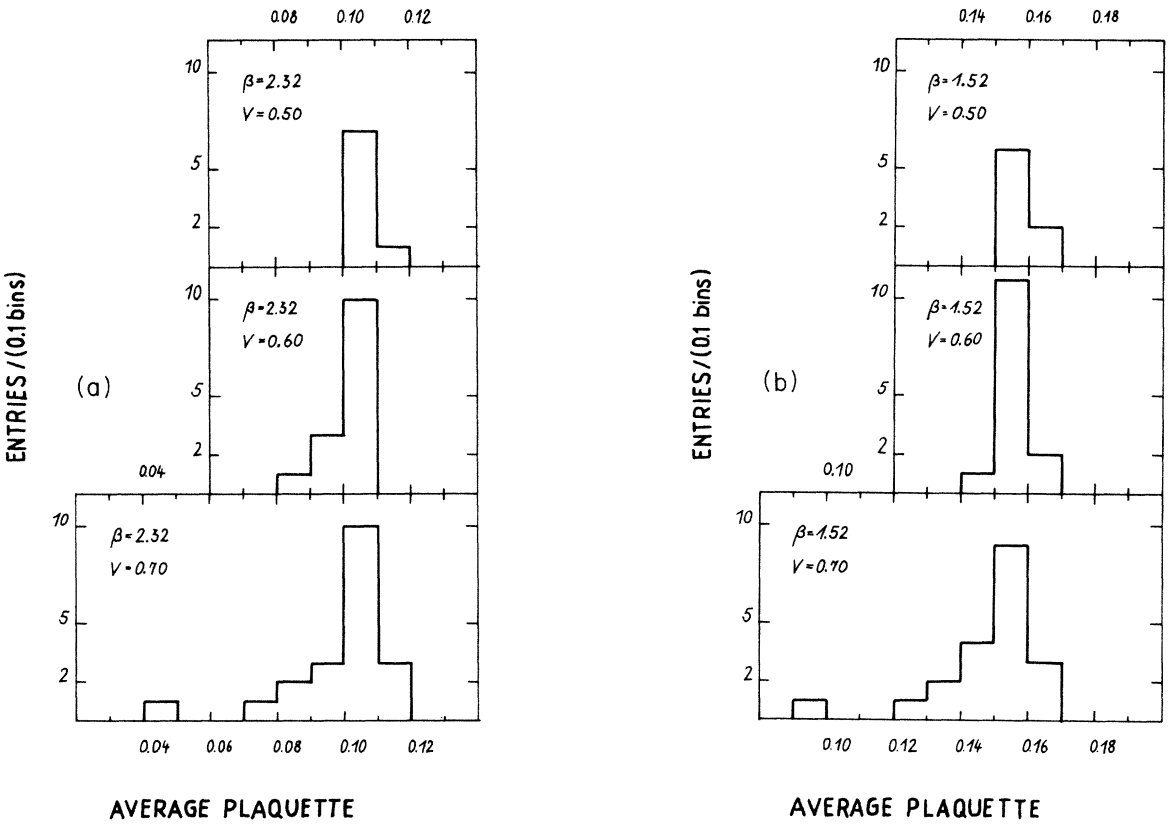

FIG. 8. Distributions of average plaquette expectation values obtained from calculations with 8 ($V=0.50$), 14 ($V=0.60$), and 20 ($V=0.70$) independent random lattices in the form of histograms for $\beta=2.32$ (a), $\beta=1.52$ (b), and $\beta=0.72$ (c).

$$Z(\beta_1\{J_\alpha\})=\int\prod_{i=1}^{N_0}d^3x_i J(\{x\})\int\prod_{l=1}^{N_1}dU_l\exp\left[-\beta A(\{x\},\{U\})-\sum_\alpha J_\alpha Q_\alpha(\{x\},\{U\})\right], \tag{12}$$

where N_0 is the number of lattice sites in the volume Ω and N_1 is the number of links. The Euclidean action is $A(\{x\},\{U\})$. The Q_α are observables and the J_α external sources. The action for a pure gauge theory is defined as

$$A(\{x\},\{U\})=\sum_{p=1}^{N_2}\omega_p\left[1-\frac{1}{n}\mathrm{Re\,Tr}U_p\right]; \tag{13}$$

p runs over all N_2 triangular plaquettes of the lattice. U_p is the product of the three link variables around the plaquette. The weight ω_p of the plaquette is given by[1]

$$\omega_p\propto\frac{l_p}{\Delta_p}, \tag{14}$$

where Δ_p is the area of the triangle and l_p is the length of its dual. We normalize ω_p as

$$\omega_p=\frac{N_2 l_p/\Delta_p}{\sum_{p=1}^{N_2} l_p/\Delta_p}. \tag{15}$$

We measure quenched expectation values[1,2] and define the Jacobian $J(\{x\})$ as

$$J(\{x\})=\left[\int\prod_{l=1}^{N_1}dU_l e^{-\beta A}\right]^{-1}. \tag{16}$$

The expectation value of a physical quantity O becomes

$$\langle O(\beta)\rangle=\frac{1}{\Omega^{N_0}}\int\prod_{i=1}^{N_0}dx_i{}^3\int\prod_{l=1}^{N_1}dU_l e^{-\beta A}O. \tag{17}$$

Here we will especially calculate the expectation value of the average plaquette action

$$O=\frac{A(\{U\})}{N_2}. \tag{18}$$

All these definitions correspond to the ones used by Ren[2] in his random-lattice calculations.

We use in our calculations in $d=3$ dimensions pseudorandom lattices with $N_0=64$ points. The lattices are derived from an initial $4\times4\times4$ cubic lattice. Our simplicial lattice has on the average $N_{1/0}=12$ links per point instead of 6 in the cubic lattice and $N_{2/0}=30$ plaquettes per point instead of 12 in the cubic lattice. In a random lattice these numbers would be $N_{1/0}\approx15.54$ and $N_{2/0}\approx40.56$.

Occasionally, there occur plaquettes in the pseudorandom lattice with very small area Δ_p. Furthermore, tetrahedra are possible with all vertices approximatively in one plane. The corresponding large radius of a circumscribing sphere yields a very large dual plaquette length l_p. This could disturb the Monte Carlo results, since such a plaquette would get an overwhelming weight ω_p. We found one of three (one of four) lattice configurations with such plaquette weights for $V=0.65$ (0.70). The number of these extreme configurations increases rapidly if $V\to1$. Therefore, we restrict the variation parameter to be

$$0.00<V\leq0.70. \tag{19}$$

IV. THE REGULAR LATTICE AS AN EXTREMUM OF THE ACTION

In Fig. 7 we show, by projecting some of the points of pseudorandom lattices into the x-y plane, how the random lattice parameter V changes the appearance of the lattice. This plot demonstrates also that we interpolate with V between the regular lattice ($V=0$) and the full pseudorandom lattice ($V=1$).

We use in our calculation the Abelian U(1) gauge group, which we approximate in the calculation by the discrete Z_{60} group. The U(1) lattice gauge theory in $d=3$ dimensions was studied in detail by Bhanot and Creutz[11] and by Ambjorn, Hey, and Otto.[12] There is no phase transition in this theory. We will not study in the present paper the properties of this theory. We calculate only the

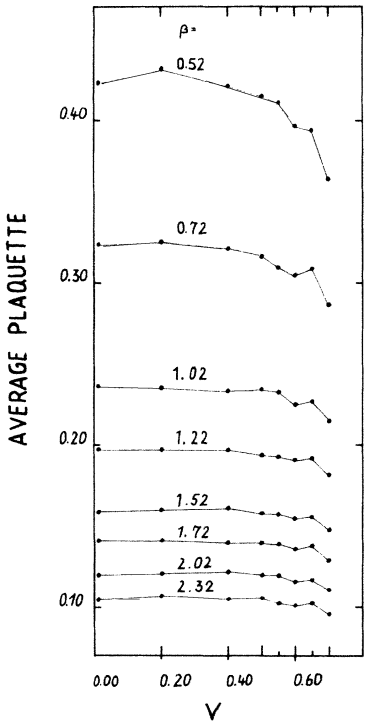

FIG. 9. Average plaquette expectation values for eight values of β as function of the random lattice parameter V. For the regular lattice we find a maximum of the action. The calculation is for the U(1) lattice gauge theory in $d=3$ dimensions.

average action per plaquette as a function of the coupling constant β and study the effects which occur if one goes from the regular simplicial lattice to the pseudorandom lattice.

There are two effects.

(i) When calculating expectation values on regular lattices of the same size, there are no systematic differences between independent calculations. Only statistical errors and possibly effects connected with the lack of thermal equilibrium lead to different results. This is not so when calculating expectation values on a random lattice or pseudorandom lattice of finite size. Depending on the actual realization of the random lattice we find systematic differences between the expectation values. We have to average the expectation values over a sufficiently large number of independent random or pseudorandom lattices as prescribed by the quenched average.[1,3] This is always true if we are using a lattice with a finite size practical for a numeric calculation. To illustrate this feature, we present in Fig. 8 distributions of plaquette expectation values obtained from calculations with 8 ($V=0.5$), 14 ($V=0.6$), and 20 ($V=0.7$) independent random lattices. The expectation values on each lattice and for each value β are calculated over 100 iterations of the lattice using the Metropolis method.

(ii) The expectation values of the average action depend in a systematic way on the random lattice parameter V, which interpolates between the regular lattice and the pseudorandom lattice. In Fig. 9 we plot average plaquette expectation values for eight different β values as functions of V. For each value of V the expectation values are the result of calculations with several independently generated random lattices (4 lattices at the lower V values up to 20 lattices at $V=0.7$). The statistical errors are hardly bigger than the points plotted. The expectation values were measured in a calculation lowering β in steps of 0.1, with 100 iterations per point and lattice.

We find from Fig. 9 that the action has an extremum (maximum) for the regular lattice. This behavior was conjectured by Cohen[9] who could show it in $d=1$ dimensions. Pertermann[10] found such a behavior in $d=2$ dimensions.

Our experience with the first application of the pseudorandom Monte Carlo study shows that this method is promising; it needs hardly more computing resources than Monte Carlo calculations on a regular hypercubic lattice. The advantages of random-lattice theory seem to be also present in pseudorandom-lattice calculations, but with the pseudorandom lattice the linking step in the construction of the lattice is done very efficiently, at least as compared with the algorithm, which we were using previously to link a random lattice. Using the pseudorandom lattice, there is nothing which would prevent the proper calculation of quenched or annealed averages over many lattices when studying lattice gauge theories.

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