PHYSICAL REVIEW D VOLUME 34, NUMBER 6 15 SEPTEMBER 1986

Systematic search of Abelian gauge theories

A. Burnel*
Physique Théorique et Mathématique, Université de Liège, Sart Tilman, B-4000 Liège 1, Belgium
(Received 15 January 1986)

We make a systematic search of all the possible field theories in n-dimensional Minkowski space with an Abelian gauge invariance and bosonic gauge fields. They include the usual cases of Maxwell theory and spontaneously broken gauge theories but also less known cases such as a conserved current-current interaction and a massive Abelian vector boson coupled to a conserved current, as well as a description of any symmetry breaking as coming from a spontaneous breakdown. Special cases in one, two, or three space-time dimensions are also considered.

I. INTRODUCTION

Gauge invariance is one of the fundamental notions in the actual description of particle physics. Based on the fact that electromagnetism is not uniquely described by the potentials, it has been erected as a dynamical principle which allows one to find the dynamics from the symmetry properties of the particles. The successes of the standard SU(3)×SU(2)×U(1) model are impressive and, at the present time, an enlargement of this symmetry either as a grand-unification group or as a supersymmetry is still an open question.

Conceptually, a gauge theory is, however, a more general notion than this dynamical principle. One can define it as a theory described by a Lagrangian which is invariant under group transformations depending on the space-time points. Associated with this invariance, there is a set of pairs of constraints.[1] In a pair, one constraint comes from the definition of canonical momenta in terms of velocities (primary constraint) while the other results from field equations written in terms of phase-space variables (secondary constraint). These pairs of constraints are first class which means that each constraint corresponds to an unphysical degree of freedom. In other words, unphysical degrees of freedom always occur by pairs in a gauge theory.

If we restrict ourselves to the Abelian case, as we will do in the following except when the contrary is explicitly stated, the best known gauge theory is the Maxwell theory described by the Lagrangian

$$\mathscr{L}=-\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu}+A_\mu j^\mu+\mathscr{L}_m(\psi,\partial_\mu\psi)\,, \qquad (1)$$

where $\mathscr{L}_m(\psi,\partial_\mu\psi)$ is the free Lagrangian for matter fields. The Higgs mechanism[2] is generally considered as a particular case of it. However, in the last few years we have seen here and there in the literature[3-11] the appearance of different Abelian gauge theories and their non-Abelian extensions, so that we can ask the question as to what extent an arbitrary gauge theory can be built up and, if it cannot, what are the Abelian gauge theories.

To this end, we will make a distinction between matter fields and gauge fields and introduce, in addition to the usual vector gauge field, a scalar gauge field. In fact, such a scalar gauge field is essentially not a new notion, since it can be encountered in the Higgs mechanism, but it is combined with an invariant matter field in a form which looks like a matter field. Let us call this scalar gauge field K while the vector gauge field is V_μ. Two gauge field combinations are invariant: $(\partial_\mu K - V_\mu)$ and $(\partial_\mu V_\nu - \partial_\nu V_\mu)$. Together with the coupling of the vector gauge field with a conserved current, they are the basis of all possible gauge theories. A special case can however also occur if the role of the scalar gauge field is played by a vector field which will be coupled to a derivative. Special considerations related to the nature of the tensor dual to $(\partial_\mu V_\nu - \partial_\nu V_\mu)$ also hold in special space-time dimensions. In one dimension, no kinetic term for the vector gauge field can be constructed. In two dimensions $(\partial_\mu V_\nu - \partial_\nu V_\mu)$ is equivalent to a pseudoscalar gauge-invariant field and, in three dimensions, the dual $\epsilon_{\mu\nu\rho}(\partial^\nu A^\rho - \partial^\rho A^\nu)$ is a conserved quantity which can be coupled to V_μ in a way which is gauge invariant up to a three-divergence. It can also be coupled to $(\partial_\mu K - V_\mu)$ in a fully gauge-invariant way. All of these special cases are also discussed in our work which is organized as follows.

Some usual assumptions, such as Lorentz invariance, the absence of derivatives of order higher than 1 in the Lagrangian, Lagrangians at most quadratic in the velocities, and the absence of couplings which are nonrenormalizable in four dimensions, are used in Sec. II to restrict the possible Abelian gauge invariances in an arbitrary space-time dimension to five generic Lagrangians. In Sec. III we discuss two particular cases which are not discussed extensively elsewhere, while in Sec. IV, the special case of one dimension is considered. The special properties of two and three space-time dimensions are discussed, respectively, in Secs. V and VI while conclusions are drawn in Sec. VII.

II. ABELIAN GAUGE-INVARIANT THEORIES FOR ANY SPACE-TIME DIMENSION

Let us begin our discussion by defining what we mean by a gauge field in contrast with an ordinary or matter field. If a gauge invariance is given, a matter field is either invariant or transforms proportionally to the fields. Moreover, it has a real mass given by the term quadratic in the fields in the Lagrangian. An example of this is the complex scalar field described by the Lagrangian

$$\mathscr{L}=\tfrac{1}{2}\partial_\mu\phi^\dagger\partial^\mu\phi-\tfrac{1}{2}m^2\phi^\dagger\phi \tag{2}$$

submitted to the transformation

$$\phi\to e^{ie\omega}\phi\ . \tag{3}$$

In contrast, a gauge field always involves in its transformation a field-independent translation. The best known example is, of course, the vector field needed for the restoration of Lagrangian invariance under point-dependent gauge transformations. It always occurs through the covariant derivative

$$D_\mu\phi=(\partial_\mu-ieV_\mu)\phi \tag{4}$$

and is submitted to the transformation

$$V_\mu\to V_\mu+\partial_\mu\omega\ . \tag{5}$$

Another example, not generally recognized as such, is given by the scalar field occurring in the Higgs mechanism.[2] If we consider, for instance, the Lagrangian

$$\mathscr{L}=\tfrac{1}{2}\partial_\mu\phi^\dagger\partial^\mu\phi-\tfrac{1}{2}a\phi^\dagger\phi-\frac{\lambda}{4}(\phi^\dagger\phi)^2 \tag{6}$$

with $a<0$, the quadratic term in ϕ cannot be considered as a mass term, so that ϕ does not satisfy our definition of a matter field. The physical interpretation of the Lagrangian (6) is obtained with the new fields ρ,θ defined by

$$\phi=\rho e^{i\theta}\ . \tag{7}$$

The massless θ field transforms as

$$\theta\to\theta+e\omega\ , \tag{8}$$

where $e\omega$ is a field-independent translation. It is a gauge field which, by the redefinition (7), is disguised into a matter field except that it has no physical interpretation.

Let K_α be a general gauge field. By definition, it is submitted to the transformation

$$K_\alpha\to K_\alpha+\lambda_\alpha\ . \tag{9}$$

The index α is arbitrary. It may be an internal-symmetry index or may refer to Minkowski space. The field K may even be of spinor nature but this possibility will explicitly be dropped out in the Lagrangian building. Except for $\alpha=\mu$ and $\lambda_\mu=\partial_\mu\lambda$, in which case $\partial_\mu K_\nu-\partial_\nu K_\mu$ is invariant, the derivative of our gauge field transforms as

$$\partial_\mu K_\alpha\to\partial_\mu K_\alpha+\partial_\mu\lambda_\alpha \tag{10}$$

and requires, for the restoration of invariance, the introduction of a compensating field $V_{\mu\alpha}$ submitted to the transformation

$$V_{\mu\alpha}\to V_{\mu\alpha}+\partial_\mu\lambda_\alpha\ . \tag{11}$$

$V_{\mu\alpha}$ is again a gauge field and the combination $\partial_\mu K_\alpha-V_{\mu\alpha}$ is invariant. If a derivative of this new gauge field is introduced, it can appear in a gauge-invariant way through the combination

$$F_{\mu\nu\alpha}=\partial_\mu V_{\nu\alpha}-\partial_\nu V_{\mu\alpha}\ . \tag{12}$$

In this equation the index α does not play, in general, any particular role, so that we will, with one notable exception, drop it in the following. This is due to the assumed Abelian nature of our gauge invariances.

To summarize, we have essentially two gauge fields, a scalar and a vector, and they can occur with the invariant combinations $\partial_\mu K-V_\mu$ and $F_{\mu\nu}=\partial_\mu V_\nu-\partial_\nu V_\mu$. They can be introduced inside the Lagrangian through the terms

$$\tfrac{1}{2}(\partial_\mu K-V_\mu)^2 \text{ and } -\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu}$$

which can be taken either separately or together. To these terms, we also add a coupling with matter fields. By the nature of matter fields, these couplings generally occur in the form $V_\mu j^\mu$ where the current is conserved. We have

$$\delta(V_\mu j^\mu)=(\delta V_\mu)j^\mu=\partial_\mu\lambda j^\mu=\partial^\mu(\lambda j_\mu)\ . \tag{13}$$

If we add a matter-field Lagrangian, its variation is compensated by $\delta(V_\mu j^\mu)$. We can, however, also consider that j_μ is an external current and use a gauge invariance up to an n divergence where n is the dimension of space-time.

It is interesting to note here that, if j_μ can be written as $\partial_\mu j$, we also have

$$\delta(A_\mu\partial^\mu j)=(\delta A_\mu)\partial^\mu j=\partial^\mu(\delta A_\mu j) \text{ if } \partial^\mu\delta A_\mu=0\ .$$

Therefore, a theory which is gauge invariant up to an n divergence can also be obtained in this way.

Except for trivial compensations between two gauge fields like K_1-K_2 with $K_i\to K_i+\lambda$, in which case the combination K_1-K_2 can always be replaced by a gauge-invariant field, it is clear that $(\partial_\mu K-V_\mu)$, $F_{\mu\nu}=\partial_\mu V_\nu-\partial_\nu V_\mu$, $A_\mu j^\mu$ with $\partial_\mu j^\mu=0$ and $K_\mu\partial^\mu j$ with $\partial^\mu\lambda_\mu=0$ are the only possible nontrivial Abelian gauge-invariant combinations. The number of Lagrangians involving these combinations and matter fields in a gauge-invariant way is rather broad. However, if we make some additional usual assumptions, such as Lorentz invariance, the absence of derivatives of order higher than 1 in the Lagrangian, Lagrangians at most quadratic in the velocities, and renormalizability of the couplings in four space-time dimensions, the number of gauge-invariant Lagrangians can be restricted to five generic cases which are given in Eqs. (14)–(18). Except for the case of the Higgs mechanism, we will always consider that K has the usual dimension of a scalar field and therefore set $V_\mu=mA_\mu$. These Lagrangians are

$$\mathscr{L}_1=-\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu}+A_\mu j^\mu\ , \tag{14}$$

$$\mathscr{L}_2=-\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu}+A_\mu j^\mu+\tfrac{1}{2}\rho^2(\partial_\mu K-A_\mu)^2+\tfrac{1}{2}\partial_\mu\rho\partial^\mu\rho-V(\rho)\ , \tag{15}$$

$$\mathscr{L}_3=-\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu}+\tfrac{1}{2}(mA_\mu-\partial_\mu K)^2+A_\mu j^\mu\ , \tag{16}$$

$$\mathscr{L}_4=\tfrac{1}{2}(mA_\mu-\partial_\mu K)^2+A_\mu j^\mu\ , \tag{17}$$

$$\mathscr{L}_5=K_\mu\partial^\mu\phi+\tfrac{1}{2}\partial_\mu\phi\partial^\mu\phi-V(\phi)\ . \tag{18}$$

$\mathscr{L}_1$ is the usual Abelian gauge theory with $n-2$ true degrees of freedom associated with the gauge fields. $\mathscr{L}_2$ is the gauge theory with the Higgs mechanism.[2] $n-1$ degrees of freedom are associated with the gauge fields while one additional degree of freedom, the Higgs boson, is associated with ρ. $\mathscr{L}_3$ is less known but corresponds to a gauge-invariant formulation of a massive vector boson coupled to a conserved current. This model, with $n-1$ degrees of freedom associated with the gauge fields is studied in detail elsewhere.[10] $\mathscr{L}_4$ will be discussed in the following section while $\mathscr{L}_5$ has also been studied elsewhere.[11] It allows one to give a spontaneous origin to any symmetry-breaking term in a Lagrangian. In principle, terms like $\frac{1}{2}(mV_{\mu\nu}-\partial_\mu K_\nu)^2$ and $-\frac{1}{4}F_{\mu\nu\lambda}F^{\mu\nu\lambda}$ could also be added to the Lagrangian $\mathscr{L}_5$. However, if a kinetic term for the K_ν field is included, a natural gauge is given by $K_\nu=0$ which makes the theory useless, while, if $-\frac{1}{4}F_{\mu\nu\lambda}F^{\mu\nu\lambda}$ is added alone, the theory looks like $\mathscr{L}_1+\mathscr{L}_5$ but no particular meaning can be given to it.

If we remove the requirement of renormalizability in four dimensions, terms like $\kappa F_{\mu\nu}j^{\mu\nu}$, $(1/m)\partial_\mu K j^\mu$ and many others like $P(\phi)F_{\mu\nu}F^{\mu\nu}$ could be added. $P(\phi)$ is any function of the matter field ϕ. It may be interesting to introduce the term $(1/m)\partial_\mu K j^\mu$ in the Lagrangian

$$\mathscr{L}_6=-\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu}+\tfrac{1}{2}(mA_\mu-\partial_\mu K)^2+\frac{1}{m}(mA_\mu-\partial_\mu K)j^\mu \tag{19}$$

to show that gauge invariance is not necessarily related to the presence of a conserved current. The Lagrangian $\mathscr{L}_6$ is indeed invariant, even if $\partial^\mu j_\mu \neq 0$. This example will be considered below. If the current j_μ is conserved, $\partial^\mu K j_\mu=\partial^\mu(Kj_\mu)$ is an n-divergence which does not affect the field equations. The theory is classically the same as if this term was removed. This can also be checked at the quantum level. When a gauge-invariant kinetic term is introduced for the K field, the secondary constraint, which is given by the Euler-Lagrange equation with respect to variation of A_0, always contains the term $\pi=\partial_0 K-A_0+\cdots$, so that K can be considered as an unphysical degree of freedom and $K=0$ is a suitable gauge. The discussion is more subtle when such a kinetic term is not included but the conclusion is the same.

If the number of gauge fields is multiplied by the introduction of an index α, the gauge field terms are generalized by $-\frac{1}{4}F^\alpha_{\mu\nu}M^\beta_\alpha F^{\mu\nu}_\beta$ and

$$\tfrac{1}{2}(\partial_\mu K-V_\mu)^\alpha N^\beta_\alpha(\partial^\mu K-V^\mu)_\beta ,$$

whatever this index is. The fields are, however, of bosonic nature. Therefore, except for special cases in special dimensions, all the fundamental Abelian gauge theories obeying our natural assumptions are given by $\mathscr{L}_1$–$\mathscr{L}_5$.

Special cases in a particular dimension are related to the properties of the tensor dual to $F_{\mu\nu}$. For $n=4$, for instance, the dual of $F_{\mu\nu}$ is a rank-two tensor ${}^*F_{\mu\nu}$ which can be coupled to $F_{\mu\nu}$ to give the CP-violating gauge-invariant term ${}^*F^{\mu\nu}F_{\mu\nu}$ in a Lagrangian. Particularities of the theory involving such terms are sufficiently well known not to be discussed further here. For $n>4$, no special property occurs. For dimensions lower than four, the discussion is made in the following sections.

III. TWO PARTICULAR THEORIES IN ARBITRARY DIMENSIONS

The cases $\mathscr{L}_1$, $\mathscr{L}_2$, $\mathscr{L}_3$, and $\mathscr{L}_5$, are either well known or discussed in detail elsewhere, so that, in this section, we will concentrate our attention to the particular case $\mathscr{L}_4$ and also to the nonrenormalizable Lagrangian $\mathscr{L}_6$.

$\mathscr{L}_4$ is characterized by the absence of a kinetic term for the vector gauge field, so that four primary constraints

$$\pi^\mu=\frac{\partial\mathscr{L}}{\partial(\partial_0 A_\mu)}=0 \tag{20}$$

are present, while

$$\pi=\frac{\partial\mathscr{L}}{\partial(\partial_0 K)}=\partial_0 K-mA_0 . \tag{21}$$

The total Hamiltonian is

$$\mathscr{H}_T=\tfrac{1}{2}\pi^2+m\pi A_0+\tfrac{1}{2}(mA_k-\partial_k K)^2 -A_0 j_0+A_k j_k+\Lambda^\mu\pi^\mu . \tag{22}$$

Associated with the primary constraints, we have the following chains of equations:

$$\pi^k=0\Longrightarrow mA_k-\partial_k K+j_k=0\Longrightarrow m\Lambda^k-\partial_k\pi-\partial_0 j_k=0 , \tag{23}$$

$$\pi^0=0\Longrightarrow m\pi-j_0=0\Longrightarrow m\partial_k(mA_k-\partial_k K)+\partial_0 j_0=0 , \tag{24}$$

where the last equation in (24) is clearly a consequence of the second equation in (23) and of the current conservation.

We clearly have a gauge theory with two first-class constraints and $2(n-1)$ second-class constraints. Because of the nature of the secondary first-class constraint [second equation in (24)], the only possible gauge condition is $K=0$. By time differentiation, it implies

$$K=0\Longrightarrow\pi+mA_0=0\Longrightarrow\partial_k(mA_k-\partial_k K)+m\Lambda^0=0 . \tag{25}$$

With the gauge conditions (25), all the constraints in (23) and (24) become second class and can be strongly realized provided that Dirac brackets instead of Poisson brackets are used. The effective Hamiltonian is

$$\mathscr{H}_{4,\text{eff}}=\frac{1}{2m^2}j_\mu j^\mu . \tag{26}$$

The physical content of $\mathscr{L}_4$ is thus a conserved current-current interaction. It is not renormalizable and there is no hope to find a renormalizable gauge, even by the introduction of additional variables because the term $-(i/m^2)g_{\mu\nu}$ of the gauge vector-boson propagator is impossible to compensate. For instance, in the Stueckelberg gauge,[12]

$$\mathscr{L}'_4=\tfrac{1}{2}(mA_\mu-\partial_\mu K)^2+A_\mu j^\mu-\tfrac{1}{2}\partial_\mu\phi\partial^\mu\phi+\tfrac{1}{2}am^2\phi^2 , \tag{27}$$

with K identified with $\phi=-S/m$, we have

$$\mathcal{L}'_{4,S}=\tfrac{1}{2}m^2A_\mu A^\mu-S\partial^\mu A_\mu+A_\mu j^\mu+\tfrac{1}{2}aS^2 \tag{28}$$

or, equivalently,

$$\mathcal{L}''_{4,S}=\tfrac{1}{2}m^2A_\mu A^\mu-\frac{1}{2a}(\partial^\mu A_\mu)^2+A_\mu j^\mu . \tag{29}$$

The propagator is

$$D_{\mu\nu}(k)=-\frac{i}{m^2}\left[g_{\mu\nu}-\frac{k_\mu k_\nu}{k^2-am^2+i\epsilon}\right] . \tag{30}$$

It still contains $1/m^2$ terms.

The case $\mathcal{L}_6$, which could also be considered without the kinetic term for the vector gauge field, possesses all the aspects of a gauge theory. It does not even require current conservation since all terms are gauge invariant. It does not come from a gauge invariance of a matter-field Lagrangian, even if the current is conserved. In this case, however, the coupling $\partial_\mu K j^\mu$ can be removed since it is an n-divergence, so that the theory is not new. Since the constraints are

$$\pi^0=0 \Longrightarrow \partial^k\pi^k+m\pi=0 , \tag{31}$$

$K=0$ is a convenient gauge and, in this gauge, the theory describes the coupling of a vector field with a nonconserved current. The theory is, however, the same in any gauge since it always involves the combination $\Lambda_\mu = mA_\mu-\partial_\mu K$ even in the interaction which, in a perturbation theory, destroys any modification of the Λ_μ propagator by the compensating $\partial_\mu K$ term. Therefore, this gauge invariance has no real effect and will not be considered further. It occurs because a gauge-invariant quantity Λ_μ can be decomposed into two gauge-dependent parts. This is quite different from usual gauge invariances where the coupling involves the noninvariant A_μ field.

IV. MODELS IN ONE DIMESION

In a geometrical language, a scalar field corresponds to a 0-form, a vector field to a 1-form, and the tensor $F_{\mu\nu}$ to a 2-form, the curvature form. If the manifold dimension is n, only p-forms with $0\le p\le n$ exist. Therefore, in one dimension, no 2-form exists and, consequently, there is no kinetic term for the vector gauge field, which, here, is also a scalar. The only gauge-invariant combination is $\dot{K}-A$ where K and A are submitted to the gauge transformations

$$K\to K+\lambda, \quad A\to A+\dot{\lambda} . \tag{32}$$

There is also no coupling with a conserved current since $\dot{J}=0$ implies that J reduces to a constant in contrast with the case of higher dimensions. $\mathcal{L}_5$ is also not a gauge-invariant Lagrangian but the physical contents are the same as in arbitrary dimensions. The only difference comes from the second-class nature of the constraints. Therefore, the only one-dimensional gauge-invariant theories are

$$L=L_r+\tfrac{1}{2}(\dot{K}-A)^\alpha N_{\alpha\beta}(\dot{K}-A)^\beta , \tag{33}$$

where L_r is some Lagrangian describing real degrees of freedom (thus excluding K and A variables). This means that, in one dimension, the gauging of an Abelian global invariance amounts to the elimination of the variables submitted to the local variation. The same is true for a non-Abelian invariance. The form of the Lagrangian (33) also means that any one-dimensional theory can be embedded into a gauge-invariant formulation. This is also true in higher dimensions but, except in the above cases, there is no coupling between matter and gauge fields.

Gauge invariances in one-dimensional space-time are interesting from the pedagogical point of view. For instance, they provide clear and simple examples for the application of the Dirac method of quantizing constrained systems. Typical examples are the motion on a straight line[5,13] or on a half line[3,4,6] described in a gauge-invariant way. The structure of the Lagrangian (33) shows that examples can be invented as one pleases.

V. SPECIAL MODELS IN TWO DIMENSIONS

Of course, our five gauge-invariant Lagrangians can be built up in two dimensions. The particular feature of this space is that a 2-form is associated to a pseudoscalar field. In other words, the dual of F is a pseudoscalar

$$\phi=\tfrac{1}{2}\epsilon_{\mu\nu}F^{\mu\nu} . \tag{34}$$

This information can be brought inside a Lagrangian by writing it as

$$\mathcal{L}=\tfrac{1}{2}\phi\epsilon_{\mu\nu}F^{\mu\nu}-\tfrac{1}{2}\phi^2+\tfrac{1}{2}(mA_\mu-\partial_\mu K)^2+A_\mu j^\mu . \tag{35}$$

The Euler-Lagrange equation corresponding to a variation of ϕ is indeed (34). An equivalent expression of $\mathcal{L}$ is

$$\mathcal{L}=\phi\epsilon_{\mu\nu}\partial^\mu A^\nu-\tfrac{1}{2}\phi^2+\tfrac{1}{2}(mA_\mu-\partial_\mu K)^2+A_\mu j^\mu . \tag{36}$$

In the $m=0$ case, Hagen[9] has recently shown how (36) is related to the first-order formulation of a scalar field theory and how it is equivalent to the Schwinger[14] model which is nothing other than our Lagrangian $\mathcal{L}_1$. We will therefore restrict our considerations to prove that (36) with $m\neq 0$ describes a two-dimensional massive vector boson, although it looks like a scalar field theory. We could easily reproduce Hagen's arguments[9] in the massive case but we prefer to prove it in a Hamiltonian way. We neglect the unessential coupling with the matter current.

When defining canonical momenta, we have ($\epsilon_{01}=-1$)

$$\pi_\phi=\frac{\partial\mathcal{L}}{\partial(\partial_0\phi)}=0, \quad \pi^1=\frac{\partial\mathcal{L}}{\partial(\partial_0 A_1)}=\phi , \qquad \pi^0=0, \quad \pi_K=\partial_0 K-mA_0 , \tag{37}$$

i.e., three primary constraints. The total Hamiltonian is

$$\mathcal{H}_T=\tfrac{1}{2}\pi_K{}^2+mA_0\pi_K-\phi\partial^1A_0+\tfrac{1}{2}(mA_1-\partial_1K)^2+\tfrac{1}{2}(\pi^1)^2 +\Lambda_\phi\pi_\phi+\Lambda^0\pi^0+\Lambda^1(\pi^1-\phi) . \tag{38}$$

The search of second-class constraints leads to the following chains of equations:

$$\pi^1-\phi=0 \Longrightarrow -m(mA_1-\partial_1K)-\Lambda_\phi=0 , \tag{39}$$

$$\pi_\phi=0 \Longrightarrow \partial^1 A_0+\Lambda^1=0\,, \tag{40}$$

$$\pi^0=0 \Longrightarrow m\pi_K+\partial^1\phi=0 \Longrightarrow m\partial_1(mA_1-\partial_1 K)+\partial_1\Lambda_\phi=0\,. \tag{41}$$

Therefore, we have a gauge theory with two first-class constraints

$$\pi^0=0,\quad m\pi_K-\partial^1\pi^1=0 \tag{42}$$

and two second-class constraints

$$\pi_\phi=\pi^1-\phi=0\,. \tag{43}$$

According to the second equation in (42), $K=0$ is a convenient gauge. It implies the chain of equations

$$K=0 \Longrightarrow \pi_K+mA_0=0 \Longrightarrow m\partial_1(A_1-\partial_1 K)+m\Lambda_0=0\,. \tag{44}$$

The strong realization of all the constraints and the gauge conditions leads to the effective Hamiltonian

$$\mathscr{H}_{\text{eff}}=\tfrac{1}{2}(\pi^1)^2+\tfrac{1}{2}m^2\phi_1{}^2+\frac{1}{2}\frac{(\partial^1\pi^1)^2}{m^2}\,, \tag{45}$$

which coincides with the effective Hamiltonian of a two-dimensional massive vector boson. There is thus no essential difference between the Lagrangian (36) and $\mathscr{L}_3$. The same is clearly true if we set $m=0$ in (36) and compare it with $\mathscr{L}_1$. The addition of $-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}$ to (35) or (36) is also possible but amounts to a simple rescaling. The constraint $\pi^1-\phi=0$ is indeed replaced by $2\pi^1+\phi=0$.

VI. SPECIAL MODELS IN THREE DIMENSIONS

The particularity of three-dimension space-time is that the dual of the curvature tensor $F_{\mu\nu}$ is a conserved pseudovector. The combination

$$\phi_\mu=\tfrac{1}{2}\epsilon_{\mu\nu\rho}(\partial^\nu A^\rho-\partial^\rho A^\nu) \tag{46}$$

indeed, obviously, satisfies

$$\partial^\mu\phi_\mu=0\,. \tag{47}$$

This means that, like a conserved current, ϕ_μ can be coupled to A_μ to give a term which is gauge invariant up to a three divergence. This term also reads

$$\tfrac{1}{2}A_\mu\epsilon^{\mu\nu\rho}\partial_\nu A_\rho \tag{48}$$

and can be added to any of our Lagrangians $\mathscr{L}_1$, $\mathscr{L}_2$, $\mathscr{L}_3$, $\mathscr{L}_4$, or considered alone with the coupling to the matter current. Neglecting the Higgs mechanism which does not give rise to any essentially different result, the most general Lagrangian including this term can be written as

$$\mathscr{L}=-\frac{1}{4m_1}F_{\mu\nu}F^{\mu\nu}+\tfrac{1}{2}A_\mu\epsilon^{\mu\nu\rho}\partial_\nu A_\rho+\frac{1}{2m_2}(\partial_\mu K-m_3A_\mu)^2+A_\mu j^\mu\,. \tag{49}$$

Three different parameters with the dimension of a mass have been introduced. We can consider four distinct cases: (a) m_1 and m_2 are arbitrary—two real degrees of freedom are associated with the gauge fields; (b) $m_1\to\infty$ and m_2 is arbitrary—one real degree of freedom is associated with the gauge fields; (c) m_1 is arbitrary and $m_2\to\infty$—one real degree of freedom is also associated with the gauge fields but the theory is different from case (b); (d) $m_1=m_2\to\infty$—no real degree of freedom is associated with the gauge fields. Cases (c) and (d) can also be obtained, up to a free field K, by taking $m_3=0$ while keeping m_2 finite. They were considered, respectively, by Deser, Jackiw, and Templeton[7] and by Hagen.[8] A non-Abelian generalization,[7,8] can also be given.

If the term (48) has a clear geometrical interpretation,[7] it does not seem to have a physical interest. Of course, it allows the introduction of a mass in a gauge-invariant way but the mechanism is restricted to three-dimensional space and it violates parity. The introduction of a gauge-invariant mass term is more natural in the Lagrangian $\mathscr{L}_3$. In the case of the Hagen model,[9] some formally interesting results[15] can be obtained concerning the Dirac method of quantizing constrained systems. We do not see any other interest, either formal or physical, to consider further here these special examples.

More interesting could be the possibility of making the term $\frac{1}{2}\epsilon^{\mu\nu\rho}\partial_\nu A_\rho A_\mu$ completely gauge invariant by including the contribution of the scalar gauge field. The addition of the three-divergence $-\frac{1}{2}\partial^\rho[\epsilon_{\mu\nu\rho}(\partial^\mu A^\nu)K]$ makes indeed the gauge invariance complete in $\frac{1}{2}\epsilon_{\mu\nu\rho}\partial^\mu A^\nu(mA^\rho-\partial^\rho K)$. Classically, such a divergence does not affect field equations but, in a quantum theory, commutation relations can be modified. In the cases when a kinetic term for the K field is included, $K=0$ is a suitable gauge and the added term is not relevant, even in the quantum case. In the absence of such a kinetic term, there may be modifications of the physical theory.

Let us take, for instance, the Lagrangian

$$\mathscr{L}=-\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu}+\tfrac{1}{2}\epsilon^{\mu\nu\rho}\partial_\mu A_\nu(mA_\rho-\partial_\rho K)\,. \tag{50}$$

In addition to the usual gauge constraints

$$\pi^0=0 \Longrightarrow \partial^k\pi^k-\frac{m}{2}\epsilon^{kl0}\partial_k A_l=0\,, \tag{51}$$

there are now two second-class constraints, a primary and a secondary one:

$$\pi+\tfrac{1}{2}\epsilon^{kl0}\partial_k A_l=0 \Longrightarrow \epsilon^{0kl}\partial_l[\pi^k-\tfrac{1}{2}\epsilon^{0kn}(mA_n-\partial_n K)]\,. \tag{52}$$

They can be realized as strong equations provided that Dirac brackets instead of Poisson brackets are used. If this is done, the fields π and K can be replaced with the help of Eqs. (52). This gives rise to a Hamiltonian theory which manifests a gauge invariance and which is nonlocal before the gauge is fixed.

Although the number of real degrees of freedom in the theories described by (50) and by the same Lagrangian with K removed is the same, although the classical theories are the same, the quantum theories could be different. The comparison between both approaches is formally interesting but is outside the scope of this paper. It is, however, worth noting that the non-Abelian extensions are completely different. For $K=0$, the completely gauge-invariant term[10] $\frac{1}{4}\epsilon^{\mu\nu\rho}F^{\alpha}_{\mu\nu}(mA^{\alpha}_{\rho}-N_{\alpha\beta}\partial_{\rho}K_{\beta})$ does indeed not reduce to the Chern-Simons density

$$\frac{m}{2}\epsilon^{\mu\nu\rho}\partial_{\mu}A^{\alpha}_{\nu}A^{\alpha}_{\rho}+g\frac{m}{6}\epsilon^{\mu\nu\rho}f_{\alpha\beta\gamma}A^{\alpha}_{\nu}A^{\beta}_{\mu}A^{\gamma}_{\rho}$$

which is invariant only up to a three-divergence.

VII. CONCLUSIONS

By considering a generally overlooked gauge field of scalar nature in addition to the usual vector gauge field, we increased the possibility of building Abelian gauge-invariant theories. Four physically interesting theories can be constructed to which we add a special case obtained with a vector gauge field submitted to index-dependent translations and coupled to a derivative. Also the special properties of one, two, and three space-time dimensions have been discussed. Most of the interesting new theories, including their non-Abelian extensions, have been discussed elsewhere, either by us or by other authors. We refer to these papers[3–11,13,15] for details on the particular subject they treat. This paper presents an overall view of the problem of building generalized Abelian gauge theories. The presentation is made on the n-dimensional Minkowski manifold with relativistic invariance. It can easily be generalized.

---

*Address all correspondence to Institut de Physique au Sart Tilman, Bâtiment B.5, B-4000 Liège 1, Belgium.

[1]The reader is assumed to be familiar with the Dirac theory of constrained systems. P. A. M. Dirac, *Lectures on Quantum Mechanics* (Yeshiva University, New York, 1964).

[2]P. W. Higgs, Phys. Lett. **12**, 132 (1964); F. Englert and R. Brout, Phys. Rev. Lett. **13**, 321 (1964); G. Guralnik, C. Hagen, and T. Kibble, *ibid.* **13**, 585 (1964); T. Kibble, Phys. Rev. **155**, 1554 (1967).

[3]N. M. Christ and T. D. Lee, Phys. Rev. D **22**, 939 (1980).

[4]A. Burnel, Eur. J. Phys. **2**, 178 (1981).

[5]A. Burnel, Phys. Rev. D **26**, 442 (1982).

[6]M. E. V. Costa and H. O. Girotti, Phys. Rev. D **24**, 3323 (1981).

[7]S. Deser, R. Jackiw, and S. Templeton, Ann. Phys. (N.Y.) **140**, 372 (1982).

[8]C. R. Hagen, Ann. Phys. (N.Y.) **157**, 342 (1984); Phys. Rev. D **31**, 331 (1985).

[9]C. R. Hagen, Phys. Rev. D **32**, 2229 (1985).

[10]A. Burnel, Phys. Rev. D **33**, 2985 (1986).

[11]A. Burnel, Liège University report, 1985 (unpublished).

[12]See Ref. 10 for details on the Stueckelberg gauge.

[13]A. Burnel, Phys. Rev. D **32**, 450 (1985).

[14]J. Schwinger, Phys. Rev. **128**, 2425 (1962).

[15]A. Burnel and M. Van Der Rest-Jaspers, Math. Phys. **26**, 3155 (1985).