PHYSICAL REVIEW D VOLUME 34, NUMBER 6 15 SEPTEMBER 1986

Time-dependent perturbation theory for quaternionic quantum mechanics, with application to *CP* nonconservation in *K*-meson decays

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(Received 23 June 1986)

We develop time-dependent perturbation theory for quaternionic quantum mechanics. We set up the analog of the Weisskopf-Wigner formalism for the decay of a degenerate set of states, identify the mass and decay matrices, and verify unitarity. The results support the conjecture that the asymptotic-state structure of quaternionic quantum field theory resides within a complex subspace of quaternionic Hilbert space. With a natural ansatz for the *CP* behavior of the quaternionic Hamiltonian, the formalism is shown to imply the existence of an effective superweak *CP* nonconservation in *K* decays.

I. INTRODUCTION

Time-dependent perturbation theory[1] plays a central role in standard, complex quantum mechanics. Through the "golden rule"[2] it describes the transition rates for decay processes, and through the Weisskopf-Wigner formalism[3] it describes the line shape and other decay dynamics of metastable states. In a recent series of papers,[4] we have formulated a quaternionic extension of standard quantum mechanics, in which the wave function Ψ is quaternion valued:

$$\Psi=\Psi_0+i\Psi_1+j\Psi_2+k\Psi_3 , \tag{1}$$

with $\Psi_{0,1,2,3}$ real and i,j,k the quaternion units[5] satisfying

$$i^2=j^2=k^2=-1 ,\quad ij=-ji=k . \tag{2}$$

The inner product and norm in the quaternionic theory are

$$(\Psi,\Phi)=\int \overline{\Psi}\Phi ,\quad ||\Psi||=(\Psi,\Psi)=\int \overline{\Psi}\Psi , \tag{3}$$

with $\overline{\Psi}=\Psi_0-i\Psi_1-j\Psi_2-k\Psi_3$ the conjugate of Ψ, and the inner-product-preserving (or unitary) dynamics is

$$\frac{\partial\Psi}{\partial t}=-\tilde{H}\Psi , \tag{4}$$

with $\tilde{H}$ a quaternion—anti-self-adjoint Hamiltonian. Our aim in this paper is to establish an analog of the standard time-dependent perturbation-theory analysis for the quaternionic quantum mechanics formulated in Eqs. (1)—(4).

To begin, let us assume that $\tilde{H}$ is the sum of an unperturbed Hamiltonian $\tilde{H}_0$ and a time-independent perturbation $\tilde{V}$, both of which are quaternion anti-self-adjoint. To simplify the formulation of the problem, we invoke the spectral theorem[6] for quaternion—anti-self-adjoint operators, which when applied to $\tilde{H}_0$ tells us that

$$\tilde{H}_0=I_0H_0 , \tag{5}$$

with $I_0{}^2=-1$, $[I_0,H_0]=0$, and H_0 a quaternion—self-adjoint operator with positive-semidefinite eigenvalues. [In other words, H_0 is the formal positive square root: $H_0=(-\tilde{H}_0{}^2)^{1/2}$.] The spectral representations for I_0 and H_0 can be written[6] in the form

$$I_0=\sum_n |n\rangle i\langle n| ,\quad H_0=\sum_n |n\rangle E_n\langle n| , \tag{6}$$

with $|n\rangle$ a complete set of eigenstates of $\tilde{H}_0$. There is no loss of generality in assuming that the kets $|n\rangle$ are real and so commute with the quaternion unit i, which implies that

$$I_0=i\sum_n |n\rangle\langle n| =i\,1=i , \tag{7}$$

and permits us to neglect the formal distinction between the operator I_0 and the quaternion unit i. (Of course, having made this assumption, the eigenkets of any operator O which does not commute with $\tilde{H}_0$ will in general be quaternion valued, since the transformation function $\langle o|n\rangle$ is quaternion valued.) Thus as the starting point for our perturbation analysis we write

$$\tilde{H}=iH_0+\tilde{V} ,\quad H_0=\sum_n |n\rangle E_n\langle n| ,\ E_n\geq 0 , \tag{8}$$

with $\tilde{V}$ a time-independent perturbation.

Let us now introduce some useful notation for matrix elements of $\tilde{V}$. Since $\langle n|\tilde{V}|l\rangle$ is a quaternion matrix, we can write

$$\langle n|\tilde{V}|l\rangle=\tilde{V}_{0nl}+i\tilde{V}_{1nl}+j\tilde{V}_{2nl}+k\tilde{V}_{3nl} , \tag{9a}$$

with $\tilde{V}_{anl}$, $a=0,1,2,3$ all real. The condition that $\tilde{V}$ be anti-self-adjoint tells us that

$$-\overline{\langle n \mid \tilde{V} \mid l \rangle} = -(\tilde{V}_{0nl} - i\tilde{V}_{1nl} - j\tilde{V}_{2nl} - k\tilde{V}_{3nl})$$
$$= \langle l \mid \tilde{V} \mid n \rangle$$
$$= \tilde{V}_{0ln} + i\tilde{V}_{1ln} + j\tilde{V}_{2ln} + k\tilde{V}_{3ln} , \tag{9b}$$

which implies that

$$\tilde{V}_{0nl} = -\tilde{V}_{0ln}, \text{ skew symmetric },$$
$$\tilde{V}_{1nl} = \tilde{V}_{1ln}, \quad \tilde{V}_{2nl} = \tilde{V}_{2ln}, \quad \tilde{V}_{3nl} = \tilde{V}_{3ln}, \text{ symmetric }. \tag{9c}$$

It will also be convenient to introduce so-called symplectic components $\tilde{V}_\alpha, \tilde{V}_\beta$ by writing

$$\langle n \mid \tilde{V} \mid l \rangle = \tilde{V}_{\alpha nl} + j\tilde{V}_{\beta nl} ,$$
$$\tilde{V}_{\alpha nl} = \tilde{V}_{0nl} + i\tilde{V}_{1nl} , \tag{10}$$
$$\tilde{V}_{\beta nl} = \tilde{V}_{2nl} - i\tilde{V}_{3nl} ,$$

so that $\tilde{V}_{\alpha nl}$ and $\tilde{V}_{\beta nl}$ are $\mathbb{C}(1,i)$ complex and, from Eq. (9c), satisfy the conditions

$$\tilde{V}^*_{\alpha nl} = -\tilde{V}_{\alpha ln}, \quad \tilde{V}_{\beta nl} = \tilde{V}_{\beta ln} , \tag{11}$$

with an asterisk denoting the $\mathbb{C}(1,i)$ conjugation $i \to -i$. Thus the starting point for our perturbation analysis takes the final form

$$\tilde{H} = iH_0 + \tilde{V}_\alpha + j\tilde{V}_\beta , \tag{12}$$

with $\tilde{V}_\alpha$ a $\mathbb{C}(1,i)$—anti-Hermitian matrix and $\tilde{V}_\beta$ a $\mathbb{C}(1,i)$—symmetric matrix in the basis $\mid n \rangle$ of real eigenkets of H_0 (which have eigenvalues $E_n \geq 0$).

II. TIME-DEPENDENT PERTURBATION THEORY

Let us now analyze the following time-dependent perturbation theory problem: Let $\mid s_a \rangle$ be a degenerate set of eigenkets of the unperturbed Hamiltonian H_0. Then to second order in the perturbation $\tilde{V}$, we wish to find at time t the state $\Psi(t)$ which obeys the quaternionic dynamics of Eq. (4), and which at $t=0$ reduces to a general quaternionic linear combination of the states $\mid s_a \rangle$:

$$\frac{\partial \Psi(t)}{\partial t} = -\tilde{H}\Psi(t) ,$$
$$\Psi(0) = \sum_a \mid s_a \rangle K_a . \tag{13}$$

The solution to this problem is greatly facilitated by the use of complex variable methods in the subspace $\mathbb{C}(1,i)$, and so it is natural to introduce a symplectic decomposition for Ψ by writing

$$\Psi = \Psi_\alpha + j\Psi_\beta , \tag{14}$$

with $\Psi_\alpha, \Psi_\beta \in \mathbb{C}(1,i)$. By analogy with standard time-dependent perturbation theory,[1] we expand Ψ on a basis of unperturbed zeroth-order eigenkets $\mid l \rangle \exp(-iE_l t)$ of the time-dependent Schrödinger equation, with time-dependent quaternionic coefficients $c_l(t)$:

$$\Psi = \sum_l \mid l \rangle e^{-iE_l t} c_l(t) . \tag{15}$$

Making a symplectic decomposition of c_l and K_a,

$$c_l(t) = c_{l\alpha}(t) + jc_{l\beta}(t) ,$$
$$K_a = K_{a\alpha} + jK_{a\beta} , \tag{16a}$$

we then find

$$\Psi_\alpha = \sum_l \mid l \rangle e^{-iE_l t} c_{l\alpha}(t) ,$$
$$\Psi_\beta = \sum_l \mid l \rangle e^{iE_l t} c_{l\beta}(t) , \tag{16b}$$
$$c_{l\alpha}(0) = \sum_a \delta_{l,s_a} K_{a\alpha}, \quad c_{l\beta}(0) = \sum_a \delta_{l,s_a} K_{a\beta} ,$$

where we have used the fact that

$$e^{-iE_l t} j = je^{iE_l t} \tag{17}$$

in moving j to the left in the β piece of the equation. Combining Eqs. (12)—(16), and making use of the quaternion algebra of Eq. (2), we get the following complex $\mathbb{C}(1,i)$ equations for the coefficients $c_{n\alpha,\beta}(t)$:

$$\frac{d}{dt} c_{n\alpha} = -\sum_l (\tilde{V}_{\alpha nl} e^{i(E_n - E_l)t} c_{l\alpha} - \tilde{V}^*_{\beta nl} e^{i(E_n + E_l)t} c_{l\beta}) + \delta(t) \sum_a \delta_{n,s_a} K_{a\alpha} ,$$
$$\frac{d}{dt} c_{n\beta} = -\sum_l (\tilde{V}_{\beta nl} e^{-i(E_n + E_l)t} c_{l\alpha} + \tilde{V}^*_{\alpha nl} e^{i(E_l - E_n)t} c_{l\beta}) + \delta(t) \sum_a \delta_{n,s_a} K_{a\beta} , \tag{18}$$
$$c_{n\alpha}(t) = c_{n\beta}(t) = 0, \quad t < 0 .$$

In writing Eq. (18), we have followed the standard procedure[7] of converting an initial-value problem on the domain $0 \leq t < \infty$ into a problem defined on $-\infty < t < \infty$, by reinterpreting the $t=0$ initial conditions as step functions at $t=0$, with a boundary condition that all c_n's vanish for $t<0$.

To solve Eq. (18) we introduce Fourier transforms with respect to t as follows:

$$c_{n\alpha}(t) = -\frac{1}{2\pi i} \int_{-\infty}^{\infty} dE\, e^{i(E_n - E)t} c_{n\alpha}(E) ,$$
$$c_{n\beta}(t) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} dE\, e^{-i(E_n + E)t} c_{n\beta}(E) , \tag{19a}$$

which can be combined into the single quaternionic formula

$$c_n(t) = c_{n\alpha}(t) + jc_{n\beta}(t)$$
$$= -\frac{1}{2\pi i} \int_{-\infty}^{\infty} dE\, e^{iE_n t} [c_{n\alpha}(E) + jc_{n\beta}(E)] e^{-iEt} . \tag{19b}$$

From Eq. (19), we see that vanishing of the c's for $t<0$ is guaranteed if $c_{n\alpha}(E)$ and $c_{n\beta}(E)$ are analytic in the upper half of the E complex plane. Substituting Eq. (19) and

$$i\delta(t) = -\frac{1}{2\pi i} \int_{-\infty}^{\infty} dE\, e^{-i(\pm E_n + E)t} \tag{20}$$

into Eq. (18) gives the following equation for $c_{n\alpha,\beta}$:

$$
\begin{aligned}
(E-E_n)c_{n\alpha}(E) &= -i\sum_l [\tilde{V}_{\alpha n l}c_{l\alpha}(E)+\tilde{V}^*_{\beta n l}c_{l\beta}(E)] \\
&\quad +\sum_a \delta_{n,s_a}K_{\alpha\alpha}\,, \\
(E+E_n)c_{n\beta}(E) &= i\sum_l [\tilde{V}_{\beta n l}c_{l\alpha}(E)-\tilde{V}^*_{\alpha n l}c_{l\beta}(E)] \\
&\quad -\sum_a \delta_{n,s_a}K_{\alpha\beta}\,.
\end{aligned}
\tag{21}
$$

To solve these equations to order $\tilde{V}^2$, we note that the c_n's for $n\neq\{s_a\}$ are $O(\tilde{V})$, whereas those for $n=\{s_a\}$ are ~ 1. Hence we can immediately solve for the former in terms of the latter, giving, for $n\neq\{s_a\}$,

$$
\begin{aligned}
c_{n\alpha}(E) &= -i\sum_b [\tilde{V}_{\alpha n s_b}c_{s_b\alpha}(E)+\tilde{V}^*_{\beta n s_b}c_{s_b\beta}(E)] \\
&\quad \times(E+i\epsilon-E_n)^{-1}+O(\tilde{V}^2)\,, \\
c_{n\beta}(E) &= i\sum_b [\tilde{V}_{\beta n s_b}c_{s_b\alpha}(E)-\tilde{V}^*_{\alpha n s_b}c_{s_b\beta}(E)] \\
&\quad \times(E+i\epsilon+E_n)^{-1}+O(\tilde{V}^2)\,,
\end{aligned}
\tag{22}
$$

where we have replaced E by $E+i\epsilon$ in the energy denominators to achieve upper-half-plane analyticity. From Eq. (21), the equations for $n=\{s_a\}$ are

$$
\begin{aligned}
(E+i\epsilon-E_s)c_{s_a\alpha}(E) &= -i\sum_l [\tilde{V}_{\alpha s_a l}c_{l\alpha}(E)+\tilde{V}^*_{\beta s_a l}c_{l\beta}(E)]+K_{\alpha\alpha} \\
&= -i\sum_b [\tilde{V}_{\alpha s_a s_b}c_{s_b\alpha}(E)+\tilde{V}^*_{\beta s_a s_b}c_{s_b\beta}(E)]+K_{\alpha\alpha} \\
&\quad -\sum_{l\neq\{s_c\}}\sum_b \{\tilde{V}_{\alpha s_a l}[\tilde{V}_{\alpha l s_b}c_{s_b\alpha}(E)+\tilde{V}^*_{\beta l s_b}c_{s_b\beta}(E)](E+i\epsilon-E_l)^{-1} \\
&\qquad +\tilde{V}^*_{\beta s_a l}[\tilde{V}^*_{\alpha l s_b}c_{s_b\beta}(E)-\tilde{V}_{\beta l s_b}c_{s_b\alpha}(E)](E+i\epsilon+E_l)^{-1}\}+O(\tilde{V}^3)\,,
\end{aligned}
\tag{23a}
$$

$$
\begin{aligned}
(E+i\epsilon+E_s)c_{s_a\beta}(E) &= i\sum_l [\tilde{V}_{\beta s_a l}c_{l\alpha}(E)-\tilde{V}^*_{\alpha s_a l}c_{l\beta}(E)]-K_{\alpha\beta} \\
&= i\sum_b [\tilde{V}_{\beta s_a s_b}c_{s_b\alpha}(E)-\tilde{V}^*_{\alpha s_a s_b}c_{s_b\beta}(E)]-K_{\alpha\beta} \\
&\quad +\sum_{l\neq\{s_c\}}\sum_b \{\tilde{V}_{\beta s_a l}[\tilde{V}_{\alpha l s_b}c_{s_b\alpha}(E)+\tilde{V}^*_{\beta l s_b}c_{s_b\beta}(E)](E+i\epsilon-E_l)^{-1} \\
&\qquad +\tilde{V}^*_{\alpha s_a l}[\tilde{V}_{\beta l s_b}c_{s_b\alpha}(E)-\tilde{V}^*_{\alpha l s_b}c_{s_b\beta}(E)](E+i\epsilon+E_l)^{-1}\}+O(\tilde{V}^3)\,,
\end{aligned}
\tag{23b}
$$

where the second line of Eqs. (23a) and (23b) is obtained from the first line by substituting Eq. (22). Grouping similar terms together, and using a summation convention for the repeated index b, Eq. (23) gives the following set of coupled linear equations which determine the occupation coefficients $c_{s_a\alpha,\beta}$ for the initial degenerate group of states:

$$
\begin{aligned}
&\left[(E+i\epsilon-E_s)\delta_{ab}+i\tilde{V}_{\alpha s_a s_b}+\sum_{l\neq\{s_c\}}\left[\tilde{V}_{\alpha s_a l}\frac{1}{E+i\epsilon-E_l}\tilde{V}_{\alpha l s_b}-\tilde{V}^*_{\beta s_a l}\frac{1}{E+i\epsilon+E_l}\tilde{V}_{\beta l s_b}\right]\right]c_{s_b\alpha}(E) \\
&\qquad +\left[i\tilde{V}^*_{\beta s_a s_b}+\sum_{l\neq\{s_c\}}\left[\tilde{V}_{\alpha s_a l}\frac{1}{E+i\epsilon-E_l}\tilde{V}^*_{\beta l s_b}+\tilde{V}^*_{\beta s_a l}\frac{1}{E+i\epsilon+E_l}\tilde{V}^*_{\alpha l s_b}\right]\right]c_{s_b\beta}(E)=K_{\alpha\alpha}\,, \\
&\left[(E+i\epsilon+E_s)\delta_{ab}+i\tilde{V}^*_{\alpha s_a s_b}-\sum_{l\neq\{s_c\}}\left[\tilde{V}_{\beta s_a l}\frac{1}{E+i\epsilon-E_l}\tilde{V}^*_{\beta l s_b}-\tilde{V}^*_{\alpha s_a l}\frac{1}{E+i\epsilon+E_l}\tilde{V}^*_{\alpha l s_b}\right]\right]c_{s_b\beta}(E) \\
&\qquad +\left[-i\tilde{V}_{\beta s_a s_b}-\sum_{l\neq\{s_c\}}\left[\tilde{V}_{\beta s_a l}\frac{1}{E+i\epsilon-E_l}\tilde{V}_{\alpha l s_b}+\tilde{V}^*_{\alpha s_a l}\frac{1}{E+i\epsilon+E_l}\tilde{V}_{\beta l s_b}\right]\right]c_{s_b\alpha}(E)=-K_{\alpha\beta}\,.
\end{aligned}
\tag{24}
$$

For the subsequent analysis, it is convenient to use the familiar formula

$$
\frac{1}{E\pm E_l+i\epsilon}=\frac{P}{E\pm E_l+i\epsilon}-i\pi\delta(E\pm E_l)
\tag{25}
$$

to split the intermediate state sums in Eq. (24) into generalized mass and decay matrices, giving

$$\left[(E+i\epsilon-E_s)\delta_{ab}-\mathcal{M}_{ab}^{\alpha\alpha}+\frac{i}{2}\Gamma_{ab}^{\alpha\alpha}\right]c_{s_b\alpha}(E)+\left[-\mathcal{M}_{ab}^{\alpha\beta}+\frac{i}{2}\Gamma_{ab}^{\alpha\beta}\right]c_{s_b\beta}(E)=K_{a\alpha}\ , \tag{26a}$$

$$\left[-\mathcal{M}_{ab}^{\beta\alpha}+\frac{i}{2}\Gamma_{ab}^{\beta\alpha}\right]c_{s_b\alpha}(E)+\left[(E+i\epsilon+E_s)\delta_{ab}-\mathcal{M}_{ab}^{\beta\beta}+\frac{i}{2}\Gamma_{ab}^{\beta\beta}\right]c_{s_b\beta}(E)=-K_{a\beta}\ . \tag{26b}$$

The coefficient matrices in Eq. (25) are given by the following formulas, in which we have made use of the anti-Hermiticity conditions of Eq. (11):

$$\mathcal{M}=\begin{bmatrix}\mathcal{M}_{ab}^{\alpha\alpha} & \mathcal{M}_{ab}^{\alpha\beta}\\ \mathcal{M}_{ab}^{\beta\alpha} & \mathcal{M}_{ab}^{\beta\beta}\end{bmatrix} =X_{ab}+\sum_{l\neq\{s_c\}}\left[Y_{ab,l}\frac{P}{E-E_l}+Z_{ab,l}\frac{P}{E+E_l}\right],$$

$$\Gamma=\begin{bmatrix}\Gamma_{ab}^{\alpha\alpha} & \Gamma_{ab}^{\alpha\beta}\\ \Gamma_{ab}^{\beta\alpha} & \Gamma_{ab}^{\beta\beta}\end{bmatrix} =2\pi\sum_{l\neq\{s_c\}}[Y_{ab,l}\delta(E-E_l)+Z_{ab,l}\delta(E+E_l)]\ ; \tag{27a}$$

$$X_{ab}=\begin{bmatrix}-i\tilde V_{\alpha s_a s_b} & -i\tilde V^*_{\beta s_a s_b}\\ i\tilde V_{\beta s_b s_a} & i\tilde V_{\alpha s_b s_a}\end{bmatrix}=X_{ba}^{*T}\ ,$$

$$Y_{ab,l}=\begin{bmatrix}\tilde V^*_{\alpha l s_a}\tilde V_{\alpha l s_b} & \tilde V^*_{\alpha l s_a}\tilde V^*_{\beta l s_b}\\ \tilde V_{\beta l s_a}\tilde V_{\alpha l s_b} & \tilde V_{\beta l s_a}\tilde V^*_{\beta l s_b}\end{bmatrix}=Y_{ba}^{*T}\ , \tag{27b}$$

$$Z_{ab,l}=\begin{bmatrix}\tilde V^*_{\beta l s_a}\tilde V_{\beta l s_b} & -\tilde V^*_{\beta l s_a}\tilde V^*_{\alpha l s_b}\\ -\tilde V_{\alpha l s_a}\tilde V_{\beta l s_b} & \tilde V_{\alpha l s_a}\tilde V^*_{\alpha l s_b}\end{bmatrix}=Z_{ba}^{*T}\ .$$

In Eq. (27b) the superscript T indicates transposition of the indicated 2×2 matrices, and we see that the generalized mass and decay matrices $\mathcal{M}$ and Γ are Hermitian. Since the coefficient matrices Y_{ab} and Z_{ab} admit the factorization

$$Y_{ab}=y_a\otimes y_b^{*T}\ ,\qquad Z_{ab}=z_a\otimes z_b^{*T}\ , \tag{28a}$$

with

$$y_a=\begin{bmatrix}\tilde V^*_{\alpha l s_a}\\ \tilde V_{\beta l s_a}\end{bmatrix},\quad z_a=\begin{bmatrix}\tilde V^*_{\beta l s_a}\\ -\tilde V_{\alpha l s_a}\end{bmatrix}, \tag{28b}$$

the decay matrix Γ is positive definite, and so contributes to Eq. (26) with the same sign as the explicit $i\epsilon$. As a result, Eq. (26) is guaranteed to give solutions for $c_{s_a\alpha,\beta}(E)$ which are upper-half-plane analytic, and thus Eqs. (19) and Eqs. (26)–(28) give the desired solution to our time-dependent perturbation-theory problem.

III. THE DECAY OF A SET OF $\mathbb{C}(1,i)$ INITIAL STATES

Let us now apply the formalism of the preceding section to the case $K_{a\beta}=0$, in which the decaying states have wave functions which are initially $\mathbb{C}(1,i)$. This will be true, for example, if the decaying states are produced from $\mathbb{C}(1,i)$ asymptotic states by physical processes which involve only the unperturbed Hamiltonian H_0. Substituting $K_{a\beta}=0$ into Eq. (26b), we can solve for $c_{s_c\beta}(E)$ in terms of $c_{s_b\alpha}(E)$, giving

$$c_{s_c\beta}(E)=\frac{i\tilde V_{\beta s_b s_c}+O(\tilde V^2)}{E+i\epsilon+E_s}c_{s_b\alpha}(E)\ . \tag{29}$$

Substituting Eq. (29) back into Eq. (26a), the second term on the left-hand side of Eq. (26a) then becomes

$$\left[-\mathcal{M}_{ac}^{\alpha\beta}+\frac{i}{2}\Gamma_{ac}^{\alpha\beta}\right]c_{s_c\beta}(E)=\left[-\sum_c\frac{\tilde V^*_{\beta s_c s_a}\tilde V_{\beta s_c s_b}}{E+i\epsilon+E_s}+O(\tilde V^3)\right]c_{s_b\alpha}(E)$$

$$=-\sum_c\frac{Z_{ab,s_c}^{\alpha\alpha}}{E+i\epsilon+E_s}c_{s_b\alpha}(E)\ , \tag{30}$$

and so it just supplies the missing $l\neq\{s_c\}$ terms in the $Z_{ab,l}^{\alpha\alpha}$ pieces of $\mathcal{M}^{\alpha\alpha}$ and $\Gamma^{\alpha\alpha}$. Thus, we can rewrite Eq. (26a) for $c_{s_b\alpha}$ as

$$\left[(E+i\epsilon-E_s)\delta_{ab}-\mathcal{M}_{ab}^{\alpha\alpha\text{tot}}(E)+\frac{i}{2}\Gamma_{ab}^{\alpha\alpha\text{tot}}(E)\right]c_{s_b\alpha}(E)=K_{a\alpha}\ ,$$

$$\mathcal{M}_{ab}^{\alpha\alpha\text{tot}}(E)=X_{ab}^{\alpha\alpha}+\sum_{l\neq\{s_c\}}Y_{ab,l}^{\alpha\alpha}\frac{P}{E-E_l}+\sum_l Z_{ab,l}^{\alpha\alpha}\frac{P}{E+E_l}\ , \tag{31}$$

$$\Gamma_{ab}^{\alpha\alpha\text{tot}}(E)=2\pi\sum_{l\neq\{s_c\}}Y_{ab,l}^{\alpha\alpha}\delta(E-E_l)+\sum_l Z_{ab,l}^{\alpha\alpha}\delta(E+E_l)\ ,$$

which has the same structure as the equation for the decaying state amplitude in complex quantum mechanics. In other words, for $\mathbb{C}(1,i)$ initial states, the quaternionic decay problem formulated in Sec. II reduces to an effective complex quantum-mechanics decay problem, given by Eq. (31).

We now follow the classic Weisskopf-Wigner[3] treatment of decaying states, and make the approximation of replacing the energy-dependent mass and decay matrices appearing in Eq. (31) by their values at the energy $E=E_s$ of the decaying group of states. This approximation is motivated by the observation that the dominant term in the coefficient of $c_{s_b\alpha}$ in Eq. (31) is the zeroth-order term $(E+i\epsilon-E_s)\delta_{ab}$, and so $c_{s_b\alpha}(E)$ is small unless $E\approx E_s$. Quantitatively, when the first-order term in the mass matrix $X_{ab}^{\alpha\alpha}$ is zero (as will be the case in our discussion of K decays below), the error of assuming constant mass and decay matrices will be of order Γ_s/E_s, with Γ_s the maximum decay rate of the group of states $\{s_c\}$. With this approximation, we have

$$\left[(E+i\epsilon-E_s)\delta_{ab}-m_{ab}+\frac{i}{2}\gamma_{ab}\right]c_{s_b\alpha}(E)=K_{a\alpha} ,$$

$$\begin{aligned} m_{ab}&=\mathcal{M}_{ab}^{\alpha\alpha\mathrm{tot}}(E_s)\\ &=X_{ab}^{\alpha\alpha}+\sum_{l\neq\{s_c\}}Y_{ab,l}^{\alpha\alpha}\frac{P}{E_s-E_l}+\sum_l Z_{ab,l}^{\alpha\alpha}\frac{1}{E_s+E_l} , \\ \gamma_{ab}&=\Gamma_{ab}^{\alpha\alpha\mathrm{tot}}(E_s)=2\pi\sum_{l\neq\{s_c\}}Y_{ab,l}^{\alpha\alpha}\delta(E_s-E_l) , \end{aligned} \tag{32}$$

or, Fourier transforming back to time as a variable,

$$\left[\left[\frac{d}{dt}+iE_s\right]\delta_{ab}+im_{ab}+\tfrac{1}{2}\gamma_{ab}\right]c_{s_b\alpha}(t)=K_{a\alpha}\delta(t) . \tag{33}$$

Equations (32) and (33) are the standard starting point for the discussion of decaying systems, with the new feature here being the presence of the $Z_{ab,l}^{\alpha\alpha}$ terms in the mass matrix.

An unusual aspect of Eq. (32) is the presence of energy denominators E_s+E_l, which are not invariant under a uniform shift $E_l\to E_l+\lambda$ of the unperturbed energies. Such a shift is induced in the complex case by multiplying the wave function by $\exp(-i\lambda t)$; in the quaternionic case, this multiplication gives

$$\begin{aligned} &\frac{\partial}{\partial t}\Psi'(t)=-\tilde{H}'\Psi'(t) ,\\ &\Psi'(t)\equiv\exp(-i\lambda t)\Psi(t) ,\\ &\tilde{H}'=\exp(-i\lambda t)\tilde{H}\exp(i\lambda t)+i\lambda\\ &\quad=i(H_0+\lambda)+\tilde{V}_\alpha+je^{2i\lambda t}\tilde{V}_\beta . \end{aligned} \tag{34}$$

Hence when the wave function is rephased, in the quaternionic case *two* things happen: the unperturbed energies are shifted according to $E_l\to E_l'=E_l+\lambda$, *and* the time-independent perturbation $\tilde{V}_\beta$ is changed to a time-dependent perturbation $\exp(2i\lambda t)\tilde{V}_\beta$. When the analysis of Sec. II is repeated for such a time-dependent perturbation, the resulting formulas are the same except for a change in the energy denominators of the $Z_{ab,l}^{\alpha\alpha}$ terms to

$$E_s'+E_l'-2\lambda=E_s+E_l . \tag{35}$$

Thus the mass and decay matrices of Eq. (32) are left invariant under the transformation of Eq. (34), as they must be under a simple change of variables which does not affect the underlying physics. The argument just given shows, however, that when we specify that the perturbation $\tilde{V}$ in Eq. (8) is *time independent*, we no longer have the usual freedom to shift the unperturbed energies by a uniform constant.[8]

A second unusual feature of Eq. (32) is the fact that the $Z_{ab,l}^{\alpha\alpha}$ terms contribute to the second-order mass matrix, but not to the decay matrix, which naively would appear to contradict unitarity. But since the quaternionic dynamics of Eq. (4) is manifestly unitary, unitarity must be satisfied order by order in $\tilde{V}$, and we shall now verify this by explicit calculation to order $\tilde{V}^2$. To simplify the analysis, let us consider the case in which the group $\{s_c\}$ contains only a single state s, so that the indices a and b can be dropped, and let us take the initial-state amplitude K_α to be unity. Equation (32) then becomes

$$\begin{aligned} &c_{s\alpha}(E)=\left[E+i\epsilon-E_s-m+\frac{i}{2}\gamma\right]^{-1} ,\\ &m=X^{\alpha\alpha}+\sum_{l\neq s}Y_l^{\alpha\alpha}\frac{P}{E_s-E_l}+\sum_l Z_l^{\alpha\alpha}\frac{1}{E_s+E_l} ,\\ &\gamma=2\pi\sum_{l\neq s}Y_l^{\alpha\alpha}\delta(E_s-E_l) ,\\ &Y_l^{\alpha\alpha}=|\tilde{V}_{\alpha ls}|^2 ,\\ &Z_l^{\alpha\alpha}=|\tilde{V}_{\beta ls}|^2 . \end{aligned} \tag{36}$$

Since the dynamics of Eq. (4) is norm preserving, substitution of Eqs. (15) and (16a) into the norm $||\Psi||$ gives, for all times $t\geq 0$, the unitarity sum rule

$$\begin{aligned} 1=||\Psi||&=\sum_l[\,|c_{l\alpha}(t)|^2+|c_{l\beta}(t)|^2]\\ &=|c_{s\alpha}(t)|^2+\sum_{l\neq s}|c_{l\alpha}(t)|^2+\sum_l|c_{l\beta}(t)|^2 . \end{aligned} \tag{37}$$

To verify perturbative unitarity, we must calculate each of the three terms on the right-hand side of Eq. (37) to order $\tilde{V}^2$ and check that their sum is unity.

We begin with the first term on the right in Eq. (37). Substituting Eq. (36) for $c_{s\alpha}(E)$ into Eq. (19a) and (for $t>0$) closing the contour down, we get

$$c_{s\alpha}(t)\approx\exp(-imt-\tfrac{1}{2}\gamma t) , \tag{38a}$$

giving

$$|c_{s\alpha}(t)|^2\approx\exp(-\gamma t)\approx 1-\gamma t . \tag{38b}$$

We turn next to the second term on the right in Eq. (37). From Eqs. (22) and (29) we get (for $l\neq s$)

$$\begin{aligned} c_{l\alpha}(E)&\approx -i\frac{\tilde{V}_{\alpha ls}c_{s\alpha}(E)}{E+i\epsilon-E_l}\\ &\approx\frac{-i\tilde{V}_{\alpha ls}}{(E+i\epsilon-E_l)(E+i\epsilon-E_s)} , \end{aligned} \tag{39a}$$

and substituting this into Eq. (19a) and closing down gives

$$c_{l\alpha}(t) \approx \frac{-i\tilde{V}_{\alpha ls}}{E_l - E_s}(1 - e^{i(E_l - E_s)t}) . \quad (39b)$$

Hence the second term on the right in Eq. (37) is given by

$$\sum_{l \neq s} |c_{l\alpha}(t)|^2 \approx 4 \sum_{l \neq s} |\tilde{V}_{\alpha ls}|^2 \frac{\sin^2[\frac{1}{2}t(E_l - E_s)]}{(E_l - E_s)^2} . \quad (39c)$$

When the set of states l forms a continuum around s, we can make the "golden rule" approximation,[2]

$$\frac{\sin^2[\frac{1}{2}t(E_l - E_s)]}{(E_l - E_s)^2} \approx \frac{1}{2}\pi t \delta(E_l - E_s) , \quad (39d)$$

and Eq. (39c) becomes

$$\sum_{l \neq s} |c_{l\alpha}(t)|^2 \approx t 2\pi \sum_{l \neq s} |\tilde{V}_{\alpha ls}|^2 \delta(E_l - E_s) = \gamma t . \quad (39e)$$

Thus the first two terms on the right in Eq. (37) exhaust the unitarity sum rule, up to the errors inherent in the Weisskopf-Wigner and golden rule analyses.

We turn our attention finally to the third term on the right in Eq. (37). From Eqs. (22) and (29) we get (this time for all l)

$$\begin{aligned} c_{l\beta}(E) &\approx \frac{i\tilde{V}_{\beta ls} c_{s\alpha}(E)}{E + i\epsilon + E_l} \\ &\approx \frac{i\tilde{V}_{\beta ls}}{(E + i\epsilon + E_l)(E + i\epsilon - E_s)} ; \end{aligned} \quad (40a)$$

substituting into Eq. (19a) and again closing down gives

$$c_{l\beta}(t) \approx \frac{i\tilde{V}_{\beta ls}}{E_l + E_s}(1 - e^{-i(E_l + E_s)t}) . \quad (40b)$$

Hence the third term on the right in Eq. (37) is given by

$$\sum_{l} |c_{l\beta}(t)|^2 = 4 \sum_{l} |\tilde{V}_{\beta ls}|^2 \frac{\sin^2[\frac{1}{2}t(E_l + E_s)]}{(E_l + E_s)^2} . \quad (40c)$$

We can estimate the sum in Eq. (40c) by noting that the time t_s characterizing the decay of the initial state s is $t_s \sim \gamma^{-1}$; for such times the argument of the sine is very large (since $E_s/\gamma \gg 1$) and the sine function is very rapidly oscillating, and so we can approximate $\sin^2[\frac{1}{2}t(E_l + E_s)]$ by its average value of $\frac{1}{2}$. So Eq. (40c) becomes

$$P_\beta \equiv \sum_{l} |c_{l\beta}(t)|^2 \sim 2 \sum_{l} \frac{|\tilde{V}_{\beta ls}|^2}{(E_l + E_s)^2} , \quad (40d)$$

showing that *the total probability in the β (or intrinsically quaternionic) amplitudes does not grow linearly with time, but rather at large times approaches the constant value of Eq. (40d).*

To estimate the magnitude of P_β, we rewrite Eq. (40d) as

$$\begin{aligned} P_\beta &\sim 2 \sum_{l \neq s} \frac{|\tilde{V}_{\beta ls}|^2}{(E_l + E_s)^2} + 2 \frac{|\tilde{V}_{\beta ss}|^2}{(2E_s)^2} \\ &\lesssim 4 \sum_{l \neq s} \frac{|\tilde{V}_{\beta ls}|^2}{(E_l + E_s)^2} , \end{aligned} \quad (40e)$$

since for l near s the individual terms in the sum are expected to be similar in size. We now express the right-hand side of Eq. (40e) in the form

$$\begin{aligned} 4 \sum_{l \neq s} \frac{|\tilde{V}_{\beta ls}|^2}{(E_l + E_s)^2} &= \frac{2}{\pi} \int_0^\infty \frac{dE}{(E + E_s)^2} \gamma_\beta(E) , \\ \gamma_\beta(E) &\equiv 2\pi \sum_{l \neq s} |\tilde{V}_{\beta ls}|^2 \delta(E_l - E) , \end{aligned} \quad (40f)$$

and approximate $\gamma_\beta(E)$ to be a constant by writing

$$\begin{aligned} \gamma_\beta(E) \sim \gamma_\beta(E_s) &= 2\pi \sum_{l \neq s} |\tilde{V}_{\beta ls}|^2 \delta(E_l - E_s) \\ &\sim \left\langle \left| \frac{\tilde{V}_\beta}{\tilde{V}_\alpha} \right|^2 \right\rangle_{\text{av}} \gamma , \end{aligned} \quad (40g)$$

giving finally

$$P_\beta \sim \frac{2}{\pi} \left\langle \left| \frac{\tilde{V}_\beta}{\tilde{V}_\alpha} \right|^2 \right\rangle_{\text{av}} \frac{\gamma}{E_s} . \quad (40h)$$

Hence P_β is of the order of the errors inherent in the Weisskopf-Wigner analysis. For example, in our application to K-meson decays in the next section we have[9]

$$\begin{aligned} &\left\langle \left| \frac{\tilde{V}_\beta}{\tilde{V}_\alpha} \right|^2 \right\rangle_{\text{av}} \sim |\eta_{+-}| \sim 2 \times 10^{-3} , \\ &\frac{\gamma}{E_s} \sim \frac{\Delta M_K}{M_K} \sim 10^{-14}, \quad P_\beta \sim 10^{-17} , \end{aligned} \quad (41)$$

and P_β is very small indeed.

The fact that P_β remains bounded and very small means that an initially $\mathbb{C}(1,i)$ state does not decay, under the influence of a quaternionic perturbation $\tilde{V}_\beta$, into an intrinsically quaternionic state. Put another way, a complex $\mathbb{C}(1,i)$ asymptotic state space is stable with respect to quaternionic perturbations; this fact lends strong support to our conjecture[10] that the asymptotic state space of a quaternionic quantum field theory resides within a complex subspace of quaternionic Hilbert space.

IV. AN APPLICATION: A MODEL FOR *CP* NONCONSERVATION IN *K* DECAYS

We proceed now to apply the calculation of the preceding section to a model which we have introduced[10] for *CP* nonconservation in *K* decays. Following Ref. 10, we identify

$$\begin{aligned} &H_0 = H_{\text{strong+electromagnetic}} , \\ &\tilde{V}_\alpha = iH_{\text{weak}} = CP \text{ even} , \\ &\tilde{V}_\beta = \tilde{V}_2 - i\tilde{V}_3, \quad \tilde{V}_2 = CP \text{ odd}, \quad \tilde{V}_3 = CP \text{ even} , \end{aligned} \quad (42)$$

so that under the quaternion-extended CP operation

$$(CP)_q \equiv jCP \tag{43a}$$

all terms in $\tilde{H}$ transform uniformly:

$$(CP)_q{}^{-1}\tilde{H}(CP)_q = -\tilde{H} . \tag{43b}$$

We take the set of states $\{s_c\}$ to be the degenerate eigenstates $|K_1\rangle$ (which is CP even) and $|K_2\rangle$ (which is CP odd) of H_0, with zeroth-order rest-frame energy $E_s = M_K$. Under the usual assumption that H_{weak} has no $\Delta S = 2$ piece, the first-order contribution $X_{ab}^{\alpha\alpha}$ to the mass matrix of Eq. (32) vanishes, and we have

$$\begin{aligned} m_{ab} &= \sum_{l \neq K_1, K_2} Y_{ab,l}^{\alpha\alpha} \frac{P}{M_K - E_l} + \sum_l Z_{ab,l}^{\alpha\alpha} \frac{1}{M_K + E_l} , \\ \gamma_{ab} &= 2\pi \sum_{l \neq K_1, K_2} Y_{ab,l}^{\alpha\alpha} \delta(M_K - E_l) . \end{aligned} \tag{44}$$

The $Y_{ab,l}^{\alpha\alpha}$ terms are the usual[11] weak contributions m_{ab}^{weak} and $\gamma_{ab}^{\text{weak}}$ to the K-meson mass and decay matrices; since no other terms contribute to γ_{ab} and since we are assuming H_{weak} to be CP even, we see that in our model no direct CP-nonconserving effects are present in K-meson (or other) decays. All CP nonconservation must arise through the $Z_{ab,l}^{\alpha\alpha}$ term in the mass matrix. To study the form of this term, we substitute $\tilde{V}_\beta = \tilde{V}_2 - i\tilde{V}_3$ into $Z_{ab,l}^{\alpha\alpha} = \tilde{V}^*_{\beta l s_a} \tilde{V}_{\beta l s_b}$, giving

$$\begin{bmatrix} Z_{K_1K_1,l}^{\alpha\alpha} & Z_{K_1K_2,l}^{\alpha\alpha} \\ Z_{K_2K_1,l}^{\alpha\alpha} & Z_{K_2K_2,l}^{\alpha\alpha} \end{bmatrix} = \begin{bmatrix} (\tilde{V}_{2lK_1})^2 + (\tilde{V}_{3lK_1})^2 & i(\tilde{V}_{3lK_1}\tilde{V}_{2lK_2} - \tilde{V}_{2lK_1}\tilde{V}_{3lK_2}) \\ i(\tilde{V}_{3lK_2}\tilde{V}_{2lK_1} - \tilde{V}_{2lK_2}\tilde{V}_{3lK_1}) & (\tilde{V}_{2lK_2})^2 + (\tilde{V}_{3lK_2})^2 \end{bmatrix} , \tag{45a}$$

where we have used the fact that the intermediate states l can be chosen to be CP eigenstates, with the consequence that

$$\tilde{V}_{2lK_1}\tilde{V}_{2lK_2} = \tilde{V}_{3lK_1}\tilde{V}_{3lK_2} = 0 , \tag{45b}$$

since one of the two factors must always vanish. Substituting Eq. (45a) into Eq. (44), we have

$$m_{ab} = m_{ab}^{\text{weak}} + \Delta m_{ab}, \quad \gamma_{ab} = \gamma_{ab}^{\text{weak}} , \tag{46a}$$

with the quaternionic contribution Δm_{ab} given by

$$\begin{bmatrix} \Delta m_{K_1K_1} & \Delta m_{K_1K_2} \\ \Delta m_{K_2K_1} & \Delta m_{K_2K_2} \end{bmatrix} = \begin{bmatrix} m_1 & im' \\ -im' & m_2 \end{bmatrix} ,$$

$$\begin{aligned} m_{1,2} &= \sum_l \frac{(\tilde{V}_{2lK_{1,2}})^2 + (\tilde{V}_{3lK_{1,2}})^2}{M_K + E_l} , \\ m' &= \sum_l \frac{\tilde{V}_{3lK_1}\tilde{V}_{2lK_2} - \tilde{V}_{2lK_1}\tilde{V}_{3lK_2}}{M_K + E_l} . \end{aligned} \tag{46b}$$

Equation (46) describes a CP nonconservation which is phenomenologically of "superweak" form,[12] but arises as a *second-order* perturbation theory effect. Physical consequences of such an interpretation of superweak CP nonconservation are discussed in Ref. 10.

ACKNOWLEDGMENT

This work was supported by the U.S. Department of Energy under Grant No. DE-AC02-76ERO-2220.

[1]By time-dependent perturbation theory we mean the use of perturbation theory to discuss time development of the state function; in most of this paper we will be considering time-independent perturbations. This is the standard terminology, see, e.g., E. Merzbacher, *Quantum Mechanics* (Wiley, New York, 1961), p. 442.

[2]Merzbacher, *Quantum Mechanics* (Ref. 1), Chap. 19, Sec. 1.

[3]V. F. Weisskopf and E. P. Wigner, Z. Phys. **63**, 54 (1930); **65**, 18 (1930); W. Heitler, *The Quantum Theory of Radiation* (Oxford University Press, Oxford, 1954), Chap. IV, Sec. 16.

[4]S. L. Adler, Phys. Rev. Lett. **55**, 783 (1985), Commun. Math. Phys. **104**, 611 (1986), and references cited therein.

[5]In Ref. 4, we denoted i,j,k by e_1, e_2, e_3.

[6]D. Finkelstein, J. M. Jauch, and D. Speiser, in *Logico-Algebraic Approach to Quantum Mechanics II*, edited by C. Hooker (Reidel, Dordrecht, 1959); Adler (Ref. 4).

[7]Heitler, *The Quantum Theory of Radiation* (Ref. 3), p. 165.

[8]I wish to thank C. P. Burgess for a helpful conversation about this. The sign reversal which leads to the $E_s + E_l$ energy denominators was previously noted by L. P. Horwitz and L. C. Biedenharn, Ann. Phys. (N.Y.) **157**, 432 (1984), Eq. (3.11).

[9]Here η_{+-} is the amplitude ratio $A(K_L \to \pi^+\pi^-)/A(K_S \to \pi^+\pi^-)$, and ΔM_K is the $K_L - K_S$ mass difference.

[10]S. L. Adler, Phys. Rev. Lett. **57**, 167 (1986).

[11]See, e.g., R. E. Marshak, Riazuddin, and C. P. Ryan, *Theory of Weak Interactions in Particle Physics* (Wiley-Interscience, New York, 1969), Secs. 6.5 and 6.6.

[12]L. Wolfenstein, Phys. Rev. Lett. **13**, 562 (1964); in *Theory and Phenomenology in Particle Physics,* edited by A. Zichichi (Academic, New York, 1969).