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Bound states and asymptotically free quarks

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We prove duality relations between bound states and asymptotically free quarks by using relativistic wave functions and couplings to currents. Our approximation is based on WKB formulas and on an ansatz on the potential at very small distances, where the instantaneous potential approximation for the Bethe-Salpeter equation is unreliable. We apply our results to two-point spectral functions of vector and axial-vector currents.

I. INTRODUCTION

During the last decade a large body of experimental evidence has been collected giving support to quantum chromodynamics (QCD) as the theory of strong-interacting particles. On the theoretical side, however, the situation is more entangled, for lack of a unique treatment of small and large distances; whereas in the former case perturbation theory provides a calculational scheme based on the asymptotic-freedom property of QCD (Ref. 1), in the latter the dynamics of quark confinement can be approached only by nonperturbative methods such as lattice QCD (Ref. 2). In this context, any approximate method which is able to grasp the two aspects of strong interactions at the same time should be welcome. In the 1970s Bramon, Etim, and Greco[3] and Sakurai[4] suggested a model of e^+e^- annihilation into hadrons based on the idea of duality between the free-quark behavior displayed by $\sigma(e^+e^-\to\text{hadrons})$ at high energy and the production of infinitely many $q\bar{q}$ bound states in the process

$$e^+e^-\to\gamma^*\to V_n\to \text{hadrons}\ , \tag{1.1}$$

where γ^* is a virtual photon.

Such a research line has been pursued in the subsequent years and some achievements have been obtained. Duality has been shown to hold for nonrelativistic quarks,[5] and in the relativistic case for both vector[6,7] and axial-vector currents.[8,9] The extension of relativistic duality to bound states made up by quarks that are asymptotically free, i.e., whose scaling laws are governed by quantum chromodynamics, is not straightforward. In order to be definite, let us consider the following relativistic wave equation for a meson having mass M:

$$[(-\hbar^2\nabla^2+m_i^{\,2})^{1/2}+(-\hbar^2\nabla^2+m_j^{\,2})^{1/2}+V(\mathbf{r})]\psi(\mathbf{r})=M\psi(\mathbf{r})\ . \tag{1.2}$$

Equation (1.2) arises from the Bethe-Salpeter equation by replacing the full interaction by an instantaneous local potential. Assuming that, for $r\to 0$, $V(\mathbf{r})$ behaves according to perturbative QCD,

$$V(\mathbf{r})\underset{r\to 0}{\sim}-\frac{4}{3}\frac{\alpha(r)}{r}\simeq-\frac{4}{3}\frac{\alpha_0}{r\ln(\tilde{r}/r)}\ , \tag{1.3}$$

one obtains, if $\psi(\mathbf{r})=Y_{lm}(\hat{\mathbf{r}})\phi_l(r)$,

$$\phi_l(r)\underset{r\to 0}{\sim} r^l[\ln(\tilde{r}/r)]^{\lambda_l} \tag{1.4}$$

with $\lambda_l>0$. This result has been obtained recently by Durand;[10] in Appendix A of the present paper we give an alternative derivation of this formula.

Such behavior is unphysical; as a matter of fact, the coupling $f_{n,0}$ of an S-wave vector meson to the electromagnetic current diverges, as it turns out to be proportional to $\phi_0(0)$ (for a proof see Appendix A). This implies that the ratio

$$R_{e^+e^-}=\frac{\sigma(e^+e^-\to\text{ hadrons})}{\sigma(e^+e^-\to\mu^+\mu^-)} \tag{1.5}$$

is divergent as well; in fact, as we shall show in Sec. IV, $R_{e^+e^-}$ is given by a sum of terms proportional to $f_{n,0}^{\;2}$ or $f_{n,2}^{\;2}$ (the coupling of a D-wave vector meson). Clearly this divergence is an artifact of the instantaneous potential approximation that, for lengths $r\lesssim 1/M$ is not reliable.

The purpose of this paper is to prove duality for relativistic quarks by assuming a potential comprising a long-range linear confining part and a small-range piece given by the perturbative QCD result, except that, for r smaller than a meson scale $r_m\lesssim 1/M$, we assume a constant potential. By exploiting this ansatz and using a WKB approximation for Eq. (1.2), we shall prove duality between bound states and asymptotically free quarks up to order α_s; moreover we shall derive canonical results for two-point spectral functions of vector and axial-vector currents.

Our result is based on the above-mentioned ansatz on the potential, so that we do not pretend to have rigorously proved duality in presence of the QCD short-range effects; nevertheless we obtain a consistent picture that can be used to get some hints on the extreme $r\to 0$ region where the approximation of an instantaneous potential fails.

The plan of the paper is as follows. In Sec. II we describe the potential and the WKB approximation; in Sec. III we give formulas of particle-current couplings. Section IV is devoted to the proof of the canonical behavior of two-point spectral functions and contains our conclusions. Finally, some details of our calculations are contained in two appendixes.

II. WKB SOLUTIONS AND THE SMALL-DISTANCE BEHAVIOR OF THE POTENTIAL

We shall consider the case of a central potential in Eq. (1.2), i.e., $V(\mathbf{r})=V(|\mathbf{r}|)$. In such a case the angular dependence in the solutions $\psi(\mathbf{r})$ can be factorized

$$\psi(\mathbf{r})=Y_{lm}(\hat{\mathbf{r}})\phi_l(r)=Y_{lm}(\hat{\mathbf{r}})\frac{u_l(r)}{r} \tag{2.1}$$

so that, using the spectral representation of the square root of the operator $(-\hbar^2\nabla^2+m^2)$, we get

$$[V(r)-M]u_l(r)+\frac{2}{\pi\hbar}\int_0^\infty dr'\int_0^\infty dk\, g(k)\times\chi_l\left[\frac{kr}{\hbar}\right]\chi_l\left[\frac{kr'}{\hbar}\right]u_l(r')=0\,, \tag{2.2}$$

where $\chi_l(x)=xj_l(x)$ are the Riccati-Bessel functions and

$$g(k)=(k^2+m_i^2)^{1/2}+(k^2+m_j^2)^{1/2}\,. \tag{2.3}$$

To start with, we recall WKB solutions of Eq. (2.2); extensive discussion can be found in Ref. 11. We remark that the following formulas are valid for potentials that are regular at the origin $r\to 0$.

Wave functions are given, in the WKB approximation, by[11]

$$\begin{aligned} u_l(r)&=Ae^{\sigma_1(r)}\chi_l[\sigma(r)],\quad 0<r<r_0\,,\\ u_l(r)&=\frac{A}{2}e^{\sigma_1(r)}\exp\left[-\int_{r_0}^r|\sigma'(x)|dx\right]\quad r_0<r<r_M\,, \end{aligned} \tag{2.4}$$

where A is a normalization constant and $\sigma(r)$, $\sigma_1(r)$ can be derived from (2.2) by using the saddle-point method; the result is

$$\begin{aligned} \sigma(r)&=\int_0^r\sigma'(r)dr\,,\\ \sigma'(r)&=\left[\frac{[M-V(r)]^2}{4}-\frac{m_i^2+m_j^2}{2}+\frac{(m_i^2-m_j^2)^2}{4[M-V(r)]^2}\right]^{1/2},\\ \sigma_1(r)&=\tfrac{1}{2}\ln\left[\frac{(\sigma'^2+m_i^2)^{1/2}(\sigma'^2+m_j^2)^{1/2}}{\sigma'(r)[M-V(r)]}\right]. \end{aligned} \tag{2.5}$$

These expressions are defined for r far from the classical turning point r_0, with

$$\sigma'(r_0)=0\,. \tag{2.6}$$

Thus the WKB approximation is applicable if $\sigma'^2 \gg m_{i,j}^2$, which implies that we can put $\sigma_1=\text{const}$ in Eq. (2.4) and neglect it altogether. It should be also stressed that, owing to the positivity of $g(k)$ [Eq. (2.3)], the WKB approximation cannot be used for $r>r_M$ (Ref. 11), with

$$V(r_M)=M\,. \tag{2.7}$$

For $r>r_M$, by other techniques, one finds a fast exponential decrease of the wave functions. The WKB method also provides the spectrum which is obtainable from

$$\int_0^{r_0}\sigma'(r)dr=\pi\left[n+\frac{l}{2}+\frac{3}{4}\right], \tag{2.8}$$

where n and l are the radial and orbital quantum numbers.

For practical purposes it is useful to have eigenfunctions in momentum space; using the formula

$$\int_0^\infty\chi_l(pr)\chi_l(kr)dr=\frac{\pi}{2}\delta(p-k) \tag{2.9}$$

one obtains

$$[g(p)-M]\tilde{u}_l(p)+\frac{2}{\pi\hbar}\int_0^\infty dp'\int_0^\infty dr\, V(r)\times\chi_l\left[\frac{pr}{\hbar}\right]\chi_l\left[\frac{p'r}{\hbar}\right]\tilde{u}_l(p')=0\,, \tag{2.10}$$

where

$$\tilde{u}_l(p)=\int_0^\infty dr\,\chi_l\left[\frac{pr}{\hbar}\right]u_l(r)\,. \tag{2.11}$$

The WKB method provides, in this case, the solution

$$\tilde{u}_l(p)=\lambda_l\chi_l(\Sigma(p)),\quad 0\le p\le p_0\,, \tag{2.12}$$

where λ_l is a normalization constant, $\Sigma(p)$ is defined as

$$\Sigma(p)=\int_0^p\Sigma'(k)dk \tag{2.13}$$

and $\Sigma'(p)$ is given by

$$V(\Sigma'(p))=M-g(p)\,. \tag{2.14}$$

p_0 is the turning point, defined by

$$\Sigma'(p_0)=0\,. \tag{2.15}$$

For $p>p_0$, $\tilde{u}_l(p)$ has a rapid exponential decrease.

As discussed in the Introduction, we shall use in the previous formulas the potential

$$\begin{aligned} V(r)&=\mu^2 r+V_C(r)\,,\\ V_C(r)&=\begin{cases}-\dfrac{4}{3}\dfrac{\alpha(r_m)}{r_m}, & r\le r_m\equiv\dfrac{k}{M}\,,\\ -\dfrac{4}{3}\dfrac{\alpha(r)}{r}, & r>r_m\,,\end{cases} \end{aligned} \tag{2.16}$$

where $\alpha(r) \underset{r\to 0}{\sim} \alpha_0/\ln(\tilde{r}/r)$ is the QCD running coupling constant and k is a constant. The Coulombic potential, arising from one-gluon exchange in perturbative QCD, has been substituted in (2.16) by $V_C(r)$ which lacks the unphysical $1/r$ singularity for $r\to 0$. As we have stated already, our assumption (2.16) will be justified later in dealing with two-point spectral functions.

By using Eqs. (2.6), (2.14), and the previous potential one obtains

$$r_0=\frac{M}{\mu^2}\left[1-\frac{[2(m_i{}^2+m_j{}^2)]^{1/2}}{M}+O(m^2/M^2)\right], \qquad (2.17)$$

$$p_0=\frac{M}{2}(1+\omega)\left[1-\frac{m_i{}^2+m_j{}^2}{M^2(1+\omega)^2}+O(m^4/M^4)\right], \qquad (2.18)$$

and

$$\omega=\frac{4}{3k}\alpha(k/M)=\frac{4}{3k}\frac{\alpha_0}{\ln(\tilde{r}M/k)}\ . \qquad (2.19)$$

The potential (2.16), together with the spectrum condition (2.8) allows us to evaluate the density of states for large n; as a matter of fact one gets

$$\frac{dn}{dM}=\frac{1}{\pi}\frac{M}{2\mu^2}+O(1/M) \qquad (2.20)$$

which shows that, for large n, M grows as $\sqrt{n}$. In fact from (2.5), (2.8), and (2.17) one gets the spectral condition in the large-n limit:

$$M^2\equiv M_{n,l}{}^2=4\mu^2\pi\left[n+\frac{l}{2}+\frac{3}{4}\right]. \qquad (2.21)$$

III. CURRENT-PARTICLE MATRIX ELEMENTS AND TWO-POINT SPECTRAL FUNCTIONS

Our analysis aims at evaluating the hadronic tensor

$$\begin{aligned}\Delta_J^{\mu\nu}(q)&=i\int d^4x\, e^{iqx}\langle 0\,|\,T^*(J_{ij}^{\mu}(x)J_{ij}^{\nu\dagger}(0))\,|\,0\rangle\\ &\equiv -g^{\mu\nu}q^2\Delta_J^{(1)}(q^2)+q^{\mu}q^{\nu}\Delta_J^{(2)}(q^2)\\ &\equiv \int\frac{dt}{t-q^2-i\epsilon}\left[-q^{\mu\nu}\rho_J^{(1)}(t)+\frac{q^{\mu}q^{\nu}}{t}\rho_J^{(2)}(t)\right],\end{aligned} \qquad (3.1)$$

where $J=V$ or A and the vector and axial-vector currents are

$$\begin{aligned}V_{ij}^{\mu}(x)&=\bar{q}_i(x)\gamma^{\mu}q_j(x)\ ,\\ A_{ij}^{\mu}(x)&=\bar{q}_i(x)\gamma^{\mu}\gamma^5q_j(x)\end{aligned} \qquad (3.2)$$

($i,j=$ flavor indices). The spectral functions $\rho_J^{(\alpha)}(t)$ ($\alpha=1,2$) can be calculated in perturbative QCD for large values of the argument; if duality between bound states and asymptotically free quarks holds, they should be also obtained as a sum of infinitely many resonances having spin 1 or 0. In this section we give expressions of spectral functions in terms of resonance masses and couplings to currents.

First of all we give the explicit expressions of currents in terms of quark and antiquark creation and annihilation operators $b^\dagger$, $d^\dagger$, b, and d:

$$\begin{aligned}V_{ij}^{\mu}(0)=\sum_{\alpha\beta}\sum_{rs}\delta_{\alpha}^{\beta}\int\frac{d^3q}{(2\pi)^{3/2}}\int\frac{d^3q'}{(2\pi)^{3/2}}\left[\frac{m_im_j}{E_iE_j}\right]^{1/2}&[\,\bar{u}_i(\mathbf{q},r)b_i^{\dagger}(\mathbf{q},r,\alpha)+\bar{v}_i(\mathbf{q},r)d_i(\mathbf{q},r,\alpha)]\gamma^{\mu}\\ &\times[u_j(\mathbf{q}',s)b_j(\mathbf{q}',s,\beta)+v_j(\mathbf{q}',s)d_j^{\dagger}(\mathbf{q}',s,\beta)]\ ,\end{aligned} \qquad (3.3)$$

where α,β are color indices, $E_i=(q^2+m_i{}^2)^{1/2}, E_j=(q'^2+m_j{}^2)^{1/2}$; the axial-vector current $A_{ij}^{\mu}(0)$ is obtained by substituting γ^{μ} with $\gamma^{\mu}\gamma^5$ in Eq. (3.3).

Physical states that can couple to these currents are ${}^3S_1$, ${}^3P_0$, ${}^3D_1$ (coupled to V^{μ}) and ${}^1S_0$, ${}^1P_1$, ${}^3P_1$ (coupled to A^{μ}); the relevant matrix elements are listed in Table I. Note that the entries of Table I should be multiplied by the flavor factor Q_{ij} [Q is the SU(N_f) matrix of the mesons $Q_{n,l}$].

In order to evaluate the couplings $f_{n,l}$ and $g_{n,l}$ in our approximation scheme, we have to express the states $|\,Q_{n,l}\rangle$ as superpositions of $q\bar{q}$ states with relative momentum $\mathbf{k}$; thus we write

TABLE I. Current-particle matrix elements for different mesons $Q_{n,l}$ having n and l as radial and angular quantum numbers. We employ the spectroscopic notation for the other quantum numbers; all the entries in the table should be multiplied by Q_{ij}, the flavor matrix belonging to the meson $Q_{n,l}$.

${}^{2S+1}L_J$	$\langle 0\,|\,V_{ij}^{\mu}(0)\,|\,Q_{n,l}(p)\rangle$	$\langle 0\,|\,A_{ij}^{\mu}(0)\,|\,Q_{n,l}(p)\rangle$
${}^3S_1$	$\epsilon^{\mu}f_{n,0}$	
${}^3P_0$	$p^{\mu}f_{n,1}$	
${}^3D_1$	$\epsilon^{\mu}f_{n,2}$	
${}^1S_0$		$ip^{\mu}g_{n,0}$
${}^3P_1$		$\epsilon^{\mu}g_{n,1}$
${}^1P_1$		$\epsilon^{\mu}\tilde{g}_{n,1}$

$$|{}^3S_1\rangle=\sum \frac{\delta_\alpha^\beta}{\sqrt{3}}Q_{ij}\int \frac{d^3k}{(2\pi)^3}\frac{(\sigma\cdot\epsilon)_{rs}}{\sqrt{2}}\phi_0(k)\times b_i^\dagger(\mathbf{k},r,\alpha)d_j^\dagger(-\mathbf{k},s,\beta)\,|\,0\rangle\;,$$

$$|{}^3P_0\rangle=\sum \frac{\delta_\alpha^\beta}{\sqrt{3}}Q_{ij}\int \frac{d^3k}{(2\pi)^3}\frac{(\sigma\cdot\hat{\mathbf{k}})_{rs}}{\sqrt{2}}\phi_1(k)\times b_i^\dagger(\mathbf{k},r,\alpha)d_j^\dagger(-\mathbf{k},s,\beta)\,|\,0\rangle\;,$$

$$|{}^3D_1\rangle=\sum \frac{\delta_\alpha^\beta}{\sqrt{3}}Q_{ij}\int \frac{d^3k}{(2\pi)^3}\tfrac{3}{2}(\hat{\mathbf{k}}^a\hat{\mathbf{k}}^b-\tfrac{1}{3}\delta^{ab})\times\epsilon^a\sigma^b_{rs}\phi_2(k)b_i^\dagger(\mathbf{k},r,\alpha)\times d_j^\dagger(-\mathbf{k},s,\beta)\,|\,0\rangle\;,\tag{3.4}$$

$$|{}^1S_0\rangle=\sum \frac{\delta_\alpha^\beta}{\sqrt{3}}Q_{ij}\int \frac{d^3k}{(2\pi)^3}\frac{\delta_{rs}}{\sqrt{2}}\phi_0(k)\times b_i^\dagger(\mathbf{k},r,\alpha)d_j^\dagger(-\mathbf{k},s,\beta)\,|\,0\rangle\;,$$

$$|{}^3P_1\rangle=\sum \frac{\delta_\alpha^\beta}{\sqrt{3}}Q_{ij}\int \frac{d^3k}{(2\pi)^3}\frac{\sqrt{3}}{2}(\sigma\times\epsilon\cdot\hat{\mathbf{k}})_{rs}\phi_1(k)\times b_i^\dagger(\mathbf{k},r,\alpha)d_j^\dagger(-\mathbf{k},s,\beta)\,|\,0\rangle\;,$$

$$|{}^1P_1\rangle=\sum \frac{\delta_\alpha^\beta}{\sqrt{3}}Q_{ij}\int \frac{d^3k}{(2\pi)^3}(\tfrac{3}{2})^{1/2}\delta_{rs}\epsilon\cdot\hat{\mathbf{k}}\phi_1(k)\times b_i^\dagger(\mathbf{k},r,\alpha)d_j^\dagger(-\mathbf{k},s,\beta)\,|\,0\rangle\;.$$

In these equations the sum runs over repeated indices; states are normalized as

$$\langle Q_{n,l}\,|\,Q_{n,l}\rangle=2M_{n,l}\;,\tag{3.5}$$

whereas the wave functions $\phi_l(k)$ are related to $\tilde{u}_l(k)$ appearing in Eq. (2.10) by

$$\phi_l(k)=\frac{\sqrt{2}\pi}{k}\tilde{u}_l(k)\;.\tag{3.6}$$

As a consequence of Eqs. (3.5) and (3.6), $\tilde{u}_l(k)$ are normalized as

$$\int_0^\infty dk\,|\,\tilde{u}_l(k)\,|^2=2M_{n,l}\;.\tag{3.7}$$

By using Eqs. (3.3)—(3.6) and canonical anticommutation relations for the operators b,d one obtains the following formulas for the coupling constants:[9]

$$\begin{aligned}
f_{n,0}&=\frac{\xi}{2\pi}\int_0^\infty dk\,\tilde{u}_0(k)k\frac{(E_i+m_i)^{1/2}(E_j+m_j)^{1/2}}{E_i^{1/2}E_j^{1/2}}\left[1+\frac{1}{3}\frac{E_i-m_i}{E_j+m_j}\right]\;,\\
f_{n,1}&=\frac{\xi}{2\pi M_{n,1}}\int_0^\infty dk\,\tilde{u}_1(k)k^2\frac{E_i-E_j+m_i-m_j}{E_i^{1/2}E_j^{1/2}(E_i+m_i)^{1/2}(E_j+m_j)^{1/2}}\;,\\
f_{n,2}&=\frac{\xi\sqrt{2}}{3\pi}\int_0^\infty dk\,\tilde{u}_2(k)\frac{k^3}{E_i^{1/2}E_j^{1/2}(E_i+m_i)^{1/2}(E_j+m_j)^{1/2}}\;,\\
g_{n,0}&=\frac{\xi}{2\pi M_{n,0}}\int_0^\infty dk\,\tilde{u}_0(k)\frac{(E_i+m_i)^{1/2}(E_j+m_j)^{1/2}}{E_i^{1/2}E_j^{1/2}}\left[1-\frac{E_i-m_i}{E_j+m_j}\right]\;,\\
g_{n,1}&=\frac{\xi}{\sqrt{6}\pi}\int_0^\infty dk\,\tilde{u}_1(k)k^2\frac{E_i+E_j+m_i+m_j}{E_i^{1/2}E_j^{1/2}(E_i+m_i)^{1/2}(E_j+m_j)^{1/2}}\;,\\
\tilde{g}_{n,1}&=\frac{\xi}{2\sqrt{3}\pi}\int_0^\infty dk\,\tilde{u}_1(k)k^2\frac{E_i-E_j+m_i-m_j}{E_i^{1/2}E_j^{1/2}(E_i+m_i)^{1/2}(E_j+m_j)^{1/2}}\;,
\end{aligned}\tag{3.8}$$

where $\xi=\sqrt{3}$ is a color factor.

The couplings $f_{n,l}$ and $g_{n,l}$ will be calculated in the next section by using WKB wave functions; for the time being we exploit the duality ansatz and write the two-point spectral functions $\rho_J^{(a)}(t)$ of Eq. (3.1) in terms of these couplings; the situation is depicted in Fig. 1. One obtains

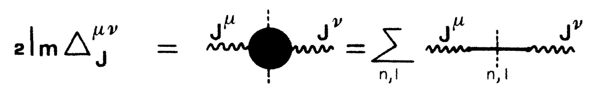

FIG. 1. The hadronic tensor $\Delta_J^{\mu\nu}(q)$ is given, according to the duality relation, by a sum over infinitely many resonances.

$$\begin{aligned}
\rho_V^{(1)}(t)&=\sum_n [f_{n,0}{}^2\delta(t-M_{n,0}{}^2)+f_{n,2}{}^2\delta(t-M_{n,2}{}^2)]\;,\\
\rho_V^{(2)}(t)&=\sum_n [f_{n,0}{}^2\delta(t-M_{n,0}{}^2)+tf_{n,1}{}^2\delta(t-M_{n,1}{}^2)\\
&\quad+f_{n,2}{}^2\delta(t-M_{n,2}{}^2)]\;,\\
\rho_A^{(1)}(t)&=\sum_n [g_{n,1}{}^2\delta(t-M_{n,1}{}^2)+\tilde{g}_{n,1}{}^2\delta(t-M_{n,1}{}^2)]\;,\\
\rho_A^{(2)}&=\sum_n [tg_{n,0}{}^2\delta(t-M_{n,0}{}^2)+g_{n,1}{}^2\delta(t-M_{n,1}{}^2)\\
&\quad+\tilde{g}_{n,1}{}^2\delta(t-M_{n,1}{}^2)]\;.
\end{aligned}\tag{3.9}$$

IV. PROOF OF RELATIVISTIC DUALITY

In the section we shall evaluate the two-point spectral functions (3.9) and shall apply the results to the calculation of the ratio $R_{e^+e^-}$ and the Weinberg sum rules.

To begin with, we evaluate the coupling constants given by Eqs. (3.8) for large n. The integrals appearing in these equations take contributions mainly from the large-k region; as a matter of fact it has been proved elsewhere[9] that for a potential consisting only of a linear confining part, the dominant contributions to such integrals come from values $k \sim M_{n,l}/2$; this feature is not destroyed by the short-range potential [$V_C(r)$ in Eq. (2.16)], which is constant for $r \to 0$ and dies off for $r \to \infty$. By expanding Eqs. (3.8) in powers of m_i/k one gets

$$f_{n,0}=\frac{\xi}{2\pi}\frac{4}{3}\left[Z_{0,0}+\frac{m_i+m_j}{4}Z_{0,1}-\frac{(m_i-m_j)^2}{8}Z_{0,2}\right]+O(m^3/M^2)\,,$$

$$\begin{aligned}
f_{n,1}&=\frac{\xi}{2\pi}\frac{m_i-m_j}{M_{n,1}}Z_{1,1}+O(m^3/M^3)\,,\\
f_{n,2}&=\frac{\xi\sqrt{2}}{3\pi}\left[Z_{2,0}-\frac{m_i+m_j}{2}Z_{2,1}-\frac{(m_i-m_j)^2}{8}Z_{2,2}\right]+O(m^3/M^3)\,,\\
g_{n,0}&=\frac{\xi}{2\pi}\frac{m_i+m_j}{M_{n,0}}Z_{0,1}+O(m^3/M^3)\,,\\
g_{n,1}&=\frac{\xi}{\sqrt{6}\pi}\left[2Z_{1,0}-\frac{(m_i+m_j)^2}{4}Z_{1,2}\right]+O(m^3/M^3)\,,\\
\tilde{g}_{n,1}&=\frac{\xi}{2\sqrt{3}\pi}(m_i-m_j)Z_{1,1}+O(m^3/M^3)\,,
\end{aligned}\tag{4.1}$$

where $Z_{l,r}$ are defined as

$$Z_{l,r}=\int_0^\infty dk\,\tilde{u}_l(k)k^{1-r}\tag{4.2}$$

and calculated in Appendix B. Using Eqs. (B6) and (B9) we obtain

$$\begin{aligned}
f_{n,0}&=\zeta\frac{4}{3}\left[p_0+\frac{m_i+m_j}{4}-\frac{(m_i-m_j)^2}{8}\left[\frac{2}{M_{n,0}}+(-1)^n\frac{\sqrt{\pi}}{\mu}\right]\right]\,,\\
f_{n,1}&=\zeta\frac{m_i-m_j}{M_{n,1}}\,,\\
f_{n,2}&=\zeta\frac{2\sqrt{2}}{3}\left[p_0-\frac{m_i+m_j}{2}-\frac{(m_i-m_j)^2}{8}\left[\frac{2}{M_{n,2}}+(-1)^n\frac{\sqrt{\pi}}{2\mu}\right]\right]\,,\\
g_{n,0}&=\zeta\frac{m_i+m_j}{M_{n,0}}\,,\\
g_{n,1}&=\zeta\frac{\sqrt{2}}{\sqrt{3}}\left[2p_0-\frac{(m_i+m_j)^2}{4}\left[\frac{2}{M_{n,1}}+(-1)^n\frac{2}{\sqrt{\pi}\mu}\right]\right]\,,\\
\tilde{g}_{n,1}&=\zeta(m_i-m_j)/\sqrt{3}\,,
\end{aligned}\tag{4.3}$$

where

$$\zeta=\frac{(-1)^n\mu\sqrt{3}}{\sqrt{2\pi}\sqrt{1+\omega}}\tag{4.4}$$

and μ is defined in (2.16), p_0 in (2.18), and ω in (2.19). From Eqs. (4.3) one can easily derive the two-point spectral functions for vector and axial-vector currents. To start with, we calculate the ratio $R_{e^+e^-}$, related to the spectral function $\rho_V^{(1)}(t)$.

Let us define

$$W_{\mu\nu}(q)=i\int d^4x\,e^{iqx}\langle 0|T^*(J_\mu^{\rm em}(x)J_\nu^{\rm em}(0))|0\rangle\,.\tag{4.5}$$

We have

$$\begin{aligned}
\Pi_{\mu\nu}(q)&=\int d^4x\,e^{iqx}\langle 0|[J_\mu^{\rm em}(x),J_\nu^{\rm em}(0)]|0\rangle\\
&=2\,{\rm Im}\,W_{\mu\nu}(q)=(g_{\mu\nu}q^2-q_\mu q_\nu)\Pi(q^2)\,.
\end{aligned}\tag{4.6}$$

The ratio $R_{e^+e^-}$ is given by

$$R_{e^+e^-}=\frac{\sigma(e^+e^-\to\text{hadrons})}{\sigma(e^+e^-\to\mu^+\mu^-)}=6\pi\Pi(q^2)\,.\tag{4.7}$$

The spectral function $\Pi(q^2)$ is related to $\rho_{\rm em}^{(1)}(q^2)$ by

$$\Pi(q^2)=\frac{2\pi}{q^2}\rho_{\rm em}^{(1)}(q^2)\tag{4.8}$$

so that (e_i = the charge of the i quark)

$$\Pi(q^2)=\frac{2\pi}{q^2}\sum_i e_i^2\sum_n [f_{n,0}{}^2\delta(q^2-M_{n,0}{}^2)+f_{n,2}{}^2\delta(q^2-M_{n,2}{}^2)]$$
$$=\frac{2\pi}{q^2}\sum_i e_i^2\int\frac{dn}{dM}dM[f_{n,0}{}^2\delta(q^2-M^2)+f_{n,2}{}^2\delta(q^2-M^2)]=\sum_i e_i^2\frac{\zeta^2}{2\mu^2q^2}\frac{8}{3}p_0{}^2=\sum_i e_i^2\frac{(1+\omega)}{2\pi}\ . \tag{4.9}$$

Thus we obtain the result

$$R_{e^+e^-}=3\sum_i e_i^2(1+\omega) \tag{4.10}$$

which coincides with the perturbative QCD result provided that

$$\omega=\frac{4}{3k}\alpha_s(q^2)=\frac{\alpha_s(q^2)}{\pi}\ . \tag{4.11}$$

This fixes the constant k in Eq. (2.16): $k=4\pi/3$.

As a further application we evaluate the spectral functions $\rho_{V-A}^{(\alpha)}=\rho_V^{(\alpha)}-\rho_A^{(\alpha)}$. The asymptotic behavior of $\rho_{V-A}^{(\alpha)}$ determines the rate of convergence of the Weinberg sum rules[12] and can be calculated in perturbative QCD (Ref. 13). Using Eqs. (3.9) and (4.3) we obtain

$$\rho_{V-A}^{(1)}(t)=\frac{3m_im_j}{2\pi^2}\left[1-\frac{\alpha_s(t)}{3\pi}\right]\ , \tag{4.12}$$

$$\rho_{V-A}^{(2)}(t)=\frac{m_im_j}{\pi^2}\frac{\alpha_s(t)}{\pi}\ . \tag{4.13}$$

These results are obtained by substituting

$$\sum_n\to\int\frac{dn}{dM}dM$$

as in Eq. (4.9); moreover terms behaving as $(-1)^n$ in $f_{n,l}{}^2$ and $g_{n,l}{}^2$ [see Eq. (4.3)] have been neglected. The reason is that in the finite width of a resonance with mass M one finds, by Eq. (2.20), a large number: $\Delta n\sim\Gamma M/2\pi\mu^2$ of states; this implies that oscillating terms are canceled out in the sum. Equations (4.12) and (4.13) agree with the parton-model and perturbative QCD results quoted in Ref. 13.

Our results, contained in Eqs. (4.10)—(4.13), show that our ansatz on the short-range potential, Eq. (2.16), provides a consistent picture for the asymptotic behavior of two-point spectral function. We have been able to prove, in this way, a relativistic version of duality between asymptotically free quarks and bound states which uses relativistic formulas for the leptonic widths and the wave functions.

A possible application of our result could be in the realm of potential models and quarkonium spectroscopy where our guess on the short-range behavior of the potential could be independently tested.

APPENDIX A

In this appendix we study the behavior of relativistic wave functions near the origin for a potential

$$V(r)\underset{r\to 0}{\sim}-\frac{4}{3}\frac{\alpha(r)}{r}=-\frac{4}{3}\frac{\alpha_0}{r\ln(\bar{r}/r)}\ . \tag{A1}$$

The relativistic wave equation (2.2),(2.3) gives, for $r\to 0$,

$$-\frac{4}{3}\frac{\alpha(r)}{r}u_l(r)+\frac{4}{\pi}\int_0^\infty dr'\int_0^\infty dk\,k\chi_l(kr)\times\chi_l(kr')u_l(r')=0 \tag{A2}$$

because the integral in k takes contributions mainly from the region $k\gg m$. By using

$$\int_0^\infty dy\,y\chi_l(y)\chi_l(zy)=\frac{1}{2z}Q_l'\left[\frac{1+z^2}{2z}\right]\ , \tag{A3}$$

where $Q_l'(x)=dQ_l(x)/dx$ and $Q_l(x)$ are the Legendre functions of the second kind, we can rewrite Eq. (A2) as

$$\tfrac{4}{3}\alpha(r)\phi_l(r)=\frac{2}{\pi}\int_0^\infty dz\,Q_l'\left[\frac{1+z^2}{2z}\right]\phi_l(zr) \tag{A4}$$

with $\phi_l(r)=u_l(r)/r$. Finally we can recast Eq. (A4) into[14]

$$\tfrac{4}{3}\alpha(r)\phi_l(r)=\frac{2}{\pi}\int_0^\infty dz\left[-\frac{\partial^2}{\partial z^2}[z\phi_l(z)]+\frac{l(l+1)}{z}\phi_l(rz)\right]\times Q_l\left[\frac{1+z^2}{2z}\right]\ . \tag{A5}$$

By inspection we find that

$$\phi_l(r)\underset{r\to 0}{\sim}r^{\nu_l}[\ln(\bar{r}/r)]^{\lambda_l} \tag{A6}$$

is a solution of (A5) provided that $\nu_l=l$ and

$$\tfrac{4}{3}\alpha_0=\frac{4}{\pi}(2l+l)\lambda_lI_l\ , \tag{A7}$$

where[14,15]

$$I_l=\tfrac{1}{2}\int_0^\infty dz\,z^{l-1}Q_l\left[\frac{1+z^2}{2z}\right]$$
$$=\int_0^\infty d\theta\,Q_l(\cosh\theta)\cosh\theta=\frac{\pi^2}{4}\frac{\Gamma(l+\frac{1}{2})}{l!\sqrt{\pi}}\ . \tag{A8}$$

Using (A8) we obtain

$$\lambda_l=\tfrac{4}{3}\alpha_0\frac{1}{2\pi}\frac{l!\sqrt{\pi}}{\Gamma(l+\frac{3}{2})} \tag{A9}$$

which coincides with the result of Ref. 10.

A consequence of Eq. (A6) is that the coupling constant $f_{n,0}$ of an S-wave vector meson to the electromagnetic current diverges.

As a matter of fact, in the asymptotic limit, from Eqs. (3.8) in the text one gets

$$f_{n,0} \simeq \frac{2\xi}{3\pi}\int_0^\infty k\tilde{u}_0(k)dk = \frac{2\xi}{3\pi}\int_0^\infty dr\, u_0(r)\int_0^\infty dk\, k \sin kr = \frac{\xi}{3}u_0'(0)=\frac{\xi}{3}\phi_0(0) \tag{A10}$$

that diverges because $\lambda_0>0$.

APPENDIX B

In this appendix we evaluate the integrals

$$Z_{l,r}=\int_0^\infty dk\, \tilde{u}_l(k)k^{1-r} \tag{B1}$$

for $l,r=0,1,2$. As discussed in Sec. II, contributions to $Z_{l,r}$ for $k>p_0$ are negligible. Thus, using Eq. (2.12) we can write

$$Z_{l,r}\simeq \int_0^{p_0} dk\, \lambda_l \chi_l(\Sigma(k))k^{1-r}\,. \tag{B2}$$

First of all we calculate λ_l. According to Eq. (3.7) it will be given by

$$\frac{1}{\lambda_l^2}\simeq\frac{1}{2M_{n,l}}\int_0^{p_0}dp\,\chi_l^2(\Sigma(p)) \simeq\frac{1}{2M_{n,l}}\int_0^{p_0}dp\,\sin^2(\Sigma(p)-l\pi/2) \simeq\frac{1}{2M_{n,l}}\left[\frac{p_0}{2}+O(1/M)\right]=\frac{1+\omega}{8}+O(m^2/M^2)\,, \tag{B3}$$

where p_0 and ω are defined in (2.18) and (2.19), respectively. For $r=0,1$, $Z_{l,r}$ in (B2) takes contributions mainly from the large-k region, thus we may write

$$Z_{l,r}\simeq\lambda_l\int_0^{p_0}dk\,\sin[\Sigma(k)-l\pi/2]k^{1-r} =\mathrm{Im}\lambda_l\int_0^{p_0}dk\,k^{1-r}\exp[i(\Sigma(k)-l\pi/2)] \simeq\frac{1}{2}\left[\frac{2\pi}{|\Sigma''(p_0)|}\right]^{1/2}p_0^{1-r}\sin[\Sigma(p_0)-l\pi/2-\pi/4]\,, \tag{B4}$$

where we have used the saddle-point method to evaluate the integral. $\Sigma(p_0)$ can be calculated from its definition (2.13) and the spectral condition (2.21), with the result

$$\Sigma(p_0)=\frac{M^2}{4\mu^2}[1+O(\alpha/M^2)]\simeq\pi\left[n+\frac{l}{2}+\frac{3}{4}\right]\,. \tag{B5}$$

Together with

$$|\Sigma''(p_0)|=2/\mu^2+O(1/M^2)$$

and Eq. (B3), we obtain ($l=0,1,2$)

$$Z_{l,0}=\frac{\sqrt{2\pi}\mu(-1)^n}{\sqrt{1+\omega}}p_0,\quad Z_{l,1}=\frac{\sqrt{2\pi}\mu(-1)^n}{\sqrt{1+\omega}}\,. \tag{B6}$$

For $r=2$, on the other hand, we have

$$Z_{l,r}\simeq\lambda_l\int_0^{p_0}\frac{dp}{p}\chi_l(\Sigma(p))\simeq\frac{2\mu^2}{M_{n,l}^2}\int_0^{\Sigma(p_0)}dy\frac{\chi_l(y)}{\sqrt{1-y/\Sigma(p_0)}[1-\sqrt{1-y/\Sigma(p_0)}]}\,. \tag{B7}$$

These integrals can be calculated for large n by using the expressions giving $\chi_l(y)$ in terms of trigonometric functions, the formulas[16]

$$\begin{aligned}\int_0^1 dx\,x^{\nu-1}(1-x)^{\mu-1}\sin ax&=-\frac{i}{2}B(\mu,\nu)[{}_1F_1(\nu,\nu+\mu;ia)-{}_1F_1(\nu,\nu+\mu;-ia)]\,,\\ \int_0^1 dx\,x^{\nu-1}(1-x)^{\mu-1}\cos ax&=\tfrac{1}{2}B(\mu,\nu)[{}_1F_1(\nu,\nu+\mu;ia)+{}_1F_1(\nu,\nu+\mu;-ia)]\end{aligned} \tag{B8}$$

and asymptotic expansions for the confluent hypergeometric functions.[17] We obtain

$$\begin{aligned}Z_{0,2}&=\frac{\sqrt{2\pi}\mu(-1)^n}{\sqrt{1+\omega}}\left[\frac{2}{M_{n,0}}+(-1)^n\frac{\sqrt{\pi}}{\mu}\right],\quad Z_{1,2}=\frac{\sqrt{2\pi}\mu(-1)^n}{\sqrt{1+\omega}}\left[\frac{2}{M_{n,1}}+(-1)^n\frac{2}{\sqrt{\pi}\mu}\right],\\ Z_{2,2}&=\frac{\sqrt{2\pi}\mu(-1)^n}{\sqrt{1+\omega}}\left[\frac{2}{M_{2,2}}+(-1)^n\frac{\sqrt{\pi}}{2\mu}\right]\,.\end{aligned} \tag{B9}$$

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