Lorentz invariance and the composite string

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It is shown that in space-times of dimension 3, 4, 6, and 10 there exists a parametrization of the string coordinates which automatically solves the nonlinear constraints of the theory and which permits the implementation of the Lorentz transformations of the string in terms of a linear action upon these parameters. Supersymmetry transformations are also discussed. The prospect of quantizing the string in terms of a set of commutators for these parameters invariant under this action is examined.

I. INTRODUCTION

In the present reincarnation of the string model, the favored method of quantization is allied to functional-integral techniques and is inspired by Polyakov's work.[1] Originally, however, the string was quantized using canonical methods. In their classic paper, Goddard, Goldstone, Rebbi, and Thorn[2] (GGRT) carry out the quantization in the light-cone gauge, using a Fock space constructed from the transverse oscillator modes to create physical states. Although rotations in the transverse space are automatically realized linearly, the full Lorentz algebra must be realized nonlinearly. The closure of this algebra requires both that the dimension of space-time is 26 and that the mass squared of the ground state is $-1/\alpha'$, where α' is the conventional slope parameter. This state is a tachyon. In a later paper, Goddard, Hanson, and Ponzano[3] repeated the calculation using a covariant formulation, employing Dirac's modification of Poisson brackets when first-class constraints are present, with the same conclusions.

The constraints of string theory arise very simply: in order to linearize the equations of motion obtained from the Nambu-Goto reparametrization-invariant Lagrangian,

$$\mathscr{L}=\alpha'\int\int\left[\left(\frac{\partial X^\mu}{\partial\sigma}\frac{\partial X_\mu}{\partial\tau}\right)^2-\left(\frac{\partial X^\mu}{\partial\sigma}\right)^2\left(\frac{\partial X^\nu}{\partial\tau}\right)^2\right]^{1/2}d\sigma\, d\tau\,, \tag{1}$$

one is led, in the now familiar fashion, to the constraints

$$\frac{\partial X^\mu}{\partial\sigma}\frac{\partial X_\mu}{\partial\sigma}+\frac{\partial X^\mu}{\partial\tau}\frac{\partial X_\mu}{\partial\tau}=0\,, \tag{2}$$

$$\frac{\partial X^\mu}{\partial\sigma}\frac{\partial X_\mu}{\partial\tau}=0\,, \tag{3}$$

which may be reexpressed as

$$\frac{\partial X^+}{\partial(\sigma\pm\tau)}\frac{\partial X^-}{\partial(\sigma\pm\tau)}=\left(\frac{\partial X^i}{\partial(\sigma\pm\tau)}\right)^2, \tag{4}$$

where $X^\pm$ denotes the light-cone components $(X^0\pm X^{d-1})$ and X^i denotes the transverse components of the coordinate vector X^μ. There is a parametrization (see Sec. II) of the constraints which exists only in space-time dimensions 3, 4, 6, and 10 based upon the identity[4,5]

$$|z||z'|=|zz'|\,, \tag{5}$$

where the z and z' belong to a division algebra. The various dimensions result from taking z and z' to be real, complex, quaternion, or octonion, respectively.

For the case of four dimensions, for complex numbers, (5) is simply

$$(a^2+b^2)(c^2+d^2)=(ac+bd)^2+(ad-bc)^2 \tag{6}$$

which permits a parametrization of (4) using

$$\begin{aligned}\frac{\partial X^+}{\partial(\sigma+\tau)}&=(a^2+b^2)\,, \quad \frac{\partial X^-}{\partial(\sigma+\tau)}=(c^2+d^2)\,,\\ \frac{\partial X^1}{\partial(\sigma+\tau)}&=(ac+bd)\,, \quad \frac{\partial X^2}{\partial(\sigma+\tau)}=(ad-bc)\end{aligned} \tag{7}$$

for the identity in $(\sigma+\tau)$. There is a similar identity in $(\sigma-\tau)$. Notice that if the variables a, b, c, and d are taken to be Grassmann, the identity (6) is only true if the minus sign in the last term is changed to a plus sign. This possibility is discussed somewhat further in Sec. III.

II. LORENTZ TRANSFORMATIONS

The advantage of the parametrization (6) over that of GGRT, which is a special case of (6), is that Lorentz transformations on X^μ may be implemented by the action of transformations of the constituent fields a,b, etc. This action may be realized in various ways; the simplest can be seen by introducing a complex variable $\xi=(X'^1-iX'^2)/(X'^0+X'^3)$, where the prime denotes partial differentiation with respect to $(\sigma+\tau)$. Then we can rewrite the identity as

$$\xi=\frac{X'^1-iX'^2}{X'^0+X'^3}=\frac{X'^0-X'^3}{X'^1+iX'^2}=\frac{c-id}{a-ib}\,. \tag{8}$$

There is a similar representation in the 6- and 10-dimensional cases, where ξ is represented as the right or left quotient of two quaternions or octonions, respectively. Notice that the parametrization is highly redundant. We observe that Lorentz transformations in (8) will be induced by SL(2, $\mathbb{C}$) transformations on the complex variable ξ of the form

$$\tilde{\xi}=\frac{\alpha\xi+\beta}{\gamma\xi+\delta}\,, \tag{9}$$

where $\alpha,\beta,\gamma,\delta\in\mathbb{C}$ and $\alpha\delta-\beta\gamma=1$. [In the three-dimensional case the transformations are induced by $SL(2,\mathbb{R})$.]

Another way, which more readily generalizes to higher dimensions, is to notice that the parametrizations can all be expressed as follows. Consider two Majorana spinors ψ and χ with either 2, 4, or 8 components and define

$$X'^{+}=\bar{\chi}\chi\,,\quad X'^{-}=\bar{\psi}\psi\,,\quad X'^{i}=\bar{\chi}\Lambda_i\psi\,, \tag{10}$$

where $\bar{\chi}$ denotes the conjugate (transposed) spinor. The matrix $\Lambda_1=I$ and the real, antisymmetric matrices Λ_i $(i=2,\ldots,d-2)$ which satisfy $\{\Lambda_i,\Lambda_j\}=-2\delta_{ij}$ are constructed as follows:[6]

$$\Lambda_2=i\sigma_2 \text{ for } d=4\,,$$

$$\{\Lambda_i\}=\begin{cases} i\sigma_2\otimes I\,, \\ i\sigma_1\otimes\sigma_2\,, & \text{for } d=6\,, \\ i\sigma_3\otimes\sigma_2\,, \end{cases} \tag{11}$$

$$\{\Lambda_i\}=\begin{cases} i\sigma_2\otimes I\otimes I\,, \\ i\sigma_3\otimes\sigma_2\otimes\sigma_3\,, \\ i\sigma_1\otimes\sigma_2\otimes I\,, \\ i\sigma_3\otimes I\otimes\sigma_2\,, & \text{for } d=10\,. \\ i\sigma_1\otimes\sigma_3\otimes\sigma_2\,, \\ i\sigma_3\otimes\sigma_2\otimes\sigma_1\,, \\ i\sigma_1\otimes\sigma_1\otimes\sigma_2\,, \end{cases}$$

An independent set can be found in the cases where $d=6$ by reversing the order of the two tensor products in each Λ. These independent sets together span the real, antisymmetric matrices in $d=6$.

The identity (5) in this representation reads

$$\bar{\chi}\chi\bar{\psi}\psi=(\bar{\chi}\Lambda_i\psi)^2\,. \tag{12}$$

The action of the Lorentz group is generated by four types of matrices,

$$I\otimes[\Lambda_i,\Lambda_j],\quad \sigma_3\otimes\Lambda_i,\quad i\sigma_2\otimes\Lambda_i,\quad \sigma_1\otimes\Lambda_i\,, \tag{13}$$

which act on the column vector

$$\begin{bmatrix}\chi\\ \psi\end{bmatrix} \tag{14}$$

in a representation where all the matrices are real. These matrices close under commutation on the Lorentz group in d dimensions: the number of matrices of the first type is $\frac{1}{2}(d-3)(d-4)$, and of each of the other types is $d-2$. This gives a total of

$$\frac{(d-3)(d-4)}{2}+3(d-2)=\frac{d(d-1)}{2}\,, \tag{15}$$

the correct dimension for $SO(1,d-1)$.

For what follows it will prove useful to exhibit the six infinitesimal transformations in the four-dimensional case explicitly:

$$\begin{bmatrix}a\\b\\c\\d\end{bmatrix}\to\begin{bmatrix}a\\b\\-c\\-d\end{bmatrix},\begin{bmatrix}b\\-a\\-d\\c\end{bmatrix},\begin{bmatrix}c\\d\\-a\\-b\end{bmatrix},\begin{bmatrix}c\\d\\a\\b\end{bmatrix},\begin{bmatrix}d\\-c\\b\\-a\end{bmatrix},\begin{bmatrix}d\\-c\\-b\\a\end{bmatrix}. \tag{16}$$

III. QUANTIZATION

The expression of the string variables as bilinears in the gauge fields χ and ψ, creating what may be called a composite string, naturally invites consideration as to how much of string theory can be recast in terms of these more basic fields. It is evident that, if χ and ψ depend upon $(\sigma+\tau)$ only, then this parametrizes the left-moving modes of X^μ, and the right movers are parametrized in terms of other fields $\tilde{\chi}(\sigma-\tau),\tilde{\psi}(\sigma-\tau)$. Since the parametrization is Lorentz invariant, unlike GGRT, the question arises as to whether it is possible to quantize the string by the imposition of commutation relations among the χ and ψ which preserves the invariance. There is a basic difficulty if χ and ψ are quantized as Majorana spinors, for then the light-cone components in (10) automatically vanish. An analogous problem arises if a particular choice of bosonic quantization is assumed. In the case of four dimensions, it is easy to verify that the commutators

$$\begin{aligned}&[a(z),c(z')]=[d(z),b(z')]\equiv f(z-z')\,,\\ &[a(z),d(z')]=[b(z),c(z')]\equiv g(z-z')\,,\end{aligned} \tag{17}$$

with all the other pairs having vanishing commutators, are invariant under the infinitesimal action of (16). (If invariance up to rescaling is permitted, then there are more possibilities.) Furthermore, by considering variations in the commutator $[a(z),a(z')]=0$ under (16) we can deduce that $f(z-z')$ and similarly $g(z-z')$ are even under interchange $z\leftrightarrow z'$. Evaluation of two typical commutators gives

$$\begin{aligned}\left[\frac{\partial X^1}{\partial z},\frac{\partial X^{1\prime}}{\partial z'}\right]&=(a'd-ad'+b'c-bc')g(z-z')\\ &\quad+(a'c-ac'-b'd+bd')f(z-z')\,,\\ \left[\frac{\partial X^1}{\partial z},\frac{\partial X^{2\prime}}{\partial z'}\right]&=(a'c+ac'-b'd-bd')g(z-z')\\ &\quad-(a'd+ad'+b'c+bc')f(z-z')\,,\end{aligned} \tag{18}$$

where a' denotes $a(z')$, etc. The evaluation of the commutators is tricky as it involves a normal-ordering ambiguity for the definition of X^μ from Eq. (7) assuming f and g to be singular as $z\to z'$. The reason that the commutators (18) are not Lorentz invariant but only covariant is that the Lorentz transformations on the X^μ are realized nonlinearly. This is the same sort of problem which one finds for the X^+ in GGRT, forcing one to 26 dimensions. Suppose one asks for the commutation relations (18) to be satisfied at the classical level. Then a consistent solution of (17) and (18) and their Lorentz transforms is

$$\begin{aligned} &a=ib,\ \ c=id\ , \\ &f=ig=(z-z')\delta'(z-z')\ . \end{aligned} \tag{19}$$

It may be significant that this solution leads back once again to $X^0=X^3=0$, the same restriction which arises using fermionic parameters.

IV. OTHER TRANSFORMATIONS

If we subject a, b, c, and d to an *arbitrary* transformation, then the vector $\partial X^\mu/\partial(\sigma+\tau)$ remains null. It is possible that this parametrization of the constraints by (7) is an avenue to explore the relationship between analytic transformations of ξ and general coordinate transformations. This point is under investigation.

An interesting subclass of transformations are those of supersymmetry where X^μ is regarded as a superfield. We consider the class of transformations which maintain the distance function:[7]

$$s=\xi_1-\xi_2-\theta_1\theta_2\ , \tag{20}$$

where θ_1 and θ_2 are Grassmann parameters. Then up to a conformal scaling factor they include the SL(2, $\mathbb{C}$) transformations (9) together with

$$\theta\to\frac{\theta}{\gamma\xi+\delta} \tag{21}$$

and also the supersymmetry transformations

$$\begin{aligned} &\xi\to\xi+\eta\xi\theta+\nu\theta+\eta\nu\xi\ , \\ &\theta\to\theta-\eta\xi-\nu\ , \end{aligned} \tag{22}$$

where η and ν are Grassmann parameters. Under these transformations the function s transforms as

$$s=s(1+\eta\theta_1)(1+\eta\theta_2)\ . \tag{23}$$

Equivalently these transformations may be realized as an action on a,b,c,d. The transformations

$$a\to a+\hat{a}\theta,\ \ b\to b+\hat{b}\theta,\ \ c\to c,\ \ d\to d\ , \tag{24}$$

correspond to (22) with $\nu=0$, while the transformations

$$a\to a,\ \ b\to b,\ \ c\to c+\hat{c}\theta,\ \ d\to d+\hat{d}\theta\ , \tag{25}$$

correspond to (22) with $\eta=0$.

V. OTHER PARAMETRIZATIONS

There exists an interesting parametrization of the classical solutions of the string equations of motion in four-dimensional Euclidean space due to Eisenhart,[8] and rediscovered by Shaw.[9] It is linear in the parameters $h(z)$ and $k(z)$ where h and k are analytic in the complex variable $z=(\tau+i\sigma)$. It is given by

$$\begin{aligned} &X^0=h-zh'+k'+\text{c.c.}\ , \\ &iX^3=h-zh'-k'-\text{c.c.}\ , \\ &X^1=k-zk'-h'+\text{c.c.}\ , \\ &iX^2=-k+zk'-h'-\text{c.c.}\ , \end{aligned} \tag{26}$$

where c.c. denotes the complex conjugate. It defines a real solution and is the analogue of an instanton for the string. There is no such linear parametrization known in Minkowski space-time.

In 26 dimensions there are hints that three octonions l,m,n can be combined to give a bilinear parametrization of the string using the identity (5) three times:

$$l^2m^2+m^2n^2+n^2l^2=|l\bar{m}|^2+|m\bar{n}|^2+|n\bar{l}|^2\ , \tag{27}$$

where l^2 denotes $|l\bar{l}|$, etc. The left-hand side of this identity may be rewritten in the canonical form

$$(l^2+m^2+n^2)^2-\frac{(2l^2-m^2-n^2)^2}{4}-\frac{3(m^2-n^2)^2}{4}\ , \tag{28}$$

thus providing a partial parametrization in terms of 24 parameters of the constraint (4) in a Lorentzian space of 27 dimensions. The possibility exists that such a space can be coordinatized by the 27-dimensional representation of a noncompact $E(6)$, or even that the 27 parameter exceptional Jordan algebra over the octonions is involved. Work is in progress in deriving a superstring from this point of view. See also Ref. 10. The difficulty is to reduce to 26 dimensions.

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