Probing a nonperturbative method on the S matrix of the nonlinear Schrödinger model in the strong-coupling regime

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A recently proposed nonperturbative scheme of calculation for the S matrix of field theories is tested for the nonlinear Schrödinger model. For the case of two-body scattering, we study the dependence of the approximate S matrix on the following approximation parameters: the momentum-space cutoff Λ, the number of nodes ν, and the scattering time T. We find convergence towards the reference value, when the approximation parameters tend to infinity, in case of a weak as well as a strong coupling constant. Also in the case of overlapping wave packets, which corresponds to an infinite effective coupling constant, we obtain acceptable convergence.

I. INTRODUCTION

Since 't Hooft discovered that QCD is a renormalizable theory,[1] much effort has been devoted to derive the observed properties of strongly interacting particles from first principles. The major achievement in that direction has been certainly Wilson's invention of lattice gauge theory.[2] This has been the starting point of a renewed interest for nonperturbative methods in field theory, and there are, presently, many proposals to improve the standard perturbation theory.[3] One of them is a time-dependent method using a discretization of the Hamiltonian in momentum space. This method has been originally conceived by one of the authors[4] for nonrelativistic few-body scattering and has proved useful for a large class of interactions. For nonrelativistic potential scattering with potentials suitably chosen to guarantee the existence of Moller wave operators, it has been proven that this scheme yields a strong approximation of the wave operators and a weak approximation of the S matrix. In field theory, the analogues of Moller wave operators are more complicated constructions, as is known from the Haag-Ruelle theory; hence one could try to generalize the method for the approximation of the S matrix, which proceeds without wave operators. Recently, we have suggested how to apply this method in field theory, where we have considered, as an example, the ϕ^4 theory.[5] As a result we found in the weak-coupling case numerical agreement of the approximated S matrix with the first-order perturbation-theory S matrix.

The aim of this paper is to test the method for a field theory, which has an analytically known S matrix, in the regime of a weak and a strong coupling constant. For the case of two-particle scattering we study the dependence of the approximate S matrix on the following approximation parameters: the momentum-space cutoff Λ, the number of nodes ν in cutoff momentum space, and the scattering time T. We find convergence towards the reference value when Λ, ν, and T tend to infinity. In particular, we obtain convergence for the following cases of the effective coupling constant: $\lambda_{\text{eff}} \ll 1$, $\lambda_{\text{eff}} \lesssim 1$, and $\lambda_{\text{eff}} \gtrsim 1$. As a general rule one can say that the smaller λ_{eff} is the faster the convergence. Also, in the case of overlapping wave packets, which corresponds to an infinite effective coupling constant, we obtain acceptable convergence. We give a criterion to minimize the errors, which can be applied when the exact solution is unknown. Finally, we observe that a scaling property of the model pertinent in the S matrix is transferred to a very high degree of accuracy to the approximation solution.

The paper is organized as follows. In Sec. II we present the formalism. In Sec. III we briefly describe the nonlinear Schrödinger model. The numerical results are presented in Sec. IV. Section V contains the conclusions.

II. DESCRIPTION OF THE METHOD

The strong approximation of the Moller (SAM) wave operator approach is a calculation scheme for wave operators and the S matrix. It has been invented to solve few-body nonrelativistic potential scattering problems with short-range and/or long-range forces.[4] Denoting by H and H^0 the full and asymptotic Hamiltonians, respectively, one introduces H_n and H_n^0, approximations of H and H^0 in the sense of strong resolvent convergence. Based on this, one introduces $\exp(iH_nT)\exp(-iH_n^0T)$ as an approximation of the Moller wave operator in the sense of strong convergence. Finally,

$$\exp(iH_n^0T)\exp(-i2H_nT)\exp(iH_n^0T)$$

is used as an approximation of the S matrix in the sense of weak convergence.

For nonrelativistic few-body scattering, one can show that this scheme is rigorous. It is also technically simple. The sequence of approximated Hamiltonian operators can be chosen to be finite dimensional; thus the problem of the calculation of $\exp(iH_nT)$ is formally equivalent to the diagonalization of H_n. An important advantage of this approach is the conservation of unitarity. A generalization of the approach to field theories has been presented for the ϕ^4 theory in Ref. 5. There, one has not been able to prove, rigorously, convergence of the approximate S matrix in a manner analogous to the case of nonrelativis-

tic potential scattering. The proof for nonrelativistic potential scattering was based on the property of the Hamiltonian being a well-defined self-adjoint operator in Hilbert space. But, in general, for local field theories H^2 does not exist in Hilbert space, so the generalization of the mathematical proof is not obvious. The solution of this problem has to be sought by looking for a suitable regularization and nonperturbative renormalization, defining the physical S matrix. However, scattering in the nonlinear Schrödinger model does not require any renormalization, so we do not discuss this point further here.

We use a generalization of the approximation scheme from nonrelativistic potential scattering, augmented by the particle-number degree of freedom. This procedure has already been presented elsewhere, so we will only recall the basic steps.

We want to calculate $\langle \Omega_{\text{out}} | S | \Omega_{\text{in}} \rangle$, where the states $| \Omega_{\text{out}} \rangle$, $| \Omega_{\text{in}} \rangle$ are wave packets normalized to unity. We introduce the following approximation procedure.

(1) We substitute H by $H_n = P_n H P_n$, where P_n is the projection operator on the subspace of the Fock space consisting of $1,2,\ldots,n$ particle states.

(2) We introduce a cutoff Λ in momentum space.

(3) The continuum of the cutoff momentum space is discretized by the introduction of a partition in ν cells characterized by the mesh of each cell. Thus we obtain a finite-dimensional Hamiltonian H_N, where $N \equiv (\Lambda, \nu, n)$ is a collective index.

(4) The time limit is substituted by a large, but finite, time T.

Finally, we evaluate the approximate S-matrix element:

$$\langle \Omega_{\text{out}} | S_N(T) | \Omega_{\text{in}} \rangle = \langle \Omega_{\text{out}} | \exp(iH_N^0 T) \exp(-2iH_N T) \exp(iH_N^0 T) | \Omega_{\text{in}} \rangle . \tag{2.1}$$

The evaluation of $\exp(iH_N T)$ is equivalent to the solution of the eigenvalue problem for the finite-dimensional Hamiltonian H_N and H_N^0, respectively. It can be solved using standard techniques or, if the number of states is too large, one can resort to statistical methods. In practice, it appears that symmetries and a suitable choice of parameters can reduce considerably the number of states which have to be taken into account. In general, one has to control the four parameters Λ, ν, n, and T and, in general, one cannot expect uniform convergence. So it would be helpful to have criteria to determine domains of the parameter space which minimize the errors between the exact and the approximate S matrix. One such criterion is based on the intertwining property

$$SH^0 = H^0 S , \tag{2.2}$$

which reflects energy conservation. Considering a state Ω_{out} on which H^0 is invertible, one can calculate

$$\Delta_{H^0} = \left| \frac{\langle \Omega_{\text{out}} | S_N(T) - (H^0)^{-1} S_N(T) H^0 | \Omega_{\text{in}} \rangle}{\langle \Omega_{\text{out}} | S_N(T) | \Omega_{\text{in}} \rangle} \right| . \tag{2.3}$$

For the analytically solvable nonlinear Schrödinger model, one can compare Δ_{H^0} with

$$\Delta = \left| \frac{\langle \Omega_{\text{out}} | S - S_N(T) | \Omega_{\text{in}} \rangle}{\langle \Omega_{\text{out}} | S | \Omega_{\text{in}} \rangle} \right| . \tag{2.4}$$

As will be shown in Sec. IV, one obtains a minimum of Δ whenever Δ_{H^0} has a minimum. Thus Δ_{H^0} might also serve for nonanalytically soluble models to narrow the domain of parameters. A condition on the mesh of the momentum-space discretization, necessary for the convergence of the S matrix, is the requirement that a momentum-space integral such as the overlap of wave packets $\langle \Omega_{\text{out}} | \Omega_{\text{in}} \rangle$ should be well approximated by the discretized sum. In general, the convergence of the S matrix with respect to the different parameters can be checked only numerically and is model dependent. An illustration will be given in the next section for the nonlinear Schrödinger model.

Finally, we would like to mention that this approach is equivalent to summing up all Feynman graphs using a regularization procedure which preserves unitarity. As in lattice theory, continuous space-time symmetries are only recovered in the limit. The question of continuous internal symmetries is deferred to future analysis.

III. APPLICATION TO THE NONLINEAR SCHRÖDINGER MODEL

The nonlinear Schrödinger model (NLSM) is a simple model of a self-interacting field theory. Sometimes refered to as the δ-function gas model, it is a one-dimensional nonrelativistic model, the Hamiltonian being

$$H = \int_{-\infty}^{\infty} \partial_x \phi^* \partial_x \phi + \lambda \phi^* \phi^* \phi \phi \, dx , \tag{3.1}$$

where

$$[\phi(x,t), \phi^*(y,t)] = \delta(x-y) . \tag{3.2}$$

A useful feature of this model is the property that the number operator $N = \int \phi^* \phi \, dx$ commutes with the Hamiltonian. Therefore each sector of the Fock space for NSLM is independent and the particle-number degree of freedom is frozen. As a consequence of this simplification, the scattering states and the S matrix can be calculated very easily.

The free Hamiltonian is taken as

$$H^0 = \int \partial_x \phi^* \partial_x \phi \, dx . \tag{3.3}$$

It has the eigenstates

$$H^0 | k_1, \ldots, k_n \rangle = \left(\sum_i k_i^2 \right) | k_1, \ldots, k_n \rangle , \tag{3.4}$$

where

$$| k_1, \ldots, k_n \rangle = a^\dagger(k_1) \cdots a^\dagger(k_n) | 0 \rangle , \tag{3.5}$$

$$a^\dagger(k) = \int dx \, e^{ikx} \phi^*(x) , \tag{3.6}$$

the vacuum state $| 0 \rangle$ being defined by $a(k) | 0 \rangle = 0$.

The $| \psi \rangle_{\text{in}}$ and $| \psi \rangle_{\text{out}}$ states are obtained in the usual resolvent approach by

$$|\psi(k_1,\ldots,k_n)\rangle_{\text{in,out}} = \lim_{\epsilon\to\pm 0}\sum_{l=0}^{\infty}[G^0(\omega+i\epsilon)H^{\text{int}}]^l\,|\,k_1,\ldots,k_n\rangle\ , \tag{3.7}$$

where

$$G^0(z)=(z-H^0)^{-1},\quad H^{\text{int}}=H-H^0\ , \tag{3.8}$$

$$\omega=\sum_i k_i{}^2\ . \tag{3.9}$$

The S matrix is defined by

$$\langle k_1',\ldots,k_n'\,|\,S\,|\,k_1,\ldots,k_n\rangle = {}_{\text{out}}\langle\psi(k_1',\ldots,k_n')\,|\,\psi(k_1,\ldots,k_n)\rangle_{\text{in}}\ . \tag{3.10}$$

For two-particle scattering the S matrix is given by Thacker:[6]

$$S=(k_2-k_1-i\lambda)/(k_2-k_1+i\lambda),\quad k_1<k_2\ . \tag{3.11}$$

We will only consider the two-body sector in the following; results on N-body states and states with a finite density of modes are given in Ref. 6. Let us note that there is no mass scale in the model; thus the parameter which plays the role of the coupling strength in perturbation theory can only be measured relatively to k_2-k_1. For wave packets with a well-defined momentum, one can define $\lambda_{\text{eff}}=\lambda/(k_2-k_1)$ as an effective coupling constant. $\lambda_{\text{eff}}\ll 1$ means weak coupling and $\lambda_{\text{eff}}\gtrsim 1$ strong coupling.

Because in our approximation we use a momentum-space cutoff Λ, it would be instructive to compare our approximate S matrix not only with the exact S matrix given by Eq. (3.11), but also with the S matrix obtained when a cutoff is imposed on the interaction. The latter can be calculated analytically in a way which is similar to the calculation of the exact S matrix. It is given by

$$S_\Lambda=\left[k_2-k_1-\frac{\lambda}{\pi}\ln\left[\frac{2k_2+\Lambda}{2k_1+\Lambda}\right]-i\lambda\right]\Big/\left[k_2-k_1-\frac{\lambda}{\pi}\ln\left[\frac{2k_2+\Lambda}{2k_1+\Lambda}\right]+i\lambda\right]\ . \tag{3.12}$$

One verifies immediately that S_Λ is unitary and leads in the limit $\Lambda\to\infty$ to the expression (3.11).

In the preceding section we have suggested the use of energy conservation, which is a general property of the S matrix, as a check for our approximate S matrix. The S matrix of the NLSM has another invariance feature, namely, scaling invariance, which could serve the same purpose. From Eq. (3.11) one immediately reads off that the S-matrix element remains invariant, if one multiplies all momenta and the coupling constant by a real number α. This also holds for the cutoff S matrix given by (3.12), if the cutoff Λ is also multiplied by α.

Now let us discuss the approximations used. Let us define the following normalized one-particle Hilbert state:

$$|\psi(k)\rangle=\int_{k-\Delta k/2}^{k+\Delta k/2}dk'\eta_k(k')\,|\,k'\rangle\ ,\qquad \eta_k(k')=(2\pi\,\Delta k)^{-1/2}\ , \tag{3.13}$$

and analogously a two-particle state, by

$$|\psi(k_1),\psi(k_2)\rangle=\int_{k_1-\Delta k/2}^{k_1+\Delta k/2}\int_{k_2-\Delta k/2}^{k_2+\Delta k/2}dk_1'dk_2'\eta_{k_1}(k_1')\eta_{k_2}(k_2')\,|\,k_1',k_2'\rangle\ . \tag{3.14}$$

The partition of the cutoff momentum-space interval $[-\Lambda,\Lambda]$ is chosen such that the cells are disjoint. The simplest choice is an equidistant partition of the ν cells ($k_j=j\,\Delta k$). We write the finite-dimensional projector:

$$P_N=\sum_{j\geq i}|\,\psi(k_i),\psi(k_j)\rangle\langle\psi(k_i),\psi(k_j)\,|\ . \tag{3.15}$$

and $H_N=P_NHP_N$. In this basis the interaction reads

$$\langle\psi(k_{i'}),\psi(k_{j'})\,|\,H^{\text{int}}\,|\,\psi(k_i),\psi(k_j)\rangle=\frac{4\lambda}{2\pi}\Delta k\,\delta_{i'+j',i+j}\ . \tag{3.16}$$

Equation (3.16) displays two properties of the NLSM interaction, which can be used to simplify the solution, i.e., the diagonalization of H_N.

(a) Firstly, Eq. (3.16) reflects total momentum conservation, which is indeed preserved in the discretization. This has the effect of simplifying the problem to the diagonalization of H_N in each subspace of total momentum contributing to the initial and final wave packet. This is not, however, a particular feature of the NLSM. Hence we use for each total momentum Q the reduced set of basis states

$$|\,\psi_I\rangle=|\,\psi(k_i)\psi(k_j),Q=k_i+k_j\rangle\ . \tag{3.17}$$

(b) Secondly, Eq. (3.16) shows that the interaction in the two-body sector is separable after splitting off the momentum-conserving δ function. Using the basis (3.17), we can write Eq. (3.16):

$$\langle\psi_I\,|\,H^{\text{int}}\,|\,\psi_J\rangle=\langle\psi_I\,|\,\chi\rangle\langle\chi\,|\,\psi_J\rangle\ ,\qquad \langle\psi_I\,|\,\chi\rangle=\left[\frac{4\lambda\,\Delta k}{2\pi}\right]^{1/2}\ , \tag{3.18}$$

which clearly displays the separability. Using this basis it is simple to diagonalize H_N. Firstly, one notes that the states $|\psi(k_i),\psi(k_j)\rangle$ diagonalize H_N^0; hence $|\psi_I\rangle$ are eigenstates of H_N^0:

$$H_N^0|\psi_I\rangle=E_I^0|\psi_I\rangle\ . \tag{3.19}$$

Then the diagonalization of H_N,

$$H_N|\phi_I\rangle=E_I|\phi_I\rangle\ , \tag{3.20}$$

is most easily obtained by the solution of the equation for the eigenvalue E_I,

$$1=\sum_J \frac{|\langle\psi_J|\chi\rangle|^2}{E_I-E_J^0}\ , \tag{3.21}$$

and for the eigenvector ϕ_I,

$$\langle\psi_J|\phi_I\rangle=\kappa\frac{\langle\psi_J|\chi\rangle}{E_I-E_J^0}\ , \tag{3.22}$$

where κ is a normalization constant.

IV. NUMERICAL RESULTS

We have calculated the following two-particle S-matrix element $\langle\Omega_1\Omega_2|S|\Omega_1\Omega_2\rangle$, where Ω_i are both one-particle Hilbert spaces given by

$$|\Omega_i\rangle=\kappa_i\int_{k_{i\,\mathrm{low}}}^{k_{i\,\mathrm{up}}}dk\{1-\cos[2\pi(k-k_{i\,\mathrm{low}})/(k_{i\,\mathrm{up}}-k_{i\,\mathrm{low}})]\}|a^\dagger(k)0\rangle\ , \tag{4.1}$$

where κ_i is chosen to normalize $|\Omega_1\Omega_2\rangle$ to unity. Ω_i represents a bell-shaped wave packet with a peak at

$$k_{ip}=(k_{i\,\mathrm{up}}+k_{i\,\mathrm{low}})/2$$

and a half-width of

$$k_{iw}=(k_{ip}-k_{i\,\mathrm{low}})/2\ .$$

First, we consider the case without overlap between Ω_1 and Ω_2, i.e.,

$$k_{1\,\mathrm{low}}<k_{1\,\mathrm{up}}\le k_{2\,\mathrm{low}}<k_{2\,\mathrm{up}}\ .$$

We take $k_{1\,\mathrm{low}}=0.25$ fm^{-1}, $k_{1\,\mathrm{up}}=0.5$ fm^{-1}, $k_{2\,\mathrm{low}}=0.75$ fm^{-1}, and $k_{2\,\mathrm{up}}=1$ fm^{-1}. We should remark here that our definition of an effective coupling constant was given for sharp momentum states. In order to give a meaningful definition for wave-packet states, we replace the sharp momentum by the peak momentum of the wave packet, i.e.,

$$\lambda_{\mathrm{eff}}=\lambda/(k_{2p}-k_{1p})\ . \tag{4.2}$$

As a first example, we study a weak-coupling case with $\lambda=2.0\times10^{-3}$ fm^{-1} corresponding to $\lambda_{\mathrm{eff}}=4.0\times10^{-3}$.

TABLE I. The approximate matrix element $\langle\Omega_1\Omega_2|S|\Omega_1\Omega_2\rangle$ dependent on the parameters T and the number ν of momentum-space cells ($\hbar=1$). The coupling constant is $\lambda=2.0\times10^{-3}$ fm^{-1} and the cutoff interval is $[0,\Lambda]$, with $\Lambda=2.0$ fm^{-1}. The reference value for the exact S matrix is $9.9997\times10^{-1}-8.0822\times10^{-3}i$, while for the cutoff-dependent S matrix it is $9.9997\times10^{-1}-8.0910\times10^{-3}i$.

T (fm^2) \ ν	100	150	200	350
20	9.9998×10^{-1} $-5.3339\times10^{-3}i$	9.9998×10^{-1} $-5.3279\times10^{-3}i$	9.9998×10^{-1} $-5.3252\times10^{-3}i$	9.9998×10^{-1} $-5.3254\times10^{-3}i$
40	9.9997×10^{-1} $-7.6221\times10^{-3}i$	9.9997×10^{-1} $-7.6202\times10^{-3}i$	9.9997×10^{-1} $-7.6204\times10^{-3}i$	9.9997×10^{-1} $-7.6205\times10^{-3}i$
60	9.9997×10^{-1} $-8.0388\times10^{-3}i$	9.9997×10^{-1} $-8.0392\times10^{-3}i$	9.9997×10^{-1} $-8.0386\times10^{-3}i$	9.9997×10^{-1} $-8.0396\times10^{-3}i$
80	9.9997×10^{-1} $-8.0818\times10^{-3}i$	9.9997×10^{-1} $-8.0825\times10^{-3}i$	9.9997×10^{-1} $-8.0829\times10^{-3}i$	9.9997×10^{-1} $-8.0829\times10^{-3}i$
100	9.9997×10^{-1} $-8.0881\times10^{-3}i$	9.9997×10^{-1} $-8.0887\times10^{-3}i$	9.9997×10^{-1} $-8.0890\times10^{-3}i$	9.9997×10^{-1} $-8.0890\times10^{-3}i$
120	9.9997×10^{-1} $-8.0897\times10^{-3}i$	9.9997×10^{-1} $-8.0901\times10^{-3}i$	9.9997×10^{-1} $-8.0903\times10^{-3}i$	9.9997×10^{-1} $-8.0903\times10^{-3}i$
140	9.9997×10^{-1} $-8.0903\times10^{-3}i$	9.9997×10^{-1} $-8.0905\times10^{-3}i$	9.9997×10^{-1} $-8.0907\times10^{-3}i$	9.9997×10^{-1} $-8.0907\times10^{-3}i$
160	9.9997×10^{-1} $-8.0908\times10^{-3}i$	9.9997×10^{-1} $-8.0907\times10^{-3}i$	9.9997×10^{-1} $-8.0908\times10^{-3}i$	9.9997×10^{-1} $-8.0908\times10^{-3}i$

In this case we have taken an asymmetric cutoff interval $[0,\Lambda]$, with $\Lambda=2$ fm^{-1}. The corresponding mesh is $\Delta k=(2/\nu)$ fm^{-1}. We find as reference value of the exact S matrix $0.999\,97-0.808\,22\times10^{-2}i$, while the value of the cutoff-dependent S matrix S_Λ is $0.999\,97-0.809\,10\times10^{-2}i$. The dependence of the approximate S matrix $S_N(T)$ on the approximation parameters ν and T is displayed in Table I. One finds convergence towards the reference value. As one expects, the agreement between $S_N(T)$ and S_Λ, which both depend on the cutoff, is better than the agreement between $S_N(T)$ and S. From Table I one reads off for $T=160$ fm^2 and $\nu=350$ that the relative error between $S_N(T)$ and S is of the order of 10^{-7}, while the relative error between $S_N(T)$ and S is of the order of 10^{-5}. In other words, among the three approximations used in $S_N(T)$, namely, the cutoff Λ, the partition in ν cells, and the finite time T, the largest error comes in this case from the cutoff Λ.

Figure 1 displays, for $\nu=200$ as a function of T, the relative error Δ between $S_N(T)$ and S as defined by Eq. (2.4), but with S replaced by S_Λ. It is compared with Δ_{H^0} given by Eq. (2.3). The function Δ_{H^0} is calculated using only the approximate S matrix $S_N(T)$ and is a measure of the violation of energy conservation introduced by the approximation. One observes that the minimum of Δ_{H^0} corresponds to a region of stability, close to the minimum of the function Δ. Thus the function Δ_{H^0} serves to determine a suitable time parameter T.

In Fig. 2 we display for the case $\lambda=10^{-1}$ fm^{-1} and $\lambda_{\text{eff}}=2\times10^{-1}$, the relative error Δ between the approximate and exact S matrix, keeping the mesh $\Delta k=2\Lambda/\nu=0.03$ fm^{-1} fixed. We display the curves corresponding to (a) $\nu=300$, $\Lambda=4.5$ fm^{-1}; (b) $\nu=500$, $\Lambda=7.5$ fm^{-1}; and (c) $\nu=700$, $\Lambda=10.5$ fm^{-1}. For all three curves their minima and region of stability nearly coincide. For $\nu=700$ the relative error is on the order of 10^{-3}. In Tables II and III we display the results for a strong-coupling case $\lambda=1$ fm^{-1}, $\lambda_{\text{eff}}=2.0$. In comparison to the weak-coupling case, we need to increase the cutoff but also keep the mesh of the cells reasonably small. In Table II we use the cutoff interval $[-\Lambda,\Lambda]$ with $\Lambda=12$

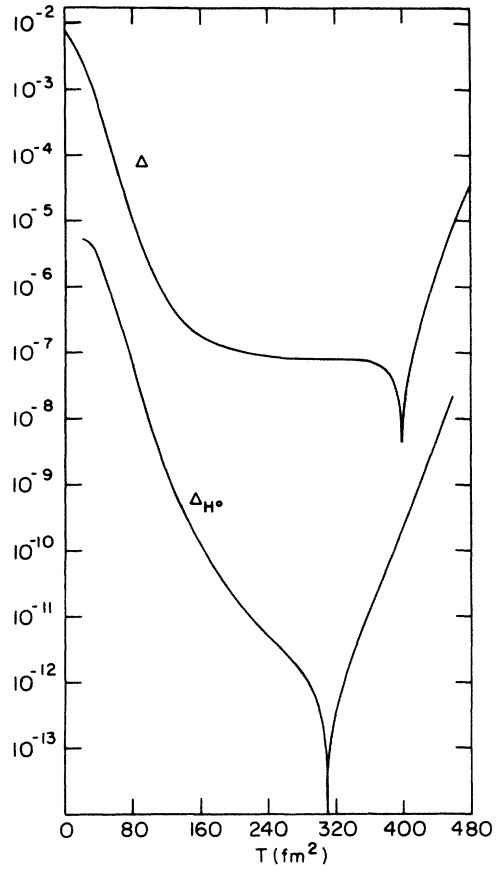

FIG. 1. Graphs of

$$\Delta_{H^0}=\left|\frac{\langle\Omega_1\Omega_2\,|\,S_N(T)-(H^0)^{-1}S_N(T)H^0\,|\,\Omega_1\Omega_2\rangle}{\langle\Omega_1\Omega_2\,|\,S_N(T)\,|\,\Omega_1\Omega_2\rangle}\right|$$

and

$$\Delta=\left|\frac{\langle\Omega_1\Omega_2\,|\,S_N(T)-S_\Lambda\,|\,\Omega_1\Omega_2\rangle}{\langle\Omega_1\Omega_2\,|\,S_\Lambda\,|\,\Omega_1\Omega_2\rangle}\right|$$

for $\lambda=2.0\times10^{-3}$ fm^{-1} and $\nu=200$ as a function of T.

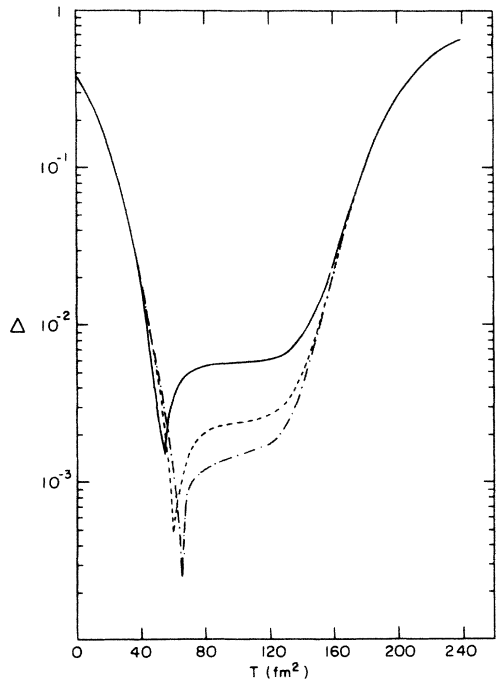

FIG. 2. Graphs of

$$\Delta=\left|\frac{\langle\Omega_1\Omega_2\,|\,S_N(T)-S\,|\,\Omega_1\Omega_2\rangle}{\langle\Omega_1\Omega_2\,|\,S\,|\,\Omega_1\Omega_2\rangle}\right|$$

for $\lambda=10^{-1}$ fm^{-1}. The solid line corresponds to $\nu=300$ and the cutoff interval $[-\Lambda,\Lambda]$ with $\Lambda=4.5$ fm^{-1}; the dashed line corresponds to $\nu=500$ and $\Lambda=7.5$ fm^{-1}; the dashed-dotted line corresponds to $\nu=700$ and $\Lambda=10.5$ fm^{-1}, all corresponding to the same mesh $\Delta k=0.03$ fm^{-1}.

TABLE II. The approximate matrix element $\langle \Omega_1\Omega_2 \mid S \mid \Omega_1\Omega_2 \rangle$ dependent on the parameters T and the number ν of momentum-space cells. The coupling constant is $\lambda=1$ fm^{-1} and the cutoff interval is $[-\Lambda,\Lambda]$, with $\Lambda=12$ fm^{-1}. The reference value for the exact S matrix is $-0.5993-0.7965i$, while for the cutoff-dependent S matrix it is $-0.6322-0.7705i$.

T (fm^2) \ ν	400	600	800	1000
20	-1.4847×10^{-2} $-5.7290\times10^{-1}i$	-1.0238×10^{-2} $-5.7181\times10^{-1}i$	-6.5990×10^{-3} $-5.7120\times10^{-1}i$	-7.0340×10^{-3} $-5.7136\times10^{-1}i$
40	-5.0691×10^{-1} $-7.5365\times10^{-1}i$	-5.0202×10^{-1} $-7.5315\times10^{-1}i$	-5.0723×10^{-1} $-7.5762\times10^{-1}i$	-5.0765×10^{-1} $-7.5761\times10^{-1}i$
60	-6.2140×10^{-1} $-7.2272\times10^{-1}i$	-5.9566×10^{-1} $-7.7782\times10^{-1}i$	-6.0401×10^{-1} $-7.8266\times10^{-1}i$	-6.0407×10^{-1} $-7.8260\times10^{-1}i$
80	-6.7256×10^{-1} $-3.6240\times10^{-1}i$	-6.0891×10^{-1} $-7.8065\times10^{-1}i$	-6.1379×10^{-1} $-7.8411\times10^{-1}i$	-6.1366×10^{-1} $-7.8408\times10^{-1}i$
100	-2.9796×10^{-1} $5.2833\times10^{-1}i$	-6.1681×10^{-1} $-7.6155\times10^{-1}i$	-6.1506×10^{-1} $-7.8411\times10^{-1}i$	-6.1498×10^{-1} $-7.8425\times10^{-1}i$
120	6.0431×10^{-1} $4.7693\times10^{-1}i$	-6.5515×10^{-1} $-6.0867\times10^{-1}i$	-6.1544×10^{-1} $-7.8309\times10^{-1}i$	-6.1525×10^{-1} $-7.8425\times10^{-1}i$
140	7.4939×10^{-1} $-2.4219\times10^{-1}i$	-3.7983×10^{-1} $1.0833\times10^{-1}i$	-6.1620×10^{-1} $-7.7276\times10^{-1}i$	-6.1533×10^{-1} $-7.8412\times10^{-1}i$
160	7.7535×10^{-1} $-3.2125\times10^{-1}i$	3.2934×10^{-1} $5.2413\times10^{-1}i$	-6.3948×10^{-1} $-6.9966\times10^{-1}i$	-6.1548×10^{-1} -7.8343×10^{-1}

fm^{-1} and vary ν between 400 and 1000. The reference value for the exact S matrix is $-0.5993-0.7965i$, while for the cutoff-dependent S matrix the value is $-0.6322-0.7705i$. For $T=160$ fm^{-2} and $\nu=1000$ the relative error between $S_N(T)$ and S is 2.5×10^{-2} and the relative error between $S_N(T)$ and S_Λ is also 2.5×10^{-2}. In Table III we use $[-\Lambda,\Lambda]$ with $\Lambda=24$ fm^{-1}, i.e., the three columns of Table III correspond to the last three columns of Table II with respect to the mesh Δk. In Table III we vary ν between 1200 and 2000. The reference value for the cutoff-dependent S matrix is now $-0.6153-0.7843i$. For $T=100$ fm^2 and $\nu=2000$, the

TABLE III. Same as Table II, with ν and the cutoff Λ doubled. The value of the cutoff-dependent S matrix is $-0.6153-0.7843i$; the value of the exact S matrix is $-0.5993-0.7965i$.

T (fm^2) \ ν	1200	1600	2000
20	-4.8076×10^{-3} $-5.7491\times10^{-1}i$	-1.1705×10^{-3} $-5.7428\times10^{-1}i$	-1.6024×10^{-3} $-5.7445\times10^{-1}i$
40	-4.9460×10^{-1} $-7.5855\times10^{-1}i$	-4.9976×10^{-1} $-7.6304\times10^{-1}i$	-5.0019×10^{-1} $-7.6303\times10^{-1}i$
60	-5.8795×10^{-1} $-7.8371\times10^{-1}i$	-5.9625×10^{-1} $-7.8858\times10^{-1}i$	-5.9631×10^{-1} $-7.8852\times10^{-1}i$
80	-6.0118×10^{-1} $-7.8661\times10^{-1}i$	-6.0601×10^{-1} $-7.9008\times10^{-1}i$	-6.0588×10^{-1} $-7.9005\times10^{-1}i$
100	-6.0937×10^{-1} $-7.6768\times10^{-1}i$	-6.0728×10^{-1} $-7.9009\times10^{-1}i$	-6.0719×10^{-1} $-7.9024\times10^{-1}i$
120	-6.4974×10^{-1} $-6.1620\times10^{-1}i$	-6.0767×10^{-1} $-7.8908\times10^{-1}i$	-6.0746×10^{-1} $-7.9023\times10^{-1}i$
140	-3.8820×10^{-1} $1.0347\times10^{-1}i$	-6.0861×10^{-1} $-7.7881\times10^{-1}i$	-6.0755×10^{-1} $-7.9011\times10^{-1}i$
160	3.1460×10^{-1} $5.3377\times10^{-1}i$	-6.3285×10^{-1} $-7.0651\times10^{-1}i$	-6.0771×10^{-1} $-7.8942\times10^{-1}i$

TABLE IV. The approximate matrix element $\langle \Omega_1\Omega_2 \mid S \mid \Omega_1\Omega_2 \rangle$ for a case of overlapping wave packets. The coupling constant is $\lambda=2.0\times10^{-3}$ fm^{-1} and the parameters of the wave packets are $k_{1\,\mathrm{low}}=0.25$ fm^{-1}, $k_{1\,\mathrm{up}}=0.5$ fm^{-1}, $k_{2\,\mathrm{low}}=0.4$ fm^{-1}, and $k_{2\,\mathrm{up}}=0.65$ fm^{-1}. The reference value of the exact S matrix is $0.998\,17-0.332\,49\times10^{-1}i$.

T (fm^2) \ ν	500	600	700
50	9.9988×10^{-1} $-1.3932\times10^{-2}i$	9.9988×10^{-1} $-1.3935\times10^{-2}i$	9.9988×10^{-1} $-1.3936\times10^{-2}i$
100	9.9972×10^{-1} $-2.2538\times10^{-2}i$	9.9972×10^{-1} $-2.2541\times10^{-2}i$	9.9972×10^{-1} $-2.2543\times10^{-2}i$
200	9.9956×10^{-1} $-2.8089\times10^{-2}i$	9.9956×10^{-1} $-2.8092\times10^{-2}i$	9.9956×10^{-1} $-2.8094\times10^{-2}i$
400	9.9948×10^{-1} $3.0119\times10^{-2}i$	9.9948×10^{-1} $-3.0123\times10^{-2}i$	9.9948×10^{-1} $-3.0125\times10^{2}i$
800	9.9942×10^{-1} $-3.0936\times10^{-2}i$	9.9942×10^{-1} $-3.0940\times10^{-2}i$	9.9942×10^{-2} $-3.0943\times10^{-2}i$
1600	9.9937×10^{-1} $-3.1400\times10^{-2}i$	9.9937×10^{-1} $-3.1404\times10^{-2}i$	9.9937×10^{-1} $-3.1407\times10^{-2}i$
3200	9.9930×10^{-1} $-3.2606\times10^{-2}i$	9.9932×10^{-1} $-3.1795\times10^{-2}i$	9.9932×10^{-1} $-3.1768\times10^{-2}i$

relative error between $S_N(T)$ and S is 1.3×10^{-2} and the relative error between $S_N(T)$ and S_Λ is also 1.3×10^{-2}. We observe in Tables II and III that the value of the approximated S matrix $S_N(T)$ for $T=160$ and $\nu=1000$ (Table II) and $\nu=2000$ (Table III) lies closer to the exact S matrix than the cutoff-dependent S matrix, which may be an accident.

In Table IV we present results for an asymptotic state with overlapping wave packets Ω_1 and Ω_2 given by $k_{1\,\mathrm{low}}=0.25$ fm^{-1}, $k_{1\,\mathrm{up}}=0.5$ fm^{-1}, $k_{2\,\mathrm{low}}=0.4$ fm^{-1}, and $k_{2\,\mathrm{up}}=0.65$ fm^{-1}. We take $\lambda=2.0\times10^{-3}$ fm^{-1} and a cutoff interval $[0,\Lambda]$ with $\Lambda=2$ fm^{-1}. We have seen in Tables I–III that keeping the approximation parameters fixed but increasing the effective coupling constant leads to increased errors, or in order to keep the error level with increased λ_{eff}, one has to increase the approximation parameters. From our original definition of the effective coupling constant $\lambda_{\mathrm{eff}}=\lambda/(k_2-k_1)$, one sees that in the case of overlapping wave packets, there are contributions from $k_2-k_1=0$, i.e., λ_{eff} is infinite. Thus one could expect the worst for the approximate S matrix. The results displayed in Table IV, however, show an acceptable convergence towards the reference value.

Finally we would like to discuss the scale invariance. The exact S matrix has a scaling property which corresponds to the following scaling property of the approximate S matrix. If we denote by $\mid \alpha\Omega \rangle$ a state generated from $\mid \Omega \rangle$ by multiplying all momenta by a common factor α, then one has

$$\langle \Omega_{\mathrm{out}} \mid S^{\lambda}(\Lambda,\nu,T) \mid \Omega_{\mathrm{in}} \rangle = \langle \alpha\Omega_{\mathrm{out}} \mid S^{\alpha\lambda}(\alpha\Lambda,\nu,T/\alpha^2) \mid \alpha\Omega_{\mathrm{in}} \rangle , \quad (4.3)$$

where S^{λ} means the S matrix corresponding to the coupling constant λ. This scaling relation for the approximate S matrix was found to be satisfied within the numerical accuracy of the calculation.

V. CONCLUSION

We have tested a nonperturbative scheme for the calculation of S-matrix elements in quantum field theories on the nonlinear Schrödinger model, which has an analytically known S matrix. We find convergence towards the reference value in the weak- and in the strong-coupling regime. Stronger coupling constants require more numerical effort; i.e., the matrices have to be larger. We have also examined the case of overlapping wave packets, which means an infinite effective coupling constant, and find acceptable convergence. Finally, scale invariance of the S matrix is also observed in the approximate S matrix.

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