# Supersymmetry in curved space

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We formulate supersymmetric quantum mechanics in a three-dimensional curved space of constant curvature. It is shown that the curvature effect reduces the vacuum energy and supersymmetry is broken dynamically. In the limit $R \to \infty$ we recover flat-space (unbroken) supersymmetry.

## I. INTRODUCTION

Supersymmetry[1] (SUSY) is a beautiful mathematical construction and has been studied from a variety of angles over the past few years. One of the thrust areas of research has been that of supersymmetry breaking,[2] since supersymmetry has to be broken at one or another energy scale if it is to describe a realistic spectrum. Another area of vigorous activity is supergravity.[3] Supergravity is the local extension of supersymmetry and such an extension becomes necessary in order to incorporate the effect of curvature.

A few years back a further extension of supersymmetry, namely, supersymmetric quantum mechanics[4] (SSQM) was introduced by Witten. SSQM serves as an excellent laboratory for testing ideas pertaining to supersymmetric field theory. In the present paper we shall formulate SSQM in a three-dimensional space of constant curvature (which is actually three-dimensional Euclidean space $E_3$ embedded in the four-dimensional Euclidean space $E_4$). The situation here closely resembles that of field theory on a sphere. It will be shown that the effect of a nonvanishing curvature reduces the vacuum energy and dynamical supersymmetry breaking takes place. It will also be shown that in the limit $R \to \infty$ we recover the flat-space ($E_3$) result. The organization of the paper is as follows: In Sec. II we describe some essential details of spherical geometry of our space and give expressions for some of the operators; in Sec. III we construct the Hamiltonian; in Sec. IV the energy spectrum is determined and supersymmetry breaking is discussed; finally Sec. V is devoted to a discussion. In the Appendix we discuss the relation between the factorization method[9] and supersymmetry.

## II. GEOMETRY OF THE CURVED SPACE

The three-dimensional curved space on which we shall consider SSQM is a three-dimensional Euclidean space $E_3$ embedded in the Euclidean four-space $E_4$. The coordinate $x_i$, $i=1,2,3,4$ are required to satisfy the following constraint:

$$x_1{}^2 = x_2{}^2 + x_3{}^2 + x_4{}^2 = R^2 . \tag{1}$$

It will be convenient for us to work in angular coordinates rather than Cartesian ones and to that end we write[5]

$$\begin{aligned} x_1 &= R \sin\chi \sin\theta \cos\phi , \\ x_2 &= R \sin\chi \sin\theta \sin\phi , \\ x_3 &= R \sin\chi \cos\phi , \\ x_4 &= R \cos\chi , \end{aligned} \tag{2}$$

where the angles vary in the following ranges:

$$0 \le \chi \le \pi, \quad 0 \le \theta \le \pi, \quad 0 \le \phi \le 2\pi . \tag{3}$$

The line element on the curved three-space is given by

$$ds^2 = R^2[d\chi^2 + \sin^2\chi(d\theta^2 + \sin^2\theta\, d\phi^2)] . \tag{4}$$

The canonical momenta are given by

$$\begin{aligned} p_\chi &= -\frac{i}{R}\frac{\partial}{\partial\chi}, \quad p_\theta = -\frac{i}{R\sin\chi}\frac{\partial}{\partial\theta} , \\ p_\phi &= -\frac{i}{R\sin\chi\sin\theta}\frac{\partial}{\partial\phi} . \end{aligned} \tag{5}$$

It then follows easily that the Laplacian is given by

$$\nabla^2 = \frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial\chi}\left[\frac{h_2h_3}{h_1}\frac{\partial}{\partial\chi}\right] + \frac{\partial}{\partial\theta}\left[\frac{h_3h_1}{h_2}\frac{\partial}{\partial\theta}\right] + \frac{\partial}{\partial\phi}\left[\frac{h_1h_2}{h_3}\frac{\partial}{\partial\phi}\right]\right] . \tag{6}$$

We also need the divergence operator and it is given by

$$\nabla\cdot\mathbf{F} = \frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial\chi}(F_\chi h_2h_3) + \frac{\partial}{\partial\theta}(F_\theta h_3h_1) + \frac{\partial}{\partial\phi}(F_\phi h_1h_2)\right] , \tag{7}$$

where $\mathbf{F}(\chi,\theta,\phi) = (F_\chi, F_\theta, F_\phi)$ and $h_1, h_2, h_3$ are given by

$$h_1 = R, \quad h_2 = R\sin\chi, \quad h_3 = R\sin\chi\sin\theta . \tag{8}$$

[Relation (8) follows from Eq. (4).]

## III. CONSTRUCTION OF THE HAMILTONIAN

In general any quantum-mechanical system whose Hamiltonian can be written as

$$H = \tfrac{1}{2}\{Q, Q^\dagger\} \tag{9}$$

is called supersymmetric, provided the operators $Q$ and

$Q^\dagger$, called supercharges, satisfy the following (anti)commutation relations:

$$\{Q,Q\}=\{Q^\dagger,Q^\dagger\}=0\ , \quad [Q,H]=[Q^\dagger,H]=0\ . \tag{10}$$

For the particular problem at hand, we construct the supercharges in the following manner:[6,7]

$$Q=(-i\mathbf{p}+\mathbf{A})\cdot\boldsymbol{\psi}^\dagger\ , \quad Q^\dagger=(i\mathbf{p}+\mathbf{A})\cdot\boldsymbol{\psi}\ , \tag{11}$$

where the function $A$ is called the superpotential and $\boldsymbol{\psi}^\dagger,\boldsymbol{\psi}$ are Grassmann parameters satisfying

$$\{\psi_i,\psi_j\}=\{\psi_i^\dagger,\psi_j^\dagger\}=0\ , \quad \{\psi_i^\dagger,\psi_j\}=\delta_{ij}\ . \tag{12}$$

Now we have to choose a suitable representation of the $\psi_i$'s and we take them to be[8]

$$\psi_i=\tfrac{1}{2}(C_i+C_{7-i}),\ \ i=1,2,3\ , \tag{13}$$

where the matrices $C_i$ are given by

$$C_1=\frac{-i}{\sqrt{2}}\begin{bmatrix}O^1 & 0\\ 0 & O^{-1}\end{bmatrix},\ C_2=\frac{-i}{\sqrt{2}}\begin{bmatrix}0 & O^1\\ O^1 & 0\end{bmatrix},$$
$$C_3=\frac{-i}{\sqrt{2}}\begin{bmatrix}0 & \Sigma_3\\ \Sigma_3 & 0\end{bmatrix},\ C_4=\frac{-i}{\sqrt{2}}\begin{bmatrix}0 & \Sigma_1\\ \Sigma_1 & 0\end{bmatrix}, \tag{14}$$
$$C_5=\frac{-i}{\sqrt{2}}\begin{bmatrix}O^2 & 0\\ 0 & O^2\end{bmatrix},\ C_6=\frac{-i}{\sqrt{2}}\begin{bmatrix}O^3 & 0\\ 0 & O^3\end{bmatrix},$$

where $\Sigma_1,\Sigma_2,\Sigma_3$ are $4\times4$ representations of Pauli matrices and

$$O^1=i\sigma_2\otimes 1,\ \ O^2=\sigma_3\otimes i\sigma_2,\ \ O^3=\sigma_1\otimes i\sigma_2\ . \tag{15}$$

Using (11) and (13) we get, after a straightforward algebra,

$$H=\tfrac{1}{2}p^2+\tfrac{1}{2}A^2+\frac{\varphi_3}{2}\nabla\cdot\mathbf{A}+i\mathbf{K}\cdot\mathbf{V}\ , \tag{16}$$

where

$$\varphi_3=\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}$$

and $V$ is given by

$$V=i(p_\theta A_\phi+p_\phi A_\theta, p_\phi A_x+p_\chi A_\phi, p_\chi A_\theta+p_\theta A_\chi) \tag{17}$$

and $K$ is a vector, whose explicit form will not be required, formed by linear combination of the matrices $C_i$.

## IV. DETERMINATION OF THE ENERGY SPECTRUM AND DYNAMICAL SUSY BREAKING

For our purpose we shall choose a superpotential which in the flat-space limit becomes the superpotential corresponding to harmonic oscillator. To this end we take

$$A_\chi=wR\tan\chi,\ \ A_\theta=0,\ \ A_\phi=0\ , \tag{18}$$

where $w$ is dimensionless.

Then from (5) and (17) we find

$$\mathbf{K}\cdot\mathbf{V}=0\ . \tag{19}$$

We note that the Hamiltonian given by (16) operates on two-component wave functions

$$f=\begin{bmatrix}f_+\\ f_-\end{bmatrix}.$$

Next we write

$$f_\pm(\chi,\theta,\phi)=F_\pm(\chi)G_\pm(\theta,\phi)\ . \tag{20}$$

Now separating the variables and using (6) and (7) one can write the eigenvalue equations for the "radial" parts $F_\pm(\chi)$ as

$$\left[-\frac{1}{\sin^2\chi}\frac{\partial}{\partial\chi}\left[\sin^2\chi\frac{\partial}{\partial\chi}\right]+\frac{l(l+1)}{\sin^2\chi}+\alpha_\pm{}^2\tan^2\chi-2R^2E'_\pm\right]F_\pm(\chi)=0\ , \tag{21}$$

where

$$\alpha_\pm{}^2=w^2R^2\pm wR\ , \tag{22}$$

$$E'_\pm=E_\pm\mp\frac{3w}{2}\ . \tag{23}$$

In obtaining Eqs. (21) we made use of the fact that the second and third terms of the operator (6) are the angular momentum terms. Our next problem is to solve Eqs. (21). To do this we perform the following transformation:

$$F_\pm(\chi)=\csc\chi K_\pm(\chi) \tag{24}$$

and obtain, from (23),

$$\left[-\frac{d^2}{d\chi^2}+\frac{l(l+1)}{\sin^2\chi}+\alpha_\pm{}^2\tan^2\chi\right]K_\pm(\chi)=\lambda_\pm K(\chi)\ , \tag{25}$$

where $\lambda_\pm$ are given by

$$\lambda_\pm=2R^2E'_\pm+1\ . \tag{26}$$

Now Eq. (25) has to be solved and first we consider the $(+)$ sector. We proceed to solve this equation by the method of factorization.[9] In this method the equation has to be resolved into two factors and we write it as

$$\left[-\frac{d}{d\chi}+P\right]\left[\frac{d}{d\chi}+P\right]K_+(\chi)=[\lambda_+-(l+1)^2-2wR^2(l+1)-(wR^2+1)]K_+(\chi)\ , \tag{27}$$

where

$$P=(wR^2+1)\tan\chi-(l+1)\cot\chi\ . \tag{28}$$

Then the energy eigenvalue can be immediately determined and following Infeld[9] we write

$$E_+^n=(n+3)w+\frac{(n+1)^2}{R^2}\ , \tag{29}$$

where $n=0,1,2,\ldots$ and it denotes the sum of principal quantum numbers.

To obtain the spectrum corresponding to the $(-)$ sector, we write it as

$$\left[-\frac{d}{d\chi}-P_1\right]\left[\frac{d}{d\chi}-P_1\right]K_-(\chi)=[\lambda_- -(l+1)^2-2(wR^2-1)(l+1)-wR^2]K_-(\chi)\ , \tag{30}$$

where

$$P_1=wR^2\tan\chi-(l+1)\cot\chi\ . \tag{31}$$

In this case the spectrum is given by

$$E_-^n=wn+\frac{n^2}{2R^2}-\frac{1}{R^2}\ . \tag{32}$$

In the flat-space limit $R\to\infty$ the energy spectra are given by

$$E_+^{Fn}=(n+3)w\ , \tag{33}$$

$$E_-^{Fn}=wn\ . \tag{34}$$

We note that (33) and (34) agree with the energy levels of a supersymmetric harmonic oscillator.[8] Now from (29) and (32)—(34) we find that

$$E_+^0=3w+1/R^2\ , \tag{35}$$

$$E_+^{F0}=3w\ , \tag{36}$$

$$E_-^0=-1/R^2\ , \tag{37}$$

$$E_-^{F0}=0\ . \tag{38}$$

From (37) it follows that in flat-space supersymmetry is unbroken while (38) tells us that the vacuum energy is reduced in the presence of curvature and supersymmetry is dynamically broken.

Now that all is said and done, a crucial question remains: Are the states of the system described by (30) and (32) acceptable as physical states? In other words, are they normalizable? The answer to this question is in the affirmative[10] and since the proof requires some algebra not connected directly with the contents of this paper, we present it in the Appendix. Incidentally, this also establishes a relation between factorization method and supersymmetry.

## V. DISCUSSION

Quantum mechanics in curved space of the type considered here was first studied by Schrödinger.[11] It was observed[11] that spectral lines in the hydrogen atoms were displaced in the presence of curvature. Here our aim was to find how supersymmetry reacts to the presence of curvature within the framework of supersymmetric quantum mechanics, and it has been shown that curvature effects cause dynamical supersymmetry breaking. The results of this paper can be summarized as follows: consider the mappings $\pi_1:E_4\to S_3$, i.e., $\pi_1:(x_1,x_2,x_3,x_4)\to x_1{}^2+x_2{}^2+x_3{}^2+x_4{}^2=R^2$; $\pi_2:S_3\to E_3$; through the first mapping, curvature is introduced and through the second mapping we come to the usual three-dimensional Euclidean space. It may also be shown[9] that the dynamical symmetry of the SUSY oscillator on $S_3$ is SU(1,1) and that as $R\to\infty$ we recover the flat-space dynamical symmetry group [viz., O(2,1)] of the radial harmonic oscillator.[9,13] Thus the change in the status of SUSY is related to the same dynamical symmetry.

A second aspect that has emerged out of this investigation is the relationship between the factorization method and supersymmetry. It has been shown[10] that those supersymmetric models which are solvable by this method are related to a host of other SSQM systems in a peculiar fashion. It appears that the factorization method is particularly suitable for solving SSQM models without performing much calculation.

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## APPENDIX

In this appendix we shall establish the normalizability of the solutions corresponding to the eigenenergies given by (29) and (32). However, this requires some consideration about one-dimensional SSQM and below we present them.

First note that the supercharges can be written as

$$Q=\left[\frac{1}{i}\frac{d}{d\chi}-iW\right]\sigma_+\ ,\qquad Q^\dagger=\left[\frac{1}{i}\frac{d}{d\chi}+iW\right]\sigma_-\ , \tag{A1}$$

where

$$\sigma_+=\begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix},\quad \sigma_-=\begin{bmatrix}0 & 0\\ 1 & 0\end{bmatrix},$$

and $W(\chi)$ is called the superpotential.

The Hamiltonian is given by

$$H=\tfrac{1}{2}p^2+\tfrac{1}{2}W^2(x)+\tfrac{1}{2}W'(x)\sigma_3\ , \tag{A2}$$

where

$$\sigma_3=\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}$$

and it operates on two-component wave functions:

$$\phi=\begin{bmatrix}\phi_+\\ \phi_-\end{bmatrix}.$$

The condition for unbroken supersymmetry is given by

$$Q\mid g\rangle=Q^\dagger\mid g\rangle=0\ . \tag{A3}$$

Using (A1) in (A3) one finds

$$\phi_+(x)=C_+\exp\left[\int^x W(t)dt\right],\qquad \phi_-(x)=C_-\exp\left[-\int^x W(t)dt\right]. \tag{A4}$$

The ground state is thus given by

$$g=\begin{bmatrix}0\\ \phi_-\end{bmatrix}. \tag{A5}$$

Notice that the bosonic sector can be written as

$$H_B=\tfrac{1}{2}Q^\dagger Q \tag{A6}$$

$$=\frac{1}{2}\left[-\frac{d}{dx}+W\right]\left[\frac{d}{dx}+W\right]. \tag{A7}$$

Now observe that both Eqs. (27) and (30) can be identified with (A7) provided we make the obvious replacements

$$P\to W,\ \ P_1\to W\ . \tag{A8}$$

[The factor $\frac{1}{2}$ appearing in (A7) can clearly by absorbed in the energy.]

We now make an interesting observation: for each of the eigenvalues given by (29) and (32), the right-hand sides of Eqs. (27) and (30), respectively, vanish identically. This implies that for each value of $l$ we have a superpotential and the solutions are the ground-state solutions of supersymmetric systems corresponding to these superpotentials. Therefore, to check renormalizability of the solutions, we have to determine whether or not the ground-state solutions behave properly. To this end, we write

$$P(l;\chi)=(wR^2+1)\tan\chi-(l+1)\cot\chi\ , \tag{A9}$$

$$P_1(l;\chi)=wR^2\tan\chi-(l+1)\cot\chi\ . \tag{A10}$$

From (A4) the ground-state solutions corresponding to (A9) and (A10) are found to be

$$K_+^0(l;\chi)=C_+\exp\left[-\int P(l;\chi)d\chi\right]$$
$$=C_+(\cos\chi)^{wR^2+1}(\sin\chi)^{l+1}\ , \tag{A11}$$

$$K_-^0(l;\chi)=C_-\exp\left[-\int P_1(l;\chi)d\chi\right]$$
$$=C_-(\cos\chi)^{wR^2}(\sin\chi)^{l+1}\ . \tag{A12}$$

Hence for each value of $l$, i.e., $l=0,1,2,\ldots$,

$$K_+^0(l;\chi)\to 0\ \text{ as } \chi\to 0,\pi/2\ , \tag{A13}$$

$$K_-^0(l;\chi)\to 0\ \text{ as } \chi\to 0,\pi/2\ . \tag{A14}$$

(We note that here we have taken $\chi\to\pi/2$, since the interval relevant for the potential under consideration is $0\le\chi\le\pi/2$.)

Hence all the states corresponding to the spectrum given by (29) and (32) are normalizable. Finally, we summarize below some important observations regarding factorization method as applied to the present problem.

(i) Equations (27) and (30), which correspond to the fermionic and bosonic sectors of the original three-dimensional supersymmetric Hamiltonian (16), are not components of a single one-dimensional supersymmetric Hamiltonian; in fact, both of them represent bosonic components of different one-dimensional supersymmetric Hamiltonian.

(ii) Each energy level of Eqs. (27) and (30) correspond to a ground state of a supersymmetric system in one dimension. In other words, in one-dimensional SSQM problems solvable by the factorization method, the ground state of a system generates excited states of another system.

[1]J Wess and J. Bagger, *Supersymmetry and Supergravity* (Princeton University Press, Princeton, NJ, 1983).

[2]E. Witten, Nucl. Phys. **B202**, 253 (1982).

[3]S. J. Gates, M. T. Grisaru, M. Rocek, and W. Siegel, *Superspace or Thousand and One Lessons in Supersymmetry* (Benjamin, Reading, Mass., 1983).

[4]E. Witten, Nucl. Phys. **B188**, 513 (1981).

[5]N. J. Vilenkin, *Special Functions and the Theory of Group Representation* (American Mathematical Society, Providence, 1968).

[6]S. Fubini and E. Rabinovici, Nucl. Phys. **B245**, 17 (1984).

[7]A. Forge and E. Rabinovici, Phys. Rev. D **32**, 927 (1985).

[8]M. de Grombrugghe and V. Rittenberg, Ann. Phys. (N.Y.) **151**, 99 (1983).

[9]L. Infeld, Phys. Rev. **59**, 737 (1941). Also see W. E. Miller, Jr., Mem. Am. Math. Soc. **50**, 1-43 (1964).

[10]See the Appendix.

[11]E. Schrödinger, Proc. R. Irish Acad. Sect. A **46**, 9 (1940).

[12]F. Cooper and B. Freedman, Ann. Phys. (N.Y.) **146**, 262 (1983).

[13]R. K. Roychoudhury and B. Ghosh, in *Group Theoretical Methods in Physics*, edited by H. Araki *et al.* (Lecture Notes in Physics, Vol. 201) (Springer, Berlin, 1983).