Photon propagators and the definition and approximation of renormalized stress tensors in curved space-time

M. R. Brown and A. C. Ottewill
University of Oxford, Department of Astrophysics, South Parks Road, Oxford OX1 3RQ, England
(Received 24 March 1986)

We present the symmetric Hadamard representation for scalar and photon Feynman Green's functions. We use these representations to give a simple definition for their associated renormalized stress tensors. We investigate the connection between the accuracy of the WKB approximation and the vanishing of the trace anomaly for these fields. We show that, although for scalars there is a direct connection, this is not true for photons, and we discuss the relevance of these results to the approximation of renormalized stress tensors in static Einstein space-times.

I. INTRODUCTION

In a recent paper,[1] we, together with Page, showed how to derive approximate values for the renormalized electromagnetic stress tensor in static Einstein space-times from a knowledge of its conformal transformation law. Of central importance to this work was the condition to be satisfied by the curvature of space-time in order that the photon Feynman Green's function be well approximated, in an appropriate sense, by its WKB approximation. We had in mind the analogous situation of conformally invariant scalar field theories. There, one knows that the requirements that the Feynman Green's function have the Hadamard form and that it be symmetric in its space-time arguments together imply that the covariant Taylor-series expansion of the regular part of this function, $W(x,x')$, contains a conserved, symmetric, second-rank tensor t^{ab}, whose trace is the curvature scalar $2v_1$ (Ref. 2). In this case, a necessary condition for the WKB approximation to be a good local approximation in a neighborhood of any point x of the space-time is that $v_1(x)$ should vanish. It is possible to restate this condition as "the trace of the renormalized stress tensor must vanish," provided we agree that the renormalized stress tensor has a trace proportional to v_1. It is generally agreed that this is so; in fact, the renormalized stress tensor is taken to be proportional to the tensor t^{ab} mentioned above. However, especially for higher-spin fields, it need not be the case that the vanishing of the trace of the renormalized stress tensor implies that the WKB approximation is good and vice versa: the trace of the renormalized stress tensor is ambiguous in that it is definition dependent whereas the condition for the WKB approximation to be exact to a given order is an unambiguous constraint on the space-time curvature. The problem confronting us in writing Ref. 1 was to find this condition for the photon Feynman Green's function. In that paper we inferred the condition from a knowledge of the trace anomaly of the renormalized photon stress tensor, as calculated by point-separation techniques, and threatened to give a direct analysis of the Feynman Green's function at a later date. Section III of this paper contains that analysis. It is presented in a way which follows closely the corresponding analysis of the scalar Feynman Green's function—this is reviewed in Sec. II.

This analysis of the photon Feynman Green's function is one reason for writing this paper. Another is to provide a much needed simplification in the definition and calculation of renormalized stress tensors. Our dissatisfaction with the current status of these objects arose from an investigation of the literature in an attempt to disentangle from the vagaries of renormalization statements about well-defined propagators. Here there are a number of comments worth making; many have been made before both by ourselves and others and we repeat them now in an attempt to set the record straight.

The essential role of a renormalized expectation value of a stress tensor operator in a given state $|A\rangle$—let us call it $T_R{}^{ab}[A]$—is that it should provide an absolute measure of the energy-momentum density of matter that is in the state $|A\rangle$. For any given state $|A\rangle$ there is an ambiguity in the definition of $T_R{}^{ab}[A]$ which can be parametrized by the addition of any symmetric, conserved, geometrical, second-rank tensor.[3] There are many such tensors—the ambiguity extends far beyond those obtained from actions quadratic in the curvature. The ambiguity of the coefficient of the $\Box R$ contribution to the vector trace anomaly[4] is merely the tip of the iceberg. In practice, this difficulty manifests itself where different methods of regularization lend themselves, more or less naturally, to different definitions of what is to constitute renormalization. Regularization techniques are themselves conditioned by the chosen representation of the Feynman Green's function—there are three serious contenders: a mode-sum representation, the DeWitt series representation based on the Schwinger-DeWitt proper-time integral, and the Hadamard series representation.

In giving a prescription for renormalization it is important that one gives a method that can be applied to all space-times. Hence one needs to give a representation of the Feynman Green's function valid in all space-times. It is very difficult to say anything about mode sums in anything other than highly symmetric space-times and we shall not discuss them further. This leaves two possible representations and here is the first simplification that can be made: We would argue strongly that the DeWitt series representation of the Feynman Green's function should be abandoned in favor of the Hadamard series representation. Both representations agree in their description of the singularity structure of the Feynman Green's function but the DeWitt series representation has the profound disadvantage that, in general, it is known not to converge.[5] Its only legitimate use is for massive theories where it provides an asymptotic solution of the wave equation, valid when the Compton wavelength of the particle is much less than the characteristic radius of curvature of the space-time; but here again this information is easily extracted from the Hadamard series. The interested reader can find the details in Ref. 6.

If it can be agreed that the DeWitt series representation be dropped then one also removes the need for dimensional and ζ-function regularization of the stress-tensor operator. These methods can be applied to the Hadamard representation[7] but the singularity structure of the Feynman Green's function is already made manifest as a function of the geodesic distance between its two space-time arguments and any further parametrization is unnecessary.

This leaves those methods of renormalization which make use of the Hadamard series representation of the Feynman propagator. Most significant among these is the work of Adler, Lieberman, and Ng,[8] as later corrected and modified by Wald.[9] What we propose as a definition of $T_R{}^{ab}[A]$ is essentially a more tidy version of this work—no disrespect is intended. Our definition is given in Secs. II and III for scalars and vectors, respectively.

In Sec. IV we shall discuss the consequences of our analysis for the approximation scheme of Ref. 1. In Sec. V we conclude with some remarks on how renormalized stress tensors should and should not be used.

Our space-time conventions follow those of Hawking and Ellis[10] and we shall work in natural units ($\hbar=G=c=k=1$).

II. SCALAR FIELD THEORY

Here we shall review the essential elements of, and describe the renormalization of the stress tensor for, a free scalar field theory in a curved space-time. The theory will be taken to have action functional

$$S[\phi]=\tfrac{1}{2}\int d^4x\, g^{1/2}\phi(\Box-\xi R-m^2)\phi\ . \tag{2.1}$$

This action gives rise to the field equation

$$g^{-1/2}\frac{\delta S}{\delta\phi}=(\Box-\xi R-m^2)\phi=0\ , \tag{2.2}$$

and the classical stress tensor T^{ab} defined by the equation

$$T^{ab}\equiv 2g^{-1/2}\frac{\delta S}{\delta g_{ab}} \tag{2.3}$$

$$=(1-2\xi)\phi^{;a}\phi^{;b}+(2\xi-\tfrac{1}{2})g^{ab}\phi_{;c}\phi^{;c}-2\xi\phi\phi^{;ab}+2\xi g^{ab}\phi\Box\phi+\xi(R^{ab}-\tfrac{1}{2}Rg^{ab})\phi^2-\tfrac{1}{2}m^2g^{ab}\phi^2\ . \tag{2.4}$$

It will prove convenient to write this tensor as

$$T^{ab}=[\,\mathcal{T}^{ab}(\phi(x)\phi(x'))]\equiv\lim_{x'\to x}\mathcal{T}^{ab}(\phi(x)\phi(x'))\ , \tag{2.5}$$

where $\mathcal{T}^{ab}=\mathcal{T}^{ab}(x,x')$ is a differential operator defined in any way so as to give the limit (2.4), for example,

$$\begin{aligned}\mathcal{T}^{ab}=&(1-2\xi)g_{b'}{}^{b}\nabla^a\nabla^{b'}+(2\xi-\tfrac{1}{2})g^{ab}g_{c'}{}^{c}\nabla^c\nabla^{c'}\\&-2\xi\nabla^a\nabla^b+2\xi g^{ab}\nabla_c\nabla^c\\&+\xi(R^{ab}-\tfrac{1}{2}Rg^{ab})-\tfrac{1}{2}m^2g^{ab}\ ,\end{aligned} \tag{2.6}$$

where $g_{b'}{}^{b}$ denotes the bivector of parallel transport, which is defined by the equation $\sigma^{;c}g_{b'}{}^{b}{}_{;c}=0$ together with the boundary condition that it be equal to the identity matrix when $x'=x$.

The inhomogeneous wave equation

$$(\Box-\xi R-m^2)G(x,x')=-\delta(x,x') \tag{2.7}$$

admits the Hadamard solutions[11,12]

$$G(x,x')=\frac{i}{8\pi^2}\left[\frac{\Delta^{1/2}}{\sigma+i\epsilon}+V\ln(\sigma+i\epsilon)+W\right]\ , \tag{2.8}$$

where $2\sigma(x,x')$ denotes the square of the geodesic distance between x and x',

$$\Delta\equiv-g^{-1/2}(x)\det(\sigma_{;ab'})g^{-1/2}(x')$$

is the biscalar form of the VanVleck-Morette determinant. $V(x,x')$ and $W(x,x')$ are regular biscalar functions possessing expansions of the form

$$\begin{aligned}V(x,x')&=\sum_0^\infty V_n(x,x')\sigma^n\ ,\\W(x,x')&=\sum_0^\infty W_n(x,x')\sigma^n\ ,\end{aligned} \tag{2.9}$$

where again $V_n(x,x')$ and $W_n(x,x')$ are regular biscalar functions. Imposing Eq. (2.7) for $x\neq x'$ it follows that V and W satisfy the equations

$$(\Box-\xi R-m^2)V=0 \tag{2.10}$$

and

$$\begin{aligned}\sigma(\Box-\xi R-m^2)W=&-2V-2\sigma^{;a}(V_{;a}-V\Delta^{-1/2}\Delta^{1/2}{}_{;a})\\&-(\Box-\xi R-m^2)\Delta^{1/2}\ .\end{aligned} \tag{2.11}$$

In turn these equations yield differential recursion relations for the coefficients V_n and W_n (Ref. 12). These recursion relations completely determine the coefficients V_n ($n\geq 0$); they also determine the coefficients W_n ($n\geq 1$) once $W_0(x,x')$ is given. $W_0(x,x')$ is undetermined and corresponds to the freedom to add to $G(x,x')$ solutions to the homogeneous wave equation.

In the quantum theory the Feynman two-point function for a unit-norm state $|\,A\,\rangle$ is defined by the equation

$$G_A(x,x')=i\langle A \mid T(\hat{\phi}(x)\hat{\phi}(x')) \mid A\rangle , \qquad (2.12)$$

where T denotes time ordering. By definition $G_A(x,x')$ must be symmetric in x and x', and satisfy the inhomogeneous wave equation (2.7). We shall consider only those states whose related Feynman two-point functions have the Hadamard form:

$$G_A=\frac{i}{8\pi^2}\left[\frac{\Delta^{1/2}}{\sigma+i\epsilon}+V\ln(\sigma+i\epsilon)+W_A\right] . \qquad (2.13)$$

The symmetry of the Green's function $G_A(x,x')$ is equivalent to the symmetry of the regular biscalar $W_A(x,x')$.

As is clear and well known, the naive expression for the expectation value of the stress-tensor operator in the state $\mid A\rangle$,

$$\langle A \mid \hat{T}^{ab} \mid A\rangle \equiv [\mathcal{T}^{ab}G_A(x,x')] ,$$

is divergent and therefore meaningless. This reflects the deeper problem that the stress-tensor *operator* has not been defined. If $\mid B\rangle$ is another unit-norm state whose Feynman two-point function has the Hadamard form then the difference in stress-energy density between the states $\mid A\rangle$ and $\mid B\rangle$ is well defined and is given by the equation

$$\begin{aligned}\langle A \mid \hat{T}^{ab} \mid A\rangle-\langle B \mid \hat{T}^{ab} \mid B\rangle &\equiv[\mathcal{T}^{ab}(G_A(x,x')-G_B(x,x'))]\\ &=[\mathcal{T}^{ab}(W_A(x,x')-W_B(x,x'))] .\end{aligned} \qquad (2.14)$$

Unlike the tensor $[\mathcal{T}^{ab}G_A(x,x')]$, the tensor

$$\tau^{ab}[W_A]\equiv[\mathcal{T}^{ab}W_A(x,x')] , \qquad (2.15)$$

is clearly well defined, and it is of interest to study its properties. Later we shall relate it to the renormalized stress tensor but here our discussion is quite independent of this.

$\tau^{ab}[W_A]$ is symmetric, but, as we shall now show, it is not, in general, conserved. Under the infinitesimal coordinate transformation

$$x^a \to x^a+\delta x^a ,$$

the metric and scalar fields transform as

$$g_{ab}\to g_{ab}-\delta x_{a;b}-\delta x_{b;a},\quad \phi\to\phi-\phi_{;a}\delta x^a .$$

The action (2.1) is invariant under this transformation, so

$$\frac{\delta S}{\delta x^a}=\frac{\delta S}{\delta g_{bc}}\frac{\delta g_{bc}}{\delta x^a}+\frac{\delta S}{\delta\phi}\frac{\delta\phi}{\delta x^a}=0 . \qquad (2.16)$$

It follows that

$$T^{ab}{}_{;b}=\phi^{;a}(\Box-\xi R-m^2)\phi . \qquad (2.17)$$

In turn this implies that

$$\tau^{ab}{}_{;b}[W_A]=[g^{aa'}\nabla_{a'}(\Box-\xi R-m^2)W_A(x,x')] . \qquad (2.18)$$

Equation (2.12) gives

$$\begin{aligned}(\Box-\xi R-m^2)W(x,x') &=-6V_1(x,x')-2V_1(x,x')_{;a}\sigma^a+O(\sigma)\\ &=-6v_1(x)+2v_1(x)_{;a}\sigma^a+O(\sigma) ,\end{aligned} \qquad (2.19)$$

where $\sigma^a\equiv\sigma^{;a}$, $v_1(x)\equiv[V_1(x,x')]$, and we have used the fact that

$$V_1(x,x')=v_1(x)-\tfrac{1}{2}v_1(x)_{;a}\sigma^a+O(\sigma) , \qquad (2.20)$$

see Appendix A [and Eq. (2.31)]. Thus, we find that

$$\tau^{ab}{}_{;b}[W_A]=-2v_1{}^{;a} . \qquad (2.21)$$

The trace of $\tau^{ab}[W_A]$ is readily computed to be

$$\tau^a{}_a[W_A]=-6v_1(x)+\tfrac{1}{2}(6\xi-1)\Box w_A(x)-m^2w_A(x) , \qquad (2.22)$$

where $w_A(x)\equiv[W_A(x,x')]$.

Elsewhere[6] we have argued that the quantum theory should properly be formulated in terms of differences such as in Eq. (2.14). Here, however, we wish to make contact with standard renormalization theory and obtain an absolute measure of the stress-energy density of the state $\mid A\rangle$. We define the renormalized stress tensor to be

$$T_R{}^{ab}[A]\equiv\frac{1}{8\pi^2}(\tau^{ab}[W_A]+2v_1g^{ab}) . \qquad (2.23)$$

Clearly this definition is closely related to that of Adler, Lieberman, and Ng[8] as modified by Wald;[9] however, we feel that it has the advantage of being more direct.

The tensor $T_R{}^{ab}[A]$ is obviously conserved; its trace is given by

$$T_R{}^a{}_a[A]=\frac{1}{8\pi^2}[2v_1(x)+\tfrac{1}{2}(6\xi-1)\Box w_A(x) -m^2w_A(x)] . \qquad (2.24)$$

For the conformally invariant theory ($\xi=\frac{1}{6},m^2=0$) we obtain the standard trace anomaly

$$T_R{}^a{}_a[A;\xi=\tfrac{1}{6},m^2=0]=\frac{1}{4\pi^2}v_1(x) . \qquad (2.25)$$

The definition of the renormalized stress tensor given here satisfies Wald's axioms[3] and so agrees with other definitions up to the possible addition of conserved geometrical tensors. Among these, a standard ambiguity lies in the choice of the implicit length scale making the argument of the logarithm in Eq. (2.8) dimensionless. In our prescription this ambiguity corresponds to adding to $W_A(x,x')$ an arbitrary multiple of $V(x,x')$ (which is a symmetric, geometrical solution to the homogeneous wave equation). The resulting additional tensor $\tau^{ab}[V]\equiv[\mathcal{T}^{ab}V(x,x')]$ is a conserved, geometrical tensor derivable as the metric variation of an action at most quadratic in the curvature. Of particular interest is the conformally invariant theory for which it can be shown that

$$\tau^{ab}[V]=-\tfrac{1}{240}B^{ab} , \qquad (2.26)$$

where B^{ab} is the Bach tensor, which is defined by the equation

$$B^{ab} \equiv g^{-1/2} \frac{\delta}{\delta g_{ab}} \int d^4x\, g^{1/2} C_{cdef} C^{cdef} \tag{2.27}$$

$$= 2C^{acdb}R_{cd} + 4C^{acdb}{}_{;cd} \ . \tag{2.28}$$

From the definition (2.27), B^{ab} is symmetric, conserved, and trace-free. We give useful alternative expressions for B^{ab} in Appendix B.

It is of interest to discuss $\tau^{ab}[W_A]$ in relation to the symmetry of the Green's function $G_A(x,x')$. Since $V(x,x')$ and $\Delta(x,x')$ are symmetric[13] the symmetry of $G_A(x,x')$ is equivalent to the symmetry of the regular biscalar $W_A(x,x')$.

An arbitrary regular biscalar $U(x,x')$ can be expanded in a covariant Taylor series:

$$U(x,x') = u(x) + u_a(x)\sigma^a + \tfrac{1}{2}u_{ab}(x)\sigma^a\sigma^b + \tfrac{1}{6}u_{abc}(x)\sigma^a\sigma^b\sigma^c + \cdots \ . \tag{2.29}$$

If the biscalar is symmetric in x and x' then the odd coefficients in this series are determined in terms of the even ones. The general symmetry constraint on the coefficients can be written

$$\sum_{r=0}^{n} \begin{pmatrix} n \\ r \end{pmatrix} u_{(a_1 a_2 \cdots a_r ; a_{r+1} \cdots a_n)} = (-1)^n u_{a_1 a_2 \cdots a_n} \ . \tag{2.30}$$

This follows from taking the symmetric nth derivative of the Taylor-series expansion of the equation $U(x,x') = U(x',x)$ and noting that $[\sigma_{;(a_1 a_2 \cdots a_r)}] = 0$ for $r \geq 3$. The lowest-order coefficients are related by the equations

$$u_a = -\tfrac{1}{2}u_{;a} \ , \tag{2.31}$$

$$u_{abc} = -\tfrac{3}{2}u_{(ab;c)} + \tfrac{1}{4}u_{;(abc)} \ . \tag{2.32}$$

Using these equations to write

$$W(x,x') = w(x) - \tfrac{1}{2}w_{;a}(x)\sigma^a + \tfrac{1}{2}w_{ab}(x)\sigma^a\sigma^b - \tfrac{1}{4}[w_{(ab;c)}(x) - \tfrac{1}{6}w_{;(abc)}(x)] \times \sigma^a\sigma^b\sigma^c + \cdots \tag{2.33}$$

one can express $\tau^{ab}[W]$ and $T_R{}^{ab}$ in terms of w and w^{ab}. This relationship can be inverted to give

$$w^{ab} = -\tau^{ab}[W] - 3v_1 g^{ab} + \tfrac{1}{2}(1-2\xi)w^{;ab} + \tfrac{1}{2}(2\xi - \tfrac{1}{2})\Box w g^{ab} + \xi R^{ab} w \tag{2.34}$$

$$= -8\pi^2 T_R{}^{ab} - v_1 g^{ab} + \tfrac{1}{2}(1-2\xi)w^{;ab} + \tfrac{1}{2}(2\xi - \tfrac{1}{2})\Box w g^{ab} + \xi R^{ab} w \ . \tag{2.35}$$

Conversely, we may ask, given a regular, symmetric biscalar, $W(x,x')$, when does the Hadamard distribution defined by Eq. (2.8) having W as its regular part satisfy the inhomogeneous wave equation (2.7) to order σ? The answer is straightforward: Using $W(x,x')$ we may define a symmetric second-rank tensor $T_R{}^{ab}$ by Eq. (2.35). Then Eq. (2.7) will be satisfied to this order if and only if $T_R{}^{ab}$ is conserved and has trace given by Eq. (2.24). In particular, for the conformally invariant theory, given a symmetric tensor t^{ab} satisfying the equations

$$t^{ab}{}_{;b} = 0 \tag{2.36}$$

and

$$t^a{}_a = \frac{1}{4\pi^2} v_1 \ , \tag{2.37}$$

there exist Hadamard Green's functions satisfying Eq. (2.7) to order σ for which the renormalized stress tensor is t^{ab}. It should be stressed that this is a local statement: it may be that there exist global obstructions to obtaining this Green's function as the vacuum expectation value of a time-ordered product of field operators for a Fock vacuum state.

We appreciate that many of the equations in this section appear elsewhere in the literature. We have presented them in a way which we feel makes most plain the logical structure of the theory. The key element in our development has been the well-defined Hadamard representation of symmetric Feynman Green's functions. The regular parts of these functions contain information about the energy-momentum content of states of massive and massless matter. However, the definition of a renormalized stress tensor is a matter of convention. Our convention, contained in Eq. (2.23), has been chosen in such a way as to be in agreement with those generally accepted,[4] but has been based as firmly as possible on the physical Green's functions of the theory in the physical space-time.

III. ELECTROMAGNETISM

Electromagnetism presents the technical complication of gauge invariance. In terms of the vector potential, $A_a(x)$, the Maxwell action is

$$S_M = -\tfrac{1}{4} \int d^4x\, g^{1/2} F_{ab} F^{ab} \ , \tag{3.1}$$

where $F_{ab} \equiv A_{b;a} - A_{a;b}$, and is invariant under the gauge transformation $A_a \to A_a + \Lambda_{;a}$ for an arbitrary scalar field Λ. The wave equation derived from this action is

$$\Box A_a - R_a{}^b A_b - \nabla_a (A_b{}^{;b}) = 0 \ . \tag{3.2}$$

Gauge invariance implies that this wave operator is singular. Hence to continue our discussion of Green's functions we follow the standard procedure of adding to the action (3.1) a gauge-breaking term and introducing a compensating complex ghost field $c(x)$. We choose these actions to have the form

$$S_{\text{GB}} = \tfrac{1}{2} \int d^4x\, g^{1/2} (A_a{}^{;a})^2 \tag{3.3}$$

and

$$S_{\text{Gh}} = -\tfrac{1}{2} \int d^4x\, c^* \Box c \ . \tag{3.4}$$

This choice of S_{GB}, corresponding to the covariant Lorentz gauge $A_a{}^{;a} = 0$, has the advantage that the wave equation derived from the combined action $S_M + S_{\text{GB}}$,

$$\Box A_a - R_a{}^b A_b = 0 \ , \tag{3.5}$$

admits Feynman Green's functions with the Hadamard form. This means that the vector Feynman Green's function $G_{ab'}(x,x')$ can be written as[12]

$$G_{ab'}=\frac{i}{8\pi^2}\left[\frac{\Delta^{1/2}g_{ab'}}{\sigma+i\epsilon}+V_{ab'}\ln(\sigma+i\epsilon)+W_{ab'}\right], \tag{3.6}$$

where $V_{ab'}(x,x')$ and $W_{ab'}(x,x')$ possess expansions similar to those for the scalar field.

The ghost field $c(x)$ is a complex scalar field satisfying the minimally coupled wave equation

$$\Box c(x)=0 . \tag{3.7}$$

We shall denote its Feynman Green's function by $G(x,x')$.

The theory defined by the total action $S\equiv S_M+S_{\rm GB}+S_{\rm Gh}$ corresponds to electromagnetism when the Green's functions $G_{ab'}$ and G are related by the equation[12]

$$G_{ab'}{}^{;a}+G_{;b'}=0 , \tag{3.8}$$

which expresses the gauge invariance of the action. Equations (3.7) and (3.8) require that G has the Hadamard form appropriate to a minimally coupled scalar field. Inserting the Hadamard expansion for these Green's functions into Eq. (3.8) one obtains the equations

$$V_{ab'}{}^{;a}+V_{;b'}=0 \tag{3.9}$$

and

$$(\Delta^{1/2}g_{ab'})^{;a}+\Delta^{1/2}{}_{;b'}+V_{ab'}\sigma^a+V\sigma_{b'}+\sigma(W_{ab'}{}^{;a}+W_{;b'})=0 , \tag{3.10}$$

where V and W are the biscalars appearing in the Hadamard representation of the ghost propagator. Equation (3.9) is an identity on the geometrical bitensors $V_{ab'}$ and V, while Eq. (3.10) is a constraint on the state-dependent bitensors $W_{ab'}$ and W.

The classical stress tensors $T_M{}^{ab}$, $T_{\rm GB}{}^{ab}$, and $T_{\rm Gh}{}^{ab}$ are defined in the standard way in terms of metric variations of their associated actions. The expectation values of the bare quantum operators in the state $|A\rangle$ can be expressed as coincidence limits of second-order differential operators acting on $G_{Aab'}$ for $T_M{}^{ab}$ and $T_{\rm GB}{}^{ab}$, and on G_A for $T_{\rm Gh}{}^{ab}$. One can show formally that Eq. (3.8) implies that[8]

$$\langle A\,|\,\hat T_{\rm GB}{}^{ab}+\hat T_{\rm Gh}{}^{ab}\,|\,A\rangle=0 .$$

Requiring that any procedure for renormalizing these stress tensors should respect gauge invariance implies that

$$\langle A\,|\,\hat T_{\rm GB}{}^{ab}+\hat T_{\rm Gh}{}^{ab}\,|\,A\rangle_{\rm ren}=0 . \tag{3.11}$$

We shall require that Eq. (3.11) be maintained. Thus we need to consider only the Maxwell stress tensor, and from here on the argument is presented so as to parallel that of Sec. II as closely as possible.

The classical Maxwell stress tensor is given by

$$\begin{aligned}T_M{}^{ab}&=F^{ac}F^b{}_c-\tfrac14 g^{ab}F^{cd}F_{cd}\\&=(A^{c;a}-A^{a;c})(A_c{}^{;b}-A^b{}_{;c})\\&\quad-\tfrac14 g^{ab}(A^{c;d}-A^{d;c})(A_{c;d}-A_{d;c}) .\end{aligned} \tag{3.12}$$

We can write

$$T_M{}^{ab}=[\vec{\tau}^{\,abcd'}(A_c(x)A_{d'}(x'))] , \tag{3.13}$$

where, for example,

$$\vec{\tau}^{\,abcd'}=\vec{f}^{\,abcd'}-\tfrac14 g^{ab}\vec{f}_e{}^{ecd'} , \tag{3.14}$$

with

$$\vec{f}^{\,abcd'}=(g_e{}^{d'}g_{b'}{}^b\nabla^{b'}-g_{ee'}g^{bd'}\nabla^{e'})\times(g^{ec}\nabla^a-g^{ac}\nabla^e) . \tag{3.15}$$

In the quantum theory, following our procedure for scalar fields, rather than consider the meaningless object $[\vec{\tau}^{\,abcd'}G_{Acd'}]$ we shall study the well-defined object

$$\tau^{ab}[W_A]\equiv[\vec{\tau}^{\,abcd'}W_{Acd'}] . \tag{3.16}$$

By definition this tensor is trace-free, but, as in the scalar theory, it is not conserved. Its divergence can be calculated by repeating the argument of Sec. II. Under the infinitesimal coordinate transformation given in that section

$$A_a\to A_a-\delta x^b{}_{;a}A_b-\delta x^b A_{a;b} .$$

From the invariance of S_M under this transformation it can be shown that

$$T_M{}^{ab}{}_{;b}=(A^{b;a}-A^{a;b})[\Box A_b-R_b{}^cA_c-\nabla_b(A_c{}^{;c})] . \tag{3.17}$$

In turn this implies that

$$\tau^{ab}{}_{;b}[W_A]=[(g^{bd'}g^{aa'}\nabla_{a'}-g^{ad'}g^{bb'}\nabla_{b'})\times(\delta_b{}^c\Box-R_b{}^c-\nabla_b\nabla^c)W_{Acd'}] . \tag{3.18}$$

The wave equation for $G_{Aab'}$ implies that

$$(\delta_b{}^c\Box-R_b{}^c)W_{Acd'}=-6V_{1bd'}-2V_{1bd';e}\sigma^e+\cdots , \tag{3.19}$$

while Eqs. (3.8) and (3.9) imply that

$$V_{1cd'}{}^{;c}=-V_{1;d'}+O(\sigma^{1/2}) \tag{3.20}$$

and

$$W_{Acd'}{}^{;c}=-W_{A;d'}-V_{1cd'}\sigma^c-V_1\sigma_{d'}+O(\sigma^{3/2}) . \tag{3.21}$$

Furthermore, it can be shown that [see Appendix A and Eq. (3.32)]

$$g^b{}_{b'}V_1{}^{ab'}(x,x')=v_1{}^{ab}(x)+[-\tfrac12 v_1{}^{ab}{}_{;c}(x)+v_1{}^{[ab]}{}_c(x)]\sigma^c+O(\sigma) . \tag{3.22}$$

Combining these equations it follows that

$$\tau^{ab}{}_{;b}[W_A]=4v_1{}^{ab}{}_{;b}-\tfrac32 v_1{}^b{}_b{}^{;a}+v_1{}^{;a} , \tag{3.23}$$

where $v_1(x)\equiv[V_1(x,x')]$.

We can now define a renormalized stress tensor by the equation

$$T_R{}^{ab}[A]\equiv\frac{1}{8\pi^2}(\tau^{ab}[W_A]-4v_1{}^{ab}+\tfrac32 v_1{}^c{}_c g^{ab}-v_1 g^{ab}) . \tag{3.24}$$

This tensor is conserved and has a trace,

$$T_R{}^a{}_a[A]=\frac{1}{4\pi^2}(v_1{}^c{}_c-2v_1)\ , \tag{3.25}$$

in agreement with the standard point-separation value for the electromagnetic trace anomaly.[4]

Again there is an ambiguity in our definition corresponding to the freedom to add to $W_{Aab'}(x,x')$ an arbitrary multiple of $V_{ab'}(x,x')$. Using the expansions of Appendix A and the geometrical identities of Appendix B, one finds

$$\tau^{ab}[V]=\tfrac{11}{240}B^{ab}\ , \tag{3.26}$$

where, as before, B^{ab} is the Bach tensor defined by Eq. (2.27).

Now we turn to a discussion of $\tau^{ab}[W_A]$ in relation to the symmetry of $W_{Aab'}$. For $G_{Aab'}(x,x')$ to represent the expectation value of a time-ordered product of field operators it must be symmetric. Since $V_{ab'}(x,x')$ is symmetric[13] this requires that the regular bivector $W_{Aab'}(x,x')$ be symmetric.

To give a Taylor-series expansion for a smooth bivector $U_{ab'}(x,x')$ one first parallel transports the primed index back to the point x (Ref. 12). The resulting object $g_b{}^{b'}U_{ab'}(x,x')$ is a second-rank tensor at x and a scalar at x'. One can now write

$$g_b{}^{b'}U_{ab'}(x,x')=u_{ab}(x)+u_{abc}(x)\sigma^c+\tfrac{1}{2}u_{abcd}(x)\sigma^c\sigma^d+\tfrac{1}{6}u_{abcde}(x)\sigma^c\sigma^d\sigma^e+\cdots\ , \tag{3.27}$$

where, by definition,

$$u_{abc\cdots e}(x)=u_{ab(c\cdots e)}(x)\ .$$

Suppose now that $U_{ab'}(x,x')$ is symmetric in the sense that $U_{ab'}(x,x')=U_{b'a}(x',x)$, then we have

$$g_b{}^{b'}U_{ab'}=g_b{}^{b'}U_{b'a}=g_b{}^{b'}g_a{}^{a'}(g_{a'}{}^cU_{b'c})\ , \tag{3.28}$$

and hence

$$u_{ab}+u_{abc}\sigma^c+\cdots=g_b{}^{b'}g_a{}^{a'}(u_{b'a'}+u_{b'a'c'}\sigma^{c'}+\cdots)\ . \tag{3.29}$$

Differentiating and taking coincidence limits we obtain a series of constraints on the expansion coefficients in the Taylor series. These constraints determine the symmetric part of the odd coefficients and the antisymmetric part of the even coefficients. The general symmetry constraint on the coefficients can be written as

$$\sum_{r=0}^{n}\binom{n}{r}u_{ab(a_1a_2\cdots a_r;a_{r+1}\cdots a_n)}=(-1)^n u_{ba(a_1a_2\cdots a_n)}\ . \tag{3.30}$$

The first four constraints are

$$u_{ab}=u_{(ab)}\ , \tag{3.31}$$

$$u_{abc}=-\tfrac{1}{2}u_{(ab);c}+u_{[ab]c}\ , \tag{3.32}$$

$$u_{abcd}=u_{(ab)cd}-u_{[ab](c;d)}\ , \tag{3.33}$$

$$u_{abcde}=-\tfrac{3}{2}u_{(ab)(cd;e)}+\tfrac{1}{4}u_{(ab);(cde)}+u_{[ab]cde}\ . \tag{3.34}$$

For convenience, we now drop the subscript A on $W_{Aab'}(x,x')$, and introduce the following notation for its Taylor-series coefficients:

$$s_{abc\cdots e}\equiv w_{(ab)c\cdots e}\ ,\qquad a_{abc\cdots e}\equiv w_{[ab]c\cdots e}\ . \tag{3.35}$$

We shall also need the Taylor-series expansion for the regular part of the symmetric ghost propagator, viz.,

$$W(x,x')=\omega-\tfrac{1}{2}\omega_{;a}\sigma^a+\tfrac{1}{2}\omega_{ab}\sigma^a\sigma^b+\cdots\ . \tag{3.36}$$

Now we can transcribe the equations of the earlier part of this section into constraints on the Taylor-series coefficients for $g_b{}^{b'}W_{ab'}$ and W. The wave equation (3.19) requires

$$a_{abc}{}^{;c}=R^c{}_{[a}s_{b]c}\ , \tag{3.37}$$

$$s_{abc}{}^c=R^c{}_{(a}s_{b)c}-6v_{1ab}\ , \tag{3.38}$$

and

$$s_b{}^b{}_{ac}{}^{;c}=\tfrac{1}{2}s_b{}^b{}_c{}^c{}_{;a}+R_a{}^{bcd}a_{cdb}+\tfrac{1}{4}(\Box s_b{}^b)_{;a}+\tfrac{1}{2}R_a{}^bs_c{}^c{}_{;b}-\tfrac{1}{2}R^{bc}s_{bc;a}+2v_{1b}{}^b{}_{;a}\ . \tag{3.39}$$

We note that Eq. (3.19) imposes further restrictions at the same order as Eq. (3.39), but Eq. (3.39) is all that we shall use in the following. Furthermore, Eq. (2.19) for the minimally coupled scalar ghost field yields the constraints

$$\omega_a{}^a=-6v_1 \tag{3.40}$$

and

$$\omega_{ab}{}^{;b}=\tfrac{1}{4}(\Box\omega)_{;a}+\tfrac{1}{2}R_a{}^b\omega_{;b}-v_{1;a}\ . \tag{3.41}$$

Finally, the gauge condition (3.21) gives the relations

$$a_{ab}{}^b=\tfrac{1}{2}\omega_{;a}+\tfrac{1}{2}s_{ab}{}^{;b} \tag{3.42}$$

and

$$s_{ac}{}^c{}_b=\tfrac{1}{4}s_{ac}{}^{;c}{}_{;b}+\tfrac{1}{2}a_{acb}{}^{;c}+\tfrac{1}{2}R_a{}^cs_{cb}+\omega_{ab}-\tfrac{1}{4}\omega_{;ab}-v_{1ab}+v_1g_{ab}\ . \tag{3.43}$$

We can express $\tau^{ab}[W]$ in terms of the above Taylor-series coefficients. Using Eqs. (3.37)–(3.43) we can write

$$\tau^{ab}[W]=\tfrac{1}{2}s_c{}^{c;ab}+\tfrac{1}{2}s^{ab;c}{}_{;c}-s_c{}^{(a;b)c}+R_c{}^{(a}s^{b)c}-s_c{}^{cab}-2a_c{}^{(ab);c}+2\omega^{ab}-\omega^{;ab}+4v_1{}^{ab}-\tfrac{1}{4}g^{ab}(\Box s_c{}^c-2s_{cd}{}^{;cd}-2\Box\omega+10v_{1c}{}^c-12v_1) \tag{3.44}$$

and, correspondingly,

$$8\pi^2 T_R{}^{ab}[W] = \tfrac{1}{2}s_c{}^{c;ab} + \tfrac{1}{2}s^{ab;c}{}_{;c} - s_c{}^{(a;b)c} + R_c{}^{(a}s^{b)c} - s_c{}^{cab} - 2a_c{}^{(ab);c} + 2\omega^{ab} - \omega^{;ab} - \tfrac{1}{4}g^{ab}(\Box s_c{}^c - 2s_{cd}{}^{;cd} - 2\Box\omega + 4v_{1c}{}^c - 8v_1) \ . \tag{3.45}$$

It can be explicitly demonstrated that $\tau^{ab}[W]$ satisfies Eq. (3.23) and that $T_R{}^{ab}[W]$ is conserved having a trace given by Eq. (3.25).

In Sec. II we showed how the tensor τ^{ab} was contained in the Taylor-series expansion of the Feynman Green's function. Here τ^{ab} appears naturally in the Taylor-series expansion of the "once-traced" field propagator

$$g_{cd'}\langle A \mid T(\hat{F}^{ac}(x)\hat{F}^{b'd'}(x')) \mid A\rangle \ .$$

We shall discuss this further in the next section.

IV. TRACE ANOMALIES AND THE WKB APPROXIMATION

Here we shall discuss the relevance of the analysis of Secs. III and IV to the approximation scheme of Ref. 1. We shall be more precise in a moment but it is worthwhile first to sketch the general ideas: In a general curved space-time the Hadamard representation separates the Feynman Green's function into two pieces, its singular and regular parts. In certain special space-times the singular part alone is a solution of the Green's function equation (for our purposes it is sufficient that it be so to order σ). The condition that this happens can be stated in a number of different ways depending on the precise circumstances: viz., the singular part be a solution, the WKB approximation be exact, and the Gaussian approximation[14] be exact (in all cases to order σ). When this condition, however stated, is satisfied one might expect that the renormalized stress tensor derived from that Green's function would be zero and, correspondingly, that the renormalized stress tensor in some state $\mid A\rangle$ would be given by $\tau^{ab}[W_A]$. This is so for scalar fields but as we shall describe in detail below it is not necessarily so for photons.

For scalars the argument runs as follows: We are interested in the condition that

$$G_{\rm sing}(x,x') \equiv \frac{i}{8\pi^2}\left[\frac{\Delta^{1/2}}{\sigma+i\epsilon} + V\ln(\sigma+i\epsilon)\right] \tag{4.1}$$

be a solution to the inhomogeneous wave equation (2.7) to order σ. It is clear that this condition is equivalent to the requirement that $W(x,x')=0$ be a solution to Eq. (2.19) and this will be the case if and only if

$$v_1(x)=0 \ . \tag{4.2}$$

We wish to stress the important point that this is a geometrical constraint on the space-time curvature and is not necessarily related to any property of the renormalized stress tensor. (For example, recall that we could have defined our renormalized stress tensor to include an arbitrary multiple of the metric variation of the action having Lagrangian density R^2. This additional conserved tensor has a trace proportional to $\Box R$.)

It is worth noting that for conformally invariant field theories on space-times conformal to Einstein space-times $V(x,x')=O(\sigma^2)$. It follows that the above argument applies equally well to the Gaussian approximation of the Feynman Green's function given by Bekenstein and Parker,[14] viz.,

$$G_{\rm Gauss}(x,x') = \frac{i}{8\pi^2}\frac{\Delta^{1/2}}{\sigma+i\epsilon} \ . \tag{4.3}$$

For photons it is desirable to phrase the analogous discussion in terms of physical, gauge-invariant objects. Of fundamental importance is the field propagator, defined for a unit-norm state $\mid A\rangle$ as

$$\langle A \mid T(\hat{F}^{ab}(x)\hat{F}^{c'd'}(x')) \mid A\rangle \ .$$

For the purposes of our discussion of stress tensors it will be sufficient to consider the "once-traced" field propagator

$$g_{bd'}\langle A \mid T(\hat{F}^{ab}(x)\hat{F}^{c'd'}(x')) \mid A\rangle \ .$$

In the classical theory we can write

$$g_{bd'}F^{ab}(x)F^{c'd'}(x') = g_b{}^{c'}\vec{f}^{\,abcd'}A_c(x)A_{d'}(x') \ , \tag{4.4}$$

where $\vec{f}^{\,abcd'}$ is defined by Eq. (3.14). In the quantum theory we shall study the regular part of the once-traced field propagator defined by

$$P^{ac'}(x,x') \equiv g_b{}^{c'}\vec{f}^{\,abcd'}W_{cd'}(x,x') \ . \tag{4.5}$$

The Maxwell equations impose constraints on the bivector $P^{ac'}(x,x')$—we have

$$\tau^{ab}[W] = [g^b{}_{c'}P^{ac'}] - \tfrac{1}{4}g^{ab}[g_{dc'}P^{dc'}] \ , \tag{4.6}$$

and by Eq. (3.23) the Maxwell equations require

$$\tau^{ab}{}_{;b}[W] = (4v_1{}^{ab} - \tfrac{3}{2}v_{1c}{}^c g^{ab} + v_1 g^{ab})_{;b} \ . \tag{4.7}$$

We are interested in the conditions under which the singular part of the field propagator alone is a solution to the Maxwell equations. Clearly, a necessary condition is that setting $P^{ac'}(x,x')$, the regular part of the once-traced field propagator, to zero be consistent with the Maxwell equations. This implies the geometrical constraint on space-time that

$$(4v_1{}^{ab} - \tfrac{3}{2}v_{1c}{}^c g^{ab} + v_1 g^{ab})_{;b} = 0 \ . \tag{4.8}$$

This equation corresponds to Eq. (4.2) for the scalar theory. Written in full it requires that the symmetric second-rank tensor,

$$4v_1{}^{ab}-\tfrac{3}{2}v_{1c}{}^{c}g^{ab}+v_1 g^{ab}=-\tfrac{1}{6}C^{cabd}R_{cd}+\tfrac{1}{6}R^{ac}R_c{}^{b}-\tfrac{1}{12}RR^{ab}-\tfrac{1}{18}R^{;ab}$$
$$+\tfrac{1}{1440}g^{ab}(13R_{cdef}R^{cdef}-148R_{cd}R^{cd}+55R^2+38\Box R)\ , \qquad (4.9)$$

be divergence-free.

There is an important difference between the scalar constraint (4.2) and the vector constraint (4.8) in that, in general, there is no reason why the vector constraint should be related to the vanishing of its associated trace anomaly. Even in the important, and relevant,[1] special case of ultrastatic space-times that are conformal to Einstein space-times there is no connection. For these space-times, using the formulas given in the Appendix of Ref. 1., it can be shown that

$$(4v_1{}^{ab}-\tfrac{3}{2}v_{1c}{}^{c}g^{ab}+v_1 g^{ab})_{;b}=-\tfrac{1}{144}(5\Box R+R^2)^{;a}\ . \qquad (4.10)$$

The point-separation photon trace anomaly in these space-times is given by

$$T_R{}^a{}_a=\frac{1}{4\pi^2}(v_1{}^c{}_c-2v_1)=-\frac{1}{192\pi^2}\Box R\ . \qquad (4.11)$$

It is rather remarkable that the right-hand side of Eq. (4.10) is the gradient of a scalar, albeit not one proportional to the trace anomaly—in a general space-time this will not be true.

This disagreement is disturbing in that it suggests that the renormalized photon stress tensor does not accurately reflect the structure of the Feynman Green's function. Not least is that the vanishing of the vector trace anomaly does not imply and is not implied by the accuracy of the WKB approximation. It may well be that the difficulty we experienced in Ref. 1 in successfully approximating the renormalized photon stress tensor is attributable to this discrepancy. It is possible that a better approximation would be obtained by finding those space-times in which the divergence of the tensor

$$4v_1{}^{ab}-\tfrac{3}{2}v_1{}^c{}_c g^{ab}+v_1 g^{ab}$$

vanishes rather than finding those space-times in which its trace vanishes. In Ref. 1 where we did manage to obtain a good approximation both of these conditions were satisfied. It is certainly the case that the vanishing of the divergence is a physical condition whereas the vanishing of the trace is a condition required by a particular renormalization ansatz.

V. SUMMARY AND CONCLUSION

In this paper we have tried to make plain the local structure of scalar and vector Feynman Green's functions which have the Hadamard form. We have not allowed ourselves the use of sometimes nonexistent integral representations such as the DeWitt series but have worked with the Hadamard series representation. The state dependence of these two-point functions is contained in the bitensor $W(x,x')$ (we suppress any tensor indices) that forms the regular part of their Hadamard representations. The necessary requirement that $W(x,x')$ be symmetric imposes conditions on the coefficients of its Taylor-series expansion [Eqs. (2.30) and (3.30)]. For scalar fields, the condition that, to lowest order, $W(x,x')$ is itself a solution of the scalar wave equation implies that the scalar $v_1(x)$ must vanish. For electromagnetism, the condition that, to lowest order, the gauge-invariant part of $W(x,x')$ is itself a solution of the appropriate vector wave equation implies that

$$(4v_1{}^{ab}-\tfrac{3}{2}v_{1c}{}^{c}g^{ab}+v_1 g^{ab})_{;b}=0\ . \qquad (5.1)$$

In Ref. 1 we assumed that this condition, the condition for the photon propagator to be well approximated by its WKB approximation, was that

$$v_{1c}{}^{c}-2v_1=0\ . \qquad (5.2)$$

The discrepancy between Eqs. (4.1) and (4.2) may well account for the difficulty we experienced in Ref. 1 in extracting accurate approximations for renormalized photon stress tensors from their conformal transformation law.

In this paper we have been careful to separate statements about properties of two-point functions from statements about renormalization. In the light of what we hope is the now transparent local structure of two-point functions, there are some comments concerning renormalization that we would like to make.

In quantum field theory, the difference between the expectation values of the stress tensor operator taken in two distinct states $|A\rangle$ and $|B\rangle$ whose associated Feynman Green's functions have the Hadamard form is an *a priori* well-defined quantity. In addition, this difference can be expressed as the difference of the tensors $\tau^{ab}[W_A]$ and $\tau^{ab}[W_B]$, which are themselves well-defined quantities. For conformally invariant scalar fields, these tensors separately must satisfy the equations

$$\tau^a{}_a=2v_1 \qquad (5.3)$$

and

$$\tau^{ab}{}_{;b}=-2v_1{}^{;a}\ . \qquad (5.4)$$

For photon fields, the analogous equations are

$$\tau^a{}_a=0 \qquad (5.5)$$

and

$$\tau^{ab}{}_{;b}=(4v_1{}^{ab}-\tfrac{3}{2}v_{1c}{}^{c}g^{ab}+v_1 g^{ab})_{;b}\ . \qquad (5.6)$$

Renormalization theory seeks to give an absolute measure of energy-momentum to any given state $|A\rangle-T_R{}^{ab}[A]$. Clearly any sensible definition of this quantity must be closely related to $\tau^{ab}[W_A]$. In addition, if it is to be thought of as a stress tensor then it must be conserved. The tensors $\tau^{ab}[W_A]$ [Eqs. (5.4) and (5.6)] are not always conserved—there is no reason why they should be; at the same time it is easy to construct from them tensors that are conserved. In Secs. II and III [Eqs. (2.23) and (3.24)] we gave the following definitions for the renormalized stress tensors.

For scalar fields

$$T_R{}^{ab}[A] \equiv \frac{1}{8\pi^2}(\tau^{ab}[W_A] + 2v_1 g^{ab}) \ . \quad (5.7)$$

For electromagnetic fields

$$T_R{}^{ab}[A] \equiv \frac{1}{8\pi^2}(\tau^{ab}[W_A] - 4v_1{}^{ab} + \tfrac{3}{2}v_{1c}{}^{c}g^{ab} - v_1 g^{ab}) \ . \quad (5.8)$$

In the spirit of Wald's axioms these definitions are equivalent to others in the literature—they can differ at most by a conserved, local geometrical tensor. We hope that they are an improvement in that they and their properties are easily accessible from known, regular, functions; they clearly satisfy the necessary, and crucial, relation

$$T_R{}^{ab}[A] - T_R{}^{ab}[B] = \frac{1}{8\pi^2}(\tau^{ab}[W_A] - \tau^{ab}[W_B]) \ . \quad (5.9)$$

Now we come to the serious question: given that we have a renormalized stress tensor, what do we do with it? Insofar as it assists in the bookkeeping of the energy-momentum content of different states of matter in given space-times it may be a valuable asset—the more so if it is an accessible quantity, and, as we have said, we hope that this paper helps in this respect. However, we know of no satisfactory justification for its being used, as it frequently is, as *the* source of gravitational energy possessed by matter in a given state in a given space-time which is to appear in the semiclassical approximation of the quantized Einstein-matter field equations. It is too optimistic to imagine that by picking one renormalized stress tensor from the many possible candidates that one has picked that one with precisely the right amount of gravitational energy. The ambiguity extends way beyond the renormalization of terms which appear in the semiclassical effective action and are at most quadratic in the curvature. For example, in Eqs. (5.7) and (5.8) we have defined our renormalized stress tensor to have traces that are in agreement with the usual values for the "trace anomalies." It should be noted that there exist equally valid definitions of renormalized stress tensors which, for conformally invariant field theories, have zero trace. They differ from our definitions here by conserved, local, geometrical tensors. The details of their construction may be found in Ref. 6.

It is our opinion that the physical role and meaning of renormalized stress tensors is to be realized in curved space-times, exactly as in flat space-times, through equations such as (5.9). In this case they are unnecessary—the tensors $\tau^{ab}[W_A]$ and $\tau^{ab}[W_B]$ carry the physically relevant information and moreover, once the states are specified, these τ tensors are essentially free from ambiguity. Further discussion of these ideas may be found in Ref. 6. It may be that one day this view will prevail; for the present it seems that renormalized stress tensors capture the imagination. That being so we hope that this paper will remove from them their largely historical connections with the mathematical devices of ζ functions, space-times having continuous dimension, etc., and instead highlight their connection with τ tensors which are obtained directly from the physical Green's functions of the theory.

ACKNOWLEDGMENTS

We would like to thank Dr. A. H. Najmi for helpful discussions. A.C.O. wishes to thank Merton College for financial support. M.R.B. wishes to thank Merton College for financial support through their Estates Office.

APPENDIX A

In this appendix we extend (and in one place correct) the covariant Taylor-series expansions given by Christensen[15] for the Hadamard coefficients of scalar and vector theories.

For scalar theories

$$V_0 = v_0 - \tfrac{1}{2}v_{0;a}\sigma^a + \tfrac{1}{2}v_{0ab}\sigma^a\sigma^b + \tfrac{1}{6}(-\tfrac{3}{2}v_{0ab;c} + \tfrac{1}{4}v_{0;(abc)})\sigma^a\sigma^b\sigma^c + \cdots \ ,$$

$$V_1 = v_1 - \tfrac{1}{2}v_{1;a}\sigma^a + \cdots \ ,$$

where

$$v_0 = \tfrac{1}{2}[(\xi - \tfrac{1}{6})R + m^2] \ ,$$

$$v_{0ab} = -\tfrac{1}{180}R_{pqra}R^{pqr}{}_b - \tfrac{1}{180}R_{apbq}R^{pq} + \tfrac{1}{90}R_{ap}R_b{}^p - \tfrac{1}{120}\Box R_{ab} + (\tfrac{1}{6}\xi - \tfrac{1}{40})R_{;ab} + \tfrac{1}{12}(\xi - \tfrac{1}{6})RR_{ab} + \tfrac{1}{12}m^2 R_{ab} \ ,$$

$$v_1 = \tfrac{1}{720}R_{pqrs}R^{pqrs} - \tfrac{1}{720}R_{pq}R^{pq} - \tfrac{1}{24}(\xi - \tfrac{1}{5})\Box R + \tfrac{1}{8}(\xi - \tfrac{1}{6})^2 R^2 + \tfrac{1}{4}m^2(\xi - \tfrac{1}{6})R + \tfrac{1}{8}m^4 \ .$$

For electromagnetism

$$V_{0ab} = v_{0ab} + (-\tfrac{1}{2}v_{0ab;c} + v_{0[ab]c})\sigma^c + \tfrac{1}{2}(v_{0(ab)cd} - v_{0[ab](c;d)})\sigma^c\sigma^d + \tfrac{1}{6}(-\tfrac{3}{2}v_{0(ab)(cd;e)} + \tfrac{1}{4}v_{0ab;(cde)} + v_{0[ab]cde})\sigma^c\sigma^d\sigma^e + \cdots \ ,$$

$$V_{1ab} = v_{1ab} + (-\tfrac{1}{2}v_{1ab;c} + v_{1[ab]c})\sigma^c + \cdots \ ,$$

where

$$v_{0ab} = \tfrac{1}{2}R_{ab} - \tfrac{1}{12}Rg_{ab} \ ,$$

$$v_{0[ab]}{}^c = \tfrac{1}{6}R^c{}_{[b;a]} \ ,$$

$$v_{0(ab)}{}^{cd}=\tfrac{1}{6}R_{ab}{}^{;(cd)}+\tfrac{1}{12}R_{ab}R^{cd}+\tfrac{1}{12}R_{(a}{}^{pqc}R_{b)pq}{}^{d}+g_{ab}\big(-\tfrac{1}{180}R_{pqr}{}^{c}R^{pqrd}$$
$$-\tfrac{1}{180}R^{c}{}_{p}{}^{d}{}_{q}R^{pq}+\tfrac{1}{90}R^{c}{}_{p}R^{dp}-\tfrac{1}{72}RR^{cd}-\tfrac{1}{40}R^{;cd}-\tfrac{1}{120}\Box R^{cd}\big)\ ,$$
$$v_{0[ab]}{}^{cde}=-\tfrac{3}{20}R^{(c}{}_{[a;b]}{}^{de)}-\tfrac{1}{12}R^{(c}{}_{[a;b]}R^{de)}-\tfrac{1}{20}R_{[a}{}^{pq(c}R_{b]pq}{}^{d;e)}-\tfrac{1}{30}R_{ab}{}^{p(c}{}_{;q}R_{p}{}^{de)q}$$
$$+\tfrac{1}{60}R_{abp}{}^{(c;d}R^{e)p}+\tfrac{1}{20}R_{abp}{}^{(c}R^{de);p}-\tfrac{1}{20}R_{ab}{}^{p(c}R_{p}{}^{d;e)}\ ,$$
$$v_{1ab}=-\tfrac{1}{48}R_{apqr}R_{b}{}^{pqr}+\tfrac{1}{8}R_{ap}R_{b}{}^{p}-\tfrac{1}{24}RR_{ab}-\tfrac{1}{24}\Box R_{ab}$$
$$+g_{ab}\big(\tfrac{1}{720}R_{pqrs}R^{pqrs}-\tfrac{1}{720}R_{pq}R^{pq}+\tfrac{1}{288}R^{2}+\tfrac{1}{120}\Box R\big)\ ,$$
$$v_{1[ab]}{}^{c}=\tfrac{1}{240}R_{[a}{}^{pqr}R_{b]pqr}{}^{;c}+\tfrac{1}{24}R_{[a}{}^{pqc}R_{b]p;q}+\tfrac{1}{120}R_{[a}{}^{pqc}R_{b]q;p}$$
$$+\tfrac{1}{120}R^{c}{}_{pq[a}R^{pq}{}_{;b]}+\tfrac{1}{24}R^{pc}{}_{;[a}R_{b]p}+\tfrac{1}{24}R^{p}{}_{[a}R_{b]}{}^{c}{}_{;p}$$
$$-\tfrac{1}{360}R^{pc}R_{p[a;b]}-\tfrac{1}{24}R^{p}{}_{[a}R_{b]p}{}^{;c}+\tfrac{1}{72}RR^{c}{}_{[a;b]}+\tfrac{1}{120}R^{c}{}_{[a;b]p}{}^{p}$$
$$-\tfrac{1}{540}R_{abpq;r}R^{pqrc}-\tfrac{1}{540}R_{abpq;r}R^{rqpc}-\tfrac{1}{540}R_{abp}{}^{c}{}_{;q}R^{pq}$$
$$+\tfrac{1}{1080}R_{ab}{}^{c}{}_{p;q}R^{pq}+\tfrac{1}{120}R_{abpq}R^{cp;q}-\tfrac{1}{120}R_{ab}{}^{pc}R_{;p}\ .$$

The corresponding DeWitt coefficients $a_n(x,x')$ (suppressing any bitensor indices) may be obtained from these expansions by using the identity

$$V_n(x,x';m^2)=(2^{r+1}r!)^{-1}\sum_{r=0}^{n}\frac{(-1)^r(m^2)^{n-r+1}}{(n-r+1)!}a_r(x,x')\ .$$

In deriving the above formulas we have used the following Taylor-series expansions:

$$\Delta^{1/2}=1+\tfrac{1}{12}R_{ab}\sigma^a\sigma^b-\tfrac{1}{24}R_{ab;c}\sigma^a\sigma^b\sigma^c+\big(\tfrac{1}{288}R_{ab}R_{cd}+\tfrac{1}{360}R_{apbq}R_{c}{}^{p}{}_{d}{}^{q}+\tfrac{1}{80}R_{ab;cd}\big)\sigma^a\sigma^b\sigma^c\sigma^d$$
$$-\big(\tfrac{1}{288}R_{ab}R_{cd;e}+\tfrac{1}{360}R_{apbq}R_{c}{}^{p}{}_{d}{}^{q}{}_{;e}+\tfrac{1}{360}R_{ab;cde}\big)\sigma^a\sigma^b\sigma^c\sigma^d\sigma^e+\cdots\ ,$$
$$\Box\Delta^{1/2}=\tfrac{1}{6}R+\big(\tfrac{1}{60}R_{pqra}R^{pqr}{}_{b}+\tfrac{1}{60}R_{apbq}R^{pq}-\tfrac{1}{30}R_{ap}R_{b}{}^{p}+\tfrac{1}{72}RR_{ab}-\tfrac{1}{120}R_{;ab}$$
$$+\tfrac{1}{40}\Box R_{ab}\big)\sigma^a\sigma^b-\big(\tfrac{1}{180}R_{pqra}R^{pqr}{}_{b;c}+\tfrac{1}{180}R_{apbq}R^{pq}{}_{;c}-\tfrac{1}{90}R_{ap}R_{b}{}^{p}{}_{;c}$$
$$+\tfrac{1}{144}RR_{ab;c}+\tfrac{1}{120}R_{ab;p}{}^{p}{}_{c}-\tfrac{1}{360}R_{;abc}\big)\sigma^a\sigma^b\sigma^c+\cdots\ ,$$
$$g_{b}{}^{b'}g_{ab';c}=-\tfrac{1}{2}R_{abcd}\sigma^d+\tfrac{1}{6}R_{abcd;e}\sigma^d\sigma^e-\tfrac{1}{24}\big(R_{abcd;ef}+R_{abpd}R^{p}{}_{ecf}\big)\sigma^d\sigma^e\sigma^f$$
$$+\big(\tfrac{1}{120}R_{abcd;efg}+\tfrac{1}{60}R_{abpd}R^{p}{}_{ecf;g}+\tfrac{7}{360}R_{abpd;e}R^{p}{}_{fcg}\big)\sigma^d\sigma^e\sigma^f\sigma^g+\cdots\ ,$$
$$g_{b}{}^{b'}g_{ab';c}{}^{c}=\tfrac{2}{3}R_{d[a;b]}\sigma^d+\big(-\tfrac{1}{6}R_{d[a;b]e}+\tfrac{1}{6}R_{abpd}R^{p}{}_{e}$$
$$-\tfrac{1}{4}R_{apqd}R_{b}{}^{pq}{}_{e}\big)\sigma^d\sigma^e+\big(\tfrac{1}{30}R_{d[a;b]ef}+\tfrac{7}{60}R_{apqd}R_{b}{}^{pq}{}_{e;f}$$
$$+\tfrac{1}{20}R_{apqd;e}R_{b}{}^{pq}{}_{f}-\tfrac{2}{45}R_{abpd;q}R^{p}{}_{e}{}^{q}{}_{f}-\tfrac{7}{90}R_{abpd;e}R^{p}{}_{f}$$
$$-\tfrac{1}{40}R_{abpd}R_{ef}{}^{;p}-\tfrac{1}{60}R_{abpd}R^{p}{}_{e;f}\big)\sigma^d\sigma^e\sigma^f+\cdots\ .$$

APPENDIX B

Here we note a number of useful geometrical identities which have been used in the text and in Appendix A:

$$R_{acde}R_{b}{}^{edc}=\tfrac{1}{2}R_{acde}R_{b}{}^{cde}\ ,$$
$$C_{acde}C_{b}{}^{cde}=\tfrac{1}{4}g_{ab}C_{cdef}C^{cdef}\ ,$$
$$R^{a}{}_{bcd;a}=R_{bd;c}-R_{bc;d}\ ,$$
$$R^{abcd}R_{ebcd;a}=\tfrac{1}{4}(R^{abcd}R_{abcd})_{;e}\ ,$$
$$R^{c}{}_{a;bc}=\tfrac{1}{2}R_{;ab}+R_{ac}R_{b}{}^{c}-R_{acbd}R^{cd}\ .$$

The Bach tensor can be written as

$$B^{ab} \equiv g^{-1/2} \frac{\delta}{\delta g_{ab}} \int d^4x\, g^{1/2} C_{cdef} C^{cdef}$$

$$= 2C^{acdb} R_{cd} + 4C^{acdb}{}_{;cd}$$

$$= 4R^{acdb} R_{cd} + \tfrac{4}{3} R R^{ab} - 2\Box R^{ab} + \tfrac{2}{3} R^{;ab} + g^{ab}(R_{cd} R^{cd} - \tfrac{1}{3} R^2 + \tfrac{1}{3} \Box R)$$

$$= -2R^{a}{}_{cde} R^{bcde} + 4R^{a}{}_{c} R^{bc} - \tfrac{2}{3} R R^{ab} - 2\Box R^{ab} + \tfrac{2}{3} R^{;ab}$$

$$+ g^{ab}(\tfrac{1}{2} R_{cdef} R^{cdef} - R_{cd} R^{cd} + \tfrac{1}{6} R^2 + \tfrac{1}{3} \Box R) .$$

[1] M. R. Brown, A. C. Ottewill, and D. N. Page, Phys. Rev. D **33**, 2840 (1986).
[2] M. R. Brown, J. Math. Phys. **25**, 136 (1984).
[3] R. M. Wald, Commun. Math. Phys. **54**, 1 (1977); Phys. Rev. D **17**, 1477 (1978).
[4] N. D. Birrell and P. C. W. Davies, *Quantum Fields in Curved Space* (Cambridge University Press, Cambridge, England, 1982); R. Endo, Prog. Theor. Phys. **71**, 1366 (1984).
[5] M. R. Brown, Class. Quantum Gravit. **2**, 535 (1985).
[6] M. R. Brown and A. C. Ottewill, Proc. R. Soc. London **A389**, 379 (1983).
[7] M. Riesz, Acta Math. **81**, 1 (1949).
[8] S. L. Adler, J. Lieberman, and Y. G. Ng, Ann. Phys. (N.Y.) **106**, 279 (1977); S. L. Adler and J. Lieberman, *ibid.* **113**, 294 (1978).
[9] R. M. Wald, Phys. Rev. D **17**, 1477 (1978).
[10] S. W. Hawking and G. F. R. Ellis, *The Large Scale Structure of Space-Time* (Cambridge University Press, Cambridge, England, 1973).
[11] J. Hadamard, *Lectures on Cauchy's Problem in Linear Parital Differential Equations* (Yale University Press, New Haven, 1923).
[12] B. S. DeWitt and R. W. Brehme, Ann. Phys. (N.Y.) **9**, 220 (1960).
[13] P. Garabedian, *Partial Differential Equations* (Wiley, New York, 1964).
[14] J. D. Bekenstein and L. Parker, Phys. Rev. D **23**, 2850 (1981).
[15] S. M. Christensen, Phys. Rev. D **14**, 2490 (1976); **17**, 946 (1978).