Hamiltonian quantization of SL(2, C) gauge theory

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A Hamiltonian formulation of the nongravitational part of the unified Yang-Mills and gravitational theory is derived. It is shown that the presence of second-class constraints does not spoil construction of a manifestly relativistic functional integral. A correct construction of a relativistic functional integral is presented.

I. INTRODUCTION

The problem of constructing quantum gravity which is at the same time renormalizable, unitary, and causal has brought to the attention all alternative theories to general relativity. Recently, a theory which is a simultaneous generalization of both Einstein and Yang-Mills theories has been proposed.[1] The generalization is based on the observation that both theories are expressed in terms of two fundamental objects of differential geometry: connections and metrics. But, while in the Einsteinian case the metric is the dynamical field, and the connection is restricted to being a function of the metric by metricity and torsion-free constraints, in Yang-Mills theory the connection is the dynamical variable and the metric is constant (δ_{ab}). So, by allowing connections and metrics to become dynamical variables in both theories, a unified action was found in a rather simple form.[2] When written in terms of the component fields the unified action splits into a purely gravitational part and an internal part.

The gravitational part consists of a term linear in the curvature, one quadratic in the curvature, and a term quadratic in the torsion and nonmetric components. The nongravitational part turns out to be a gauge theory of the SL(N,C) group proposed earlier by Cahill.[3]

In this paper we discuss the Hamiltonian quantization of the nongravitational sector of this theory and present a correct construction of a relativistic functional integral. The theory we deal with is similar to the massive Yang-Mills theory,[4] and the appearance of second-class constraints does not prevent the construction of a relativistic functional integral.

II. HAMILTONIAN FORMULATION AND CONSTRUCTION OF THE FUNCTIONAL INTEGRAL

We shall briefly review the Hamiltonian formulation of the SL(2,C) theory before proceeding to the construction of the functional integral.

The Lagrangian density of the theory is given by

$$\mathscr{L}=-\frac{1}{4g^2}(G_{\mu\nu}G^{\mu\nu}+W_{\mu\nu}W^{\mu\nu})+\frac{M^2}{2}B_\mu B^\mu\ , \tag{1}$$

where

$$\begin{aligned}G_{\mu\nu}&=\partial_\mu A_\nu-\partial_\nu A_\mu+[A_\mu,A_\nu]+[B_\mu,B_\nu]\ ,\\ W_{\mu\nu}&=\partial_\mu B_\nu-\partial_\nu B_\mu+[A_\mu,B_\nu]-[A_\nu,B_\mu]\ .\end{aligned} \tag{2}$$

First, one should find the canonical momenta

$$\begin{aligned}\pi_a^i&=\frac{\delta\mathscr{L}}{\delta\dot A_i^a}=-\frac{1}{g^2}G_a^{0i}\ ,\\ P_a^i&=\frac{\delta\mathscr{L}}{\delta\dot B_i^a}=-\frac{1}{g^2}W_a^{0i}\ .\end{aligned} \tag{3}$$

By inspecting (1) one concludes that the Lagrangian density does not depend on velocities $\dot A_0$ and $\dot B_0$. Thus,

$$\begin{aligned}\pi_a^0&=\frac{\delta\mathscr{L}}{\delta\dot A_0^a}=0\ ,\\ P_a^0&=\frac{\delta\mathscr{L}}{\delta\dot B_0^a}=0\ .\end{aligned} \tag{4}$$

As usual the canonical Hamiltonian is given by

$$\mathscr{H}_c=\pi_a^i\dot A_i^a+P_a^i\dot B_i^a-\mathscr{L}\ . \tag{5}$$

Using Eqs. (3) one can explicitly find velocities $\dot A_i^a$ and $\dot B_i^a$ and replacing them in (5) one obtains, for the canonical Hamiltonian,

$$\begin{aligned}\mathscr{H}_c=&-\tfrac{1}{2}g^2(\pi_a^i\pi_i^a+P_a^iP_i^a)+\pi_a^i(\partial_iA_0^a+\epsilon^{abc}A_0^bA_i^c-\epsilon^{abc}B_0^bB_i^c)\\ &+P_a^i(\partial_iB_0^a+\epsilon^{abc}A_0^bB_i^c-\epsilon^{abc}A_i^bB_0^c)+\frac{1}{4g^2}(G_{ij}^aG_a^{ij}+W_{ij}^aW_a^{ij})-\tfrac{1}{2}M^2B_0^aB_a^0-\tfrac{1}{2}M^2B_i^aB_a^i\ .\end{aligned} \tag{6}$$

Following the Dirac systematic method[5] for the system with constraints, we find that the only primary constraints are given by (4). To calculate the secondary constraints one needs to examine the consistency conditions imposed on primary constraints:

$$C^a \equiv \frac{d}{dt}\pi_0^a(x) = \{\pi_0^a, H_T\} \approx 0 \; , \qquad D^a \equiv \frac{d}{dt}P_0^a(x) = \{P_0^a, H_T\} \approx 0 \; , \tag{7}$$

where the total Hamiltonian is defined by

$$H_T = \int d^3y\,(\mathscr{H}_c + u_0^a\pi_0^a + \tilde{u}_0^a P_0^a) \tag{8}$$

The secondary constraints are then

$$C^a = \partial_i \pi_a^i + \epsilon^{abc}\pi_b^i A_i^c + \epsilon^{abc}P_b^i B_i^c \; , \qquad D^a = \partial_i P_a^i - \epsilon^{abc}P_c^i A_i^b - \epsilon^{abc}\pi_b^i B_i^c + M^2 B_0^a \; . \tag{9}$$

It should be noted that further consistency conditions do not lead to any new constraints. They determine $\tilde{u}_0^a$, but leave u_0^a as an arbitrary function of time. This further means that π_0^a is the first-class constraint and hence the variable A_0^a can vary arbitrarily with time. In order to see what kind of constraints we have obtained, one should work out all Poisson brackets for the constraints (4) and (9).

In a straightforward manner one obtains

$$\{C_a^*, \pi_b^0\} \approx 0 \; , \quad \{C_a^*, P_b^0\} \approx 0 \; , \quad \{D_a, \pi_b^0\} \approx 0 \; , \quad \{D_a, P_b^0\} \approx M^2\delta_{ab} \; , \tag{10}$$

where

$$C_a^* \equiv C^a - \epsilon^{abc}B_0^b P_0^c \; . \tag{11}$$

Since the C^a generate the SU(2) gauge transformations one should expect C^a to be a first-class constraint. This is true only after one constructs Dirac brackets, when all second-class constraints become strong equalities. Thus we have found (10) that C_a^* and π_a^0 are the first-class constraints while D_a and P_a^0 are the second-class ones.

To formulate the quantum theory one has to express the Hamiltonian in terms of physical variables by solving all of the constraints. Next, we fix a gauge ($\chi^a=0$) for every first-class constraint. And finally we replace the Poisson brackets with Dirac brackets. As this can be done in a straightforward way[1] we will focus our attention instead on the construction of the functional integral.

The first thought is to write the functional integral in terms of physical variables:

$$\langle \text{out} \,|\, S \,|\, \text{in} \rangle = \int dP^* dq^* \exp\left[i \int d^4x [P^*\dot{q}^* - h^*(q^*, P^*)] \right] , \tag{12}$$

where P^*, q^* are the canonical variables and their canonically conjugate momenta on the physical subspace.

Unfortunately this turns out to be a bad idea because the physical Hamiltonian $h^*(q^*,P^*)$ is a highly nonlinear function of q^* and P^*. The usual way is to derive the functional integral over the entire phase space which means that using constraints (4) and (9) one rewrites integral (12) in terms of the original fields.

For a better understanding of this procedure let us first write explicitly the set of physical variables (P^*,q^*). We solve all the constraints (4) and (9):

$$\pi_a^0 = 0, \quad P_a^0 = 0 \; ,$$

$$C_a^* = 0, \quad D_a = 0 \; .$$

Besides the trivial solutions for π_a^0 and P_a^0 we can use the constraints $D_a=0$ to express the variables B_0^a in terms of other variables, and to eliminate B_0^a from the Hamiltonian.

To solve the constraints $C_a=0$ we may proceed in the usual way of decomposing the remaining fields into transverse and longitudinal parts. We find the solution of C_a for π_a^{iL} giving $C_a(\bar{\pi}_a^{iL})=0$.

The next step is to fix a gauge for every first-class constraint. For $\pi_a^0=0$ it is convenient to use $A_0^a=0$, while for C_a, as in the Yang-Mills case, we shall choose $\chi^a \equiv \partial_i A^{ia} = 0$ ($A_a^{iL}=0$).

Now we are able to write the set of physical variables (P^*,q^*) as

$$(P^*, q^*) = (\pi_a^{iT}, P_a^{iT}, P_a^{iL}, A_i^{aT}, B_i^{aT}, B_i^{aL}) \; . \tag{13}$$

As we have already mentioned one has a system of massless and massive spin-one particles. So the degrees of freedom which we obtained could have been expected from the beginning.

In order to write integral (12) in terms of the original fields one must reintroduce unphysical degrees A_i^{aL}, π_a^{iL}, and B_0^a back into (12):

$$\langle \text{out} \,|\, S \,|\, \text{in} \rangle = \int dP^* dq^* dA_i^{aL} d\pi_a^{iL} dB_0^a \delta(A_i^{aL}) \delta(\pi_a^{iL} - \bar{\pi}_a^{iL}) \times \delta(B_0^a - \bar{B}_0^a) \exp\left[i \int d^4x [P^*\dot{q}^* + \pi_a^{iL}\dot{A}_i^{aL} - \mathscr{H}(P^*, q^*, A_i^{aL}, \pi_a^{iL}, B_0^a)] \right] . \tag{14}$$

By changing variables in the δ functionals $\pi_a^{iL} \rightarrow C_a$, $A_i^{aL} \rightarrow \chi^a$, and $B_0^a \rightarrow D^a$ one obtains

$$dA_i^{aL}\delta(A_i^{aL}) = dA_i^{aL}\delta(\chi^a)\det\left|\frac{\partial\chi^a}{\partial A_i^{bL}}\right| ,$$

$$d\pi_a^{iL}\delta(\pi_a^{iL}-\bar{\pi}_a^{iL}) = d\pi_a^{iL}\delta(C_a)\det\left|\frac{\partial C_a}{\partial \pi_b^{iL}}\right| , \tag{15}$$

$$dB_0^a\delta(B_0^a-\bar{B}_0^a) = dB_0^a\delta(D_a)\det\left|\frac{\partial D_a}{\partial B_0^b}\right| .$$

By simple inspection of our constraints and the gauge conditions one finds that $\det|\partial\chi^a/\partial A_i^{bL}|$ contributes a constant to the functional integral:

$$\det\left|\frac{\partial C_a}{\partial \pi_b^{iL}}\right| = \det|[\chi^a, C_b]| \tag{16}$$

and

$$\det\left|\frac{\partial D_a}{\partial B_0^b}\right| = \det M^2 . \tag{17}$$

The integral (14) becomes

$$\langle \text{out}|S|\text{in}\rangle = \int dA_i^a d\pi_a^i dB_\mu^a dP_a^i \delta(\chi^a)\det|[\chi^a,C_b]| \times\delta(C_a)\delta(D_a)\exp\left[i\int d^4x\,[\pi_a^i\dot{A}_i^a + P_a^i\dot{B}_i^a - \mathscr{H}(A_i^a,B_\mu^a,\pi_a^i,P_a^i)]\right] . \tag{18}$$

We now exponentiate $\delta(C_a)$ as $dA_0^a e^{iA_0^a C_a}$ and finally obtain

$$\langle \text{out}|S|\text{in}\rangle = \int \exp\left[i\int d^4x(\pi_a^i\dot{A}_i^a + P_a^i\dot{B}_i^a - \mathscr{H}_c)\right] \times dA_\mu^a dB_\mu^a d\pi_a^i dP_a^i \delta(\chi^a)\det|\{C^a,\chi^b\}|\,\delta(D_a) . \tag{19}$$

Starting from this expression, the authors of Ref. 1 could not get a manifestly relativistic functional integral, so they claim that the presence of the second-class constraints in the measure through the δ function prevent us from constructing a manifestly relativistic functional integral. To cure this problem they propose a modification of the definition of the functional integral: namely, modification in the usual limiting procedure.

But in fact this is not true. The presence of second-class constraints does not spoil manifest relativistic invariance of the functional integral. Their presence only means that we cannot use directly (blindly) the Faddeev-Popov method.[4] We shall show how to obtain a manifestly relativistic functional integral starting from (19). The integral (19) can obviously be rewritten as

$$\langle \text{out}|S|\text{in}\rangle = \int \exp\left[i\int d^4x(\pi_a^i\dot{A}_i^a + P_a^i\dot{B}_i^a - \mathscr{H}_c + \lambda^a D_a)\right] \times dA_\mu^a dB_\mu^a d\pi_a^i dP_a^i \delta(\chi^a)\det|\{C^a,\chi^b\}|\,d\lambda^a . \tag{20}$$

Using (6) and (9) and changing variables $B_0^a \rightarrow B_0^a - \lambda^a$ into (20) one obtains

$$\langle \text{out}|S|\text{in}\rangle = \int \exp\left[i\int d^4x\left[\pi_a^i(\partial_0 A_i^a - \partial_i A_0^a - \epsilon^{abc}A_0^b A_i^c + \epsilon^{abc}B_0^b B_i^c) + P_a^i(\partial_0 B_i^a - \partial_i B_0^a - \epsilon^{abc}A_0^b B_i^c + \epsilon^{abc}A_i^b B_0^c) - \tfrac{1}{2}g^2\pi_a^i\pi_a^i - \tfrac{1}{2}g^2 P_a^i P_a^i - \frac{1}{4g^2}(G_{ij}G^{ij} + W_{ij}W^{ij}) + \frac{M^2}{2}B_a^0 B_a^0 + \frac{M^2}{2}B_i^a B_a^i - \frac{M^2}{2}\lambda^{a^2}\right]\right] \times dA_\mu^a dB_\mu^a d\pi_a^i dP_a^i d\lambda^a \delta(\chi^a)\det|\{C^a,\chi^b\}| . \tag{21}$$

Now we can easily perform Gaussian integration over λ^a, P_a^i, and π_a^i

$$\langle \text{out}|S|\text{in}\rangle = \int \exp\left[i\int d^4x\left[\frac{1}{4g^2}(2G_{0i}{}^2 - G_{ij}G^{ij} + 2W_{0i}{}^2 - W_{ij}W^{ij}) + \tfrac{1}{2}M^2 B_\mu B^\mu\right]\right] dA_\mu^a dB_\mu^a \delta(\chi^a)\det|\{C^a,\chi^b\}| . \tag{22}$$

So, we obtain in the exponential nothing but the Lagrangian (1):

$$\langle \text{out}|S|\text{in}\rangle = \int \exp\left[i\int d^4x\,\mathscr{L}(x)\right] dA_\mu^a dB_\mu^a \delta(\chi^a)\det|\{C^a,\chi^b\}| . \tag{23}$$

We have shown that the proposed theory, when expressed in the Hamiltonian form, is relativistically invariant and that modification in the definition of the functional integral is not needed. The gravitational part will be discussed elsewhere.

Note added in proof. Some time after this paper was accepted for publication a work[6] with similar results came to my attention.

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