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Relativistic scattering coherence

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Wave propagation through inhomogeneous, turbulent media is investigated for the case where the signal and inhomogeneities move relativistically. Although in classical treatments the mean-square angular deviations grow as the path length, this is found not to be true relativistically. Special attention is given to the problem of light propagating through a cosmological background of gravitational waves.

I. INTRODUCTION

In classical treatments of wave propagation through inhomogeneous, turbulent media, such as those of Chernov[1] and Tatarskii,[2] the mean-square angular deviations of a signal from its unperturbed path grow as the path length L. This is just the well-known $\sqrt{N}$ law for rms deviations where N is the number of scatterings suffered by the propagating wave. As we will show here, however, the situation is more complicated in the relativistic case.

In Sec. II we present the expression for the angular deviation and introduce the autocorrelation function of the inhomogeneities. Section III explicitly shows the calculation of the deviations, demonstrates the vanishing of the "L effect" in the fully relativistic case, and investigates the leading-order surviving terms. Section IV generalizes the calculations to arbitrary signal and inhomogeneity velocities, discusses the role of the coherence of the scattering, and, as an application of the ideas of this paper, calculates the red-shift induced in light traversing a gravitational-wave background.

II. ANGULAR DEVIATIONS—THEORY

The inhomogeneities of the medium give rise to variations ϵ in the index of refraction:

$$n(x,t)=\bar{n}[1+\epsilon(x,t)] \ . \tag{1}$$

We will assume $|\epsilon| \ll 1$ and that the path length the signal traverses is much greater than the scale of the inhomogeneities: $L \gg \lambda$.

Using Fermat's principle or the geodesic equation we find that the angular deviation for a signal originally propagating along the x axis is

$$\theta_j=\int_0^L \epsilon_{,j}dx \ , \tag{2}$$

where $\epsilon_{,j}=\partial\epsilon/\partial j$ and $j=y$ or z; by symmetry $\theta_y=\theta_z$. In Sec. IV C we will discuss θ_t. In Eq. (2) we have neglected terms of order ϵ^2 and also end-point terms, which cannot give rise to a dependence on path length. However, since we show in Sec. III that the path-length dependence vanishes in some cases, end-point terms will be important numerically (see Sec. IV D).

The mean-square angular deviation is

$$\langle \theta_j{}^2\rangle=\left\langle \int_0^L\int_0^L \epsilon_{,j}(x)\epsilon_{,j}(x')dx\,dx'\right\rangle \ . \tag{3}$$

Now we recognize $\langle \epsilon_{,j}(x)\epsilon_{,j}(x')\rangle$ as the autocorrelation function of the inhomogeneities that, for a medium homogeneous on large scales, is dependent only on the absolute difference of the coordinates:

$$R_{\epsilon_{,j}}(x,x')=\langle \epsilon_{,j}(x)\epsilon_{,j}(x')\rangle=R_{\epsilon_{,j}}(|x-x'|) \ . \tag{4}$$

This is equivalent to choosing a stationary, zero mean probability distribution for $\epsilon_{,j}$.

It is useful to switch to relative coordinates

$$x_r=x-x',\quad x_c=(x+x')/2 \ .$$

By geometric analysis and interchange of integration variables we may rewrite an integral of the form of Eq. (3) as

$$\begin{aligned}\int_0^L\int_0^L R(x,x')dx\,dx'&=\int_0^{L/2}dx_c\int_{-2x_c}^{2x_c}dx_rR[x_c,x_r]+\int_{L/2}^{L}dx_c\int_{-2L+2x_c}^{2L-2x_c}dx_rR[x_c,x_r]\\&=\int_0^L dx_r\int_{x_r/2}^{L-x_r/2}dx_cR[x_c,x_r]+\int_{-L}^{0}dx_r\int_{-x_r/2}^{L+x_r/2}dx_cR[x_c,x_r] \ .\end{aligned} \tag{5}$$

Since the correlation function has the property that $R[x_c,x_r]=R[\,|x_r|\,]$ then Eq. (3) becomes

$$\langle \theta_j{}^2\rangle=2L\int_0^L dx_r(1-x_r/L)R_{\epsilon_{,j}}(x_r) \ . \tag{6}$$

Classically, the integral over the correlation function is roughly independent of L due to the incoherence of the scattering; i.e., the scattering in one place is independent of or uncorrelated with the scattering elsewhere (see Sec. IV B). Thus, the mean-square deviations are proportional to the path length; this is referred to here as the "L effect."

As we will prove in the next section, as the velocity v of the inhomogeneities becomes relativistic, the constant terms of the integral vanish, leaving only terms inversely proportional to powers of L, and so the L effect vanishes.

III. ANGULAR DEVIATIONS—CALCULATION

A. The L effect

Without loss of generality we take the variations ϵ in the index of refraction to be harmonic; we would always construct the true ϵ by an appropriate superposition. In addition, the inhomogeneities are frequently due to waves, e.g., a cosmological background of gravitational waves, so it is convenient as well to write

$$\begin{aligned}\epsilon(x,t)&=\mathrm{Re}(\epsilon_0 e^{ik_\mu x^\mu})\\&=(\epsilon_0 e^{ik_\mu x^\mu}+\epsilon_0^* e^{-ik_\mu x^\mu})/2\\&=\mathrm{Re}\epsilon_0\cos(\mathbf{k}\cdot\mathbf{r}-\omega t)-\mathrm{Im}\epsilon_0\sin(\mathbf{k}\cdot\mathbf{r}-\omega t)\ .\end{aligned} \tag{7}$$

Writing $\mathbf{k}\cdot\mathbf{r}-\omega t\equiv kr\cos\theta-kvt$ and specializing to signals propagating with the speed of light ($c=1$), e.g., light from distant galaxies, gives $\mathbf{k}\cdot\mathbf{r}-\omega t=kr(\cos\theta-v)$. So Eq. (3) becomes

$$\langle\theta_j{}^2\rangle=\left\langle\left[\frac{k_j}{k}\right]^2(\cos\theta-v)^{-2}\sin^{2s}\theta[\mathrm{Re}^2\epsilon_+(\cos^2\alpha-2\cos\alpha+1)+\mathrm{Im}^2\epsilon_+\sin^2\alpha-2\,\mathrm{Re}\epsilon_+\mathrm{Im}\epsilon_+\sin\alpha(\cos\alpha-1)]\right\rangle\ , \tag{8}$$

where $\alpha\equiv kL(\cos\theta-v)$ and s is the spin of the field responsible for the inhomogeneities (i.e., $s=0$ denotes a scalar, $s=1$ denotes a vector, and $s=2$ denotes a tensor; in our special case of a gravitational-wave background the perturbations to the index of refraction are tensor: $\epsilon=\frac{1}{2}h_{xx}$). The factors $\sin^s\theta$ and ϵ_+ arise from the Euler transformation of a wave propagating in an arbitrary direction to components along the coordinate axes. The notation reflects the " + " polarization used in discussing gravitational waves. (We assume a stochastic background so both polarizations enter equally.)

Since $\mathrm{Re}\epsilon_+$ and $\mathrm{Im}\epsilon_+$ differ only by a random-phase factor, we may write

$$\langle\mathrm{Re}^2\epsilon_+\rangle=\langle\mathrm{Im}^2\epsilon_+\rangle,\quad\langle\mathrm{Re}\epsilon_+\mathrm{Im}\epsilon_+\rangle=0\ . \tag{9}$$

Now $k_y=k\sin\theta\cos\phi$ and $k_z=k\sin\theta\sin\phi$, so averaging over ϕ yields

$$\begin{aligned}\langle\theta_j{}^2\rangle&=\langle\mathrm{Re}^2\epsilon_+\rangle\langle\sin^{2+2s}\theta(\cos\theta-v)^{-2}(1-\cos\alpha)\rangle\\&=\tfrac{1}{2}\langle\mathrm{Re}^2\epsilon_+\rangle\int_{-1}^{+1}dx(1-x^2)^{1+s}(x-v)^{-2}(1-\cos\alpha)\\&=\tfrac{1}{2}kL\langle\mathrm{Re}^2\epsilon_+\rangle\int_{-kL(1+v)/2}^{kL(1-v)/2}dy\frac{\sin^2y}{y^2}\left[(1-v^2)-\frac{4v}{kL}y-\frac{4}{k^2L^2}y^2\right]^{1+s}\\&\equiv\tfrac{1}{2}kL\langle\mathrm{Re}^2\epsilon_+\rangle I\ ,\end{aligned} \tag{10}$$

where $x=\cos\theta$ and $y=\frac{1}{2}\alpha=\frac{1}{2}kL(x-v)$.

The problem is thus reduced to determining the dependence of the integral I on the path length L. We can write I as the sum $I=\sum_{n=-2}^{2s}a_nI_n$ where a_n is a constant independent of kL and

$$I_n=\int_{-kL(1+v)/2}^{kL(1-v)/2}dy\frac{\sin^2y}{y^2}\left[\frac{y}{kL}\right]^{n+2} \tag{11a}$$

$$=(kL)^{-n-2}\int_{-kL(1+v)/2}^{kL(1-v)/2}dy\,y^n\sin^2y\ . \tag{11b}$$

We examine I_n for $n\geq0$. The maximum value the integrand in (11b) can assume is of order $(kL)^n$ so the maximum of I_n is of order $(kL)^{-1}$; thus those terms in I corresponding to $n\geq0$ cannot cause an L effect in $\langle\theta_j{}^2\rangle$. That is, these terms do not vary as any positive power of L.

We are left with only the $n=-1$ and -2 terms. Expanding the term in large parentheses in Eq. (10) gives, for the relevant terms $J_n\equiv kLI_n$,

$$J_{-2}=kL(1-v^2)^{1+s}\int_{-kL(1+v)/2}^{kL(1-v)/2}dy\,y^{-2}\sin^2y\ , \tag{12a}$$

$$J_{-1}=-4(1+s)v(1-v^2)^s\int_{-kL(1+v)/2}^{kL(1-v)/2}dy\,y^{-1}\sin^2y\ . \tag{12b}$$

Since the integrands have no singularities we see that as $v\to1$ the L effect in J_{-2} vanishes and for $s>0$ the J_{-1} term likewise vanishes.

Thus we have proven for $s>0$ (in particular, this is the case for a gravitational-wave background, $s=2$), and will prove for $s=0$ in the next section, that the mean-square angular deviations are not directly proportional to the path length (or any positive power of it) as the inhomogeneities become highly relativistic. Figure 1 demon-

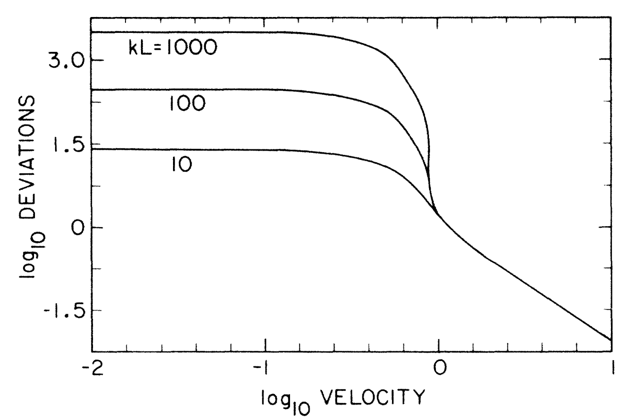

FIG. 1. The $\log_{10}$ of the mean-square angular deviations $\langle\theta^2\rangle/\langle\mathrm{Re}^2\epsilon_+\rangle$ is plotted vs the $\log_{10}$ of the reduced velocity $U=v/V$ for the case $s=2$. The curves are labeled by the reduced path length kL.

strates this clearly in a plot of the deviations versus the velocity for several different path lengths. (As introduced in Sec. IV A, V is the signal velocity, currently taken to equal 1.)

The lack of an L effect for the gravitational-wave case was first noted by Zipoy[3] and discussed by Bertotti and Catenacci;[4] the general case where $v \neq 0$ or $s \neq 2$ and $v \neq 1$ does not seem to have been previously investigated.

B. Leading-order terms

Now that we have shown that no L effect survives for $s>0$ as $v \to 1$, we examine the $s=0$ case and the leading-order terms that survive for $s>0$. For $s=0$ we are left with J_{-1} and must see how it depends on L.

We have used up our tricks and must resort to actual integration to evaluate J_{-1}. We could, of course, have derived the results of part (a) by explicit integration; doing this does indeed verify the conclusions made. Now

$$\int dy\, y^{-1}\sin^2 y = \tfrac{1}{2}\ln y - \tfrac{1}{2}\mathrm{ci}(2y) ,$$

where

$$\mathrm{ci}(x) \equiv -\int_x^\infty \frac{\cos t}{t} dt$$

is the cosine integral.

Thus, for $s=0$ the leading order term from J_{-1} gives

$$\langle \theta_j^2 \rangle = \langle \mathrm{Re}^2 \epsilon_+ \rangle v \times \{ -\ln[kL(1-v)] + \mathrm{ci}[kL(1-v)] + \ln[kL(1+v)] - \mathrm{ci}[kL(1+v)] \} \tag{13a}$$

or

$$\lim_{v \to 1} \langle \theta_j^2 \rangle = \langle \mathrm{Re}^2 \epsilon_+ \rangle [C + \ln(2kL) - \mathrm{ci}(2kL)] , \tag{13b}$$

where $C = 0.577\,215\ldots$ is Euler's constant. Overall (not just from J_{-1}) as kL gets large

$$\lim_{kL \to \infty} \lim_{v \to 1} \langle \theta_j^2 \rangle \sim \langle \mathrm{Re}^2 \epsilon_+ \rangle \ln(kL) \quad (s=0) , \tag{13c}$$

so although there is no L effect, the mean angular deviations do grow logarithmically for $s=0$.

Straightforward but tedious calculations give

$$\lim_{v \to 1} \langle \theta_j^2 \rangle = \tfrac{4}{3} \langle \mathrm{Re}^2 \epsilon_+ \rangle \quad (s=1) , \tag{14a}$$

$$\lim_{v \to 1} \langle \theta_j^2 \rangle = \tfrac{4}{5} \langle \mathrm{Re}^2 \epsilon_+ \rangle \quad (s=2) , \tag{14b}$$

and in general

$$\lim_{v \to 1} \langle \theta_j^2 \rangle = \langle \mathrm{Re}^2 \epsilon_+ \rangle 4^s \sum_{n=0}^{1+s} (-1)^n \frac{1}{n+s} \begin{bmatrix} 1+s \\ n \end{bmatrix} \quad (s>0) , \tag{14c}$$

for the leading-order terms (the other terms vanish as kL gets large). The $s=2$ case agrees with separate calculations[5] on scattering by a gravitational-wave background.

IV. DISCUSSION

A. Generalization of velocities

If we take the signal to propagate at velocity V rather than the speed of light then the harmonic factor $\mathbf{k}\cdot\mathbf{r} - \omega t = kr(\cos\theta - v/V)$ rather than $kr(\cos\theta - v)$. Thus we need but substitute $U = v/V$ for v everywhere in our previous calculation, and be a bit careful about the limits of the integrals.

For $U \leq 1$ the problem is identical to that just calculated and so those results hold. For $U > 1$ the limits on the integral in Eq. (10) are both negative and so the region of integration does not include zero. Once again examining I_n we see that the maximum value the integrand can attain is of order $(kL)^{-n-2}(kL)^n = (kL)^{-2}$ since y is always of order kL. Thus $\langle \theta_j^2 \rangle$ is of order $(kL)^0$ or less, i.e., there is no L effect for $U > 1$ from any of the terms (for all n). In fact, for $v \gg V$ the mean-square angular deviations die off as $(v/V)^{-2}$, as can be seen in Fig. 1.

Note that the vanishing of the L effect is not a relativistic effect *per se*. It does not depend on v or $V \to 1$, but rather $v \to V$, so it is a matter of relative velocities rather than relativistic ones. We discuss this further in the next section.

B. Coherence

The results of Secs. III A and IV A can be explained physically by examining the correlation between the signal and the medium motion, i.e., the coherence of what the signal experiences while propagating through the inhomogeneities. The characteristics discussed in the following paragraphs are exhibited in Fig. 2 where we plot the correlation function versus separation for various velocities.

For $U \ll 1$ the signal effectively sees a frozen field of variations in the index of refraction and it random walks through them. The correlation function is appreciably different from zero only for small separations, and there it is always positive, so one sees from Eq. (6) or the usual theory of random walks that the mean-square angular deviations grow as the path length.

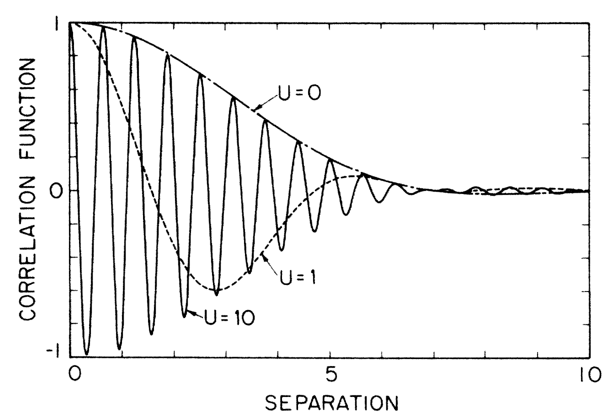

FIG. 2. The correlation function of the inhomogeneities, $R_{\epsilon_j}(y)/R_{\epsilon_j}(0)$, is plotted vs reduced separation, $y=kx$, for $s=2$. The curves are labeled by the reduced velocity U.

For $U \approx 1$ the signal almost "rides" the inhomogeneity waves and so alternating coherence and anticoherence occurs in the scattering; thus there is no secular increase in the deviations. An analogy exists in interference phenomena, where the superposition of phasors in alternately constructive and destructive interference with the first results in an intensity that does not grow with the number of superpositions.

When $U \gg 1$, so many variations affect the signal in a given path interval that their effects are blurred out to give an averaged homogeneous medium and so again there is no increase in the angular deviations with path length.

C. Calculation of induced red-shift

As a demonstration of and a check on the methods used here, we calculate the mean and mean-square red-shift induced in a light ray propagating through a gravitational-wave background, neglecting cosmological expansion and curvature effects.

For weak gravitational waves $\epsilon = \frac{1}{2} h_{xx}$, where $h_{\mu\nu} = g_{\mu\nu} - \eta_{\mu\nu}$ is the deviation of the metric from the Minkowski metric. From either Fermat's principle or the geodesic equation for the photon four-momentum we find the red-shift

$$z = \frac{\delta E}{E} = \theta_t = -\int_0^L \epsilon_{,t} dx$$
$$= -\frac{1}{2} \int_0^L h_{xx,t} dx \ , \tag{15}$$

where

$$h_{xx} = \sin^2\theta [\mathrm{Re}\, h_+ \cos(\mathbf{k}\cdot\mathbf{r} - \omega t) - \mathrm{Im} h_+ \sin(\mathbf{k}\cdot\mathbf{r} - \omega t)]$$

and E is the photon energy.

Our cosmological background is assumed stochastic; the gravitational waves are propagating in all directions with all polarizations and phases so

$$\langle \mathrm{Re} h_+ \rangle = \langle \mathrm{Im} h_+ \rangle = 0 \ .$$

Thus

$$\langle z \rangle = 0 \ , \tag{16}$$

as expected, since the light gets red-shifted climbing out of a potential well by the same amount that it was blue-shifted going in.

Now

$$\langle z^2 \rangle = \frac{1}{4} \left\langle \int_0^L h_{xx,t}(r,t) dx \int_0^L h_{xx,t}(r',t') dx' \right\rangle \ . \tag{17a}$$

Calculating the integrals and averages reveals

$$\langle z^2 \rangle = \tfrac{2}{3} \langle \mathrm{Re}^2 h_+ \rangle \ . \tag{17b}$$

It is convenient to express this in terms of the energy density of the gravitational waves ($G = c = 1$):

$$T_{00} = \frac{1}{32\pi} \sum_{i,j} \langle h_{ij,0} \rangle \ , \tag{18}$$

where i,j run over x, y, and z and ,0 is the same as ,t.

A tedious calculation involving the properties of $h_{\mu\nu}$ gives

$$T_{00} = \frac{\pi}{2} \nu^2 \langle \mathrm{Re}^2 h_+ \rangle \ , \tag{19}$$

where $\nu = \omega / 2\pi$ is the frequency of the wave. Thus

$$\langle z^2 \rangle = \frac{4}{3\pi} \nu^{-2} T_{00} \ , \tag{20}$$

which agrees exactly with the expression of Burke,[6] obtained using a Killing-vector—group-theory approach, thus providing an independent test of our method.

D. End-point terms

As mentioned in Sec. II, the expression for the angular deviations, Eq. (2), is not wholly correct. Now that we have seen that the terms proportional to path length vanish as $U \to 1$, we must insert the end-point terms in Eq. (2) to obtain the full result. The red-shift expression, Eq. (15), is not affected.

For general spin fields, we can write the index of refraction as

$$n(x,t) = \bar{n} \left[1 + m_{a_1 \cdots a_s} \frac{dx^{a_1}}{dt} \cdots \frac{dx^{a_s}}{dt} \right] ,$$

so, to first order in the deviations,

$$n = \bar{n} \left[1 + m_{1 \cdots 11} + s m_{1 \cdots 1j} \frac{dx^j}{dt} \right] ,$$

where $1 \cdots 1$ contains $s-1$ ones and $j = 2,3$ $[(t,x,y,z) = (0,1,2,3)]$. By Eq. (1) this defines ϵ.

The full expression for the angular deviations is

$$\theta_j = -\left. \frac{d\epsilon}{d\left[\frac{dx^j}{dt}\right]} \right|_0^L + \int_0^L \epsilon_{,j} dx$$
$$= -s m_{1 \cdots 1j} \big|_0^L + \int_0^L m_{1 \cdots 11,j} dx \ . \tag{21}$$

Note that scalar ($s = 0$) fields have no end-point contributions.

In the mean-square angular deviations the cross terms vanish, giving

$$\langle \theta_j^2 \rangle = 2 s^2 \langle m_{1 \cdots 1j}^2(0) \rangle + \left\langle \int_0^L m_{1 \cdots 11,j} dx \int_0^L m_{1 \cdots 11,j} dx' \right\rangle . \tag{22}$$

The second (path) term is what we have calculated in the preceding sections. Evaluating the first (end-point) term's autocorrelation function, one finds it equal to one-half of Eq. (14c). Thus,

$$\langle \theta_j^2 \rangle_{\text{total}} = \langle \theta_j^2 \rangle_{\text{path}} + \langle \theta_j^2 \rangle_{\text{end point}}$$
$$= \langle \theta_j^2 \rangle_{\text{path}} + s^2 \langle \theta_j^2 \rangle_{\text{path}\,(U=1, kL \to \infty)} \ . \tag{23}$$

For $U \geq 1$ we see the end-point term provides the dominant contribution to the angular deviations (for $s \neq 0$). Note that in Fig. 1 the end-point contribution is not included, to aid in discerning the limiting behaviors of $\langle \theta^2 \rangle_{\text{path}}$.

V. CONCLUSIONS

We have investigated the propagation of a signal through an inhomogeneous, turbulent medium and found that the classical behavior, where the mean-square angular deviations grow linearly with path length L, does not always hold. It is obeyed only when the velocity of the inhomogeneities v is much less than the signal velocity V. When the variations in the index of refraction move at speeds comparable to or greater than the signal ($v/V \geq 1$), there is no L effect.

As $v \to V$ the proportionality of the angular deviations to the path length vanishes as $[1-v^2/V^2]^{1+s}$, where s is the spin of the field responsible for the inhomogeneities. In particular, for the problem of light propagating through a cosmological background of gravitational waves we found

$$\lim_{v\to 1} \langle \theta^2 \rangle \sim \langle \mathrm{Re}^2 \epsilon_+ \rangle \left[\frac{\pi}{2}(1-v^2)^3 kL + O((1-v^2)^2 \ln(kL)) + 8 \right] \tag{24}$$

and

$$\langle z^2 \rangle = \tfrac{2}{3} \langle \mathrm{Re}^2 h_+ \rangle \tag{25a}$$

$$= \frac{4}{3\pi} \nu^{-2} T_{00} \,, \tag{25b}$$

for the mean-square angular deviation $\langle \theta^2 \rangle = \langle \theta_y{}^2 \rangle + \langle \theta_z{}^2 \rangle$ and the mean-square red-shift. For $v \gg V$ the mean-square deviations die off as $(v/V)^{-2}$.

Note that the quantities calculated in this paper were monochromatic, i.e., assuming inhomogeneities of a single frequency ω. If the inhomogeneities at different frequencies are statistically independent then we simply replace θ^2 by $\theta^2(\omega)$ and $\langle \mathrm{Re}^2 \epsilon_+(\omega) \rangle$ by $2\omega^{-2} S_e(\omega)$, where $S_e(\omega)$ is the power spectral density, to find

$$\langle \theta^2 \rangle = \int_0^\infty dw \, \theta^2(\omega) \,.$$

A similar argument holds for a range of velocities v.

Finally, the use of correlation functions and the concept of the coherence of the scattering provide valuable insights into the behavior of the deviations in various regimes.

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