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Five-dimensional gravitational super Chern-Simons terms

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We consider the Chern-Simons five-form C_5(CS) for the five-dimensional anti–de Sitter superalgebra SU(2,2 | 1), whose exterior derivative is equal to $d_{ABC}R^CR^BR^A$ where R^A are the full group curvatures and d_{ABC} are the invariant d symbols of SU(2,2 | 1). Under the simple local five-dimensional supersymmetry transformations, C_5(CS) is not invariant. Its relation to the invariant action of simple five-dimensional supergravity in the group-manifold formulation is given.

Gravitational Chern-Simons (CS) terms play an important role in supergravity,[1] but at present, the existence of a supersymmetric extension of them is still an open question. It is known that CS terms in odd dimensions, $d=2n+1$, can be obtained from invariant polynomials P in one dimension higher, $d=2n+2$, constructed from the curvature two-forms $F=F^a\lambda_a$, where λ_a are the generators of some Lie group.[2]

For example in $d=6$, one can start from

$$P=\mathrm{tr}FFF=d_{abc}F^aF^bF^c \ , \qquad (1)$$

where d_{abc} is the completely symmetric three-index invariant tensor of the Lie algebra considered. It is well known that d_{abc} is nonzero only for SU(N) with $N\geq 3$. Since $dP=0$, because of the Bianchi identities $DF=0$, one has locally

$$P=dC_5 \ . \qquad (2)$$

The five-form C_5 is then the CS form, and is in this case given by

$$C_5=\mathrm{tr}(F^2A-\tfrac{1}{2}FA^3+\tfrac{1}{10}A^5) \qquad (3)$$

with $F=dA+AA$ and A being the Yang-Mills gauge connection.

Under a transformation $A\to A+\delta A$, the corresponding variation of the integrated CS term is the integral of δA times a covariant expression, which is proportional to the covariant anomaly in one dimension lower, namely, in $d=2n$. In our example we have

$$\delta\int C_5=\int \mathrm{tr}(\delta A\,3FF)=\int 3d_{abc}F^aF^b\delta A^c \ , \qquad (4)$$

where the four-form FF is indeed proportional to the covariant anomaly in four dimensions.[2]

On the other hand, if one performs a gauge transformation $A\to A+\delta A$, with $\delta A=d\Lambda+[A,\Lambda]$, where Λ is the gauge parameter, the variation of the CS term is locally closed and yields the consistent anomaly in $d=2n$ dimensions. Using again our explicit example, we have

$$\begin{aligned}\delta(\text{gauge})C_5&=\mathrm{tr}[d\Lambda\, d(A\,dA+\tfrac{1}{2}AAA)]\\&=d\,\mathrm{tr}[\Lambda\, d(A\,dA+\tfrac{1}{2}AAA)] \ ,\end{aligned} \qquad (5)$$

where $d(A\,dA+\frac{1}{2}AAA)$ is the consistent anomaly for a system of chiral fermions coupled to Yang-Mills field in four-dimensions.[2]

Because of the closure of $\delta(\text{gauge})C_5$, the integrated CS term in odd dimensions is a gauge-invariant expression in a trivial space-time with fields falling off sufficiently fast at infinity, despite the explicit appearance of the bare connections A. Exploiting this latter property, CS forms have been used to build gauge-invariant supergravity actions in odd dimensions.[3] In this paper we extend these ideas to superalgebras, instead of ordinary algebras.

It was shown in Ref. 4 that the action of $d=3$ simple conformal supergravity can be written as

$$I_3=\int(\gamma_{AB}R^B\omega^A+\tfrac{1}{6}f_{ABC}\omega^C\omega^B\omega^A) \ , \qquad (6)$$

where R^A and ω^A are the curvature two-forms and connection one-forms of the superalgebra Osp(1|4), γ_{AB} its Killing metric, and $f_{ABC}=\gamma_{AD}f^D{}_{BC}$ its structure constants. This is the CS form I_3 (CS) belonging to Osp(1|4), because $dI_3=\int\gamma_{AB}R^BR^A$.

This example suggests further extensions. For instance, in $d=6$ we can start from the six-form

$$P=d_{ABC}R^CR^BR^A \ , \qquad (7)$$

where d_{ABC} are the d-symbols and R^A the curvatures of the superalgebra SU(2,2|1). We are interested in SU(2,2|1) because it is the superalgebra of simple anti–de Sitter supergravity in five dimensions.[5,6] Its maximal bosonic subgroup is SU(2,2)⊗U(1), where SU(2,2) is locally isomorphic to the anti–de Sitter group SO(4,2) in five dimensions. Since $dP=0$, because of the Bianchi identities, locally $P=dC_5$ with

$$\begin{aligned}C_5=\ &d_{ABC}R^CR^B\omega^A+e_{ABCD}R^D\omega^C\omega^B\omega^A\\&+f_{ABCDE}\omega^E\omega^D\omega^C\omega^B\omega^A \ ,\end{aligned} \qquad (8)$$

where ω^A are the one-form connections and e_{ABCD} and f_{ABCDE} are suitable combinations of the d-symbols and the structure constants of SU(2,2|1). The expression given by (8) is the CS five-form belonging to the superalgebra SU(2,2|1) and since its integral is gauge invariant, it can be used as an action of a theory in five dimensions.

In this paper we will examine the possibility that (8) be invariant under the transformations of five-dimensional simple supergravity, in which case it could be added to the action of this theory. As we shall discuss, the transformation rules of the fields of this supergravity model (vielbein, gravitino, and photon) are the sum of gauge transformations and curvature terms. We shall use second-order formalism with a dependent spin connection.

The study of the possibility that the CS form be supersymmetric is quite interesting for three reasons: (i) Its variation is simply the product of two curvatures and a varied connection, and only the nongauge part of the varied connection contributes, as we shall see; (ii) only the spin connection and gravitino have nonvanishing such variations, and these extra variations are themselves proportional to curvatures; (iii) the curvatures are not all independent, but $R(P)$ is linearly related to $R(B)$, see below. Thus, all one has to do is to consider the various products of three curvatures in the variation of C_5 and determine whether their coefficients vanish. The question of invariance of C_5 reduces thus to a purely algebraic problem.

A similar analysis has been performed in Ref. 7 but using the Noether method. These authors constructed an action invariant under local five-dimensional Poincaré supergravity, at the two lowest orders in the gravitational coupling constant. Using our geometrical approach, we find that same structure, plus other terms that complete the CS five-form of SU(2,2|1). If it would have turned out that this purely geometrical construction yielded a supersymmetric extension of the bosonic CS terms, it would have been useful for the analogous ten-dimensional problem.[8]

We now turn to the CS form (8) and investigate its invariance properties under supersymmetry transformations. We will directly analyze its variation, namely,

$$\delta \int C_5 = \int 3 d_{ABC} R^C R^B \delta\omega^A \; . \tag{9}$$

If T_A are the generators of the superalgebra SU(2,2|1), the d symbols are defined as

$$d_{ABC} = \mathrm{str}(T_A \{ T_B, T_C]) \; . \tag{10}$$

The symbol $\{\ ]$ is a shorthand notation for $[T_A, T_B]$ if T_A and T_B are both fermionic and for $\{T_A, T_B\}$ if T_A and/or T_B are bosonic. The d_{ABC} are an invariant tensor of SU(2,2|1) and by construction are completely supersymmetric, in the sense that $d_{ABC} = (-1)^{AB} d_{BAC} = (-1)^{BC} d_{ACB}$, where A is the grading ($A=0$ for a bosonic generator and $A=1$ for a fermionic one).

In order to perform the explicit computations for the d symbols, we choose for T_A the representation

$$\begin{gathered} T_A = \{ T_a, T_{ab}, T_\otimes ; T_\alpha \} \; , \\ T_a = \begin{bmatrix} \frac{1}{2}\Gamma_a & 0 \\ 0 & 0 \end{bmatrix}, \quad T_{ab} = \begin{bmatrix} \Sigma_{ab} & 0 \\ 0 & 0 \end{bmatrix}, \\ T_\otimes = \begin{bmatrix} \frac{1}{4} 1_4 & 0 \\ 0 & 1 \end{bmatrix}, \quad T_\alpha = \begin{bmatrix} 0 & \xi_\alpha \\ -\bar{\xi}_\alpha & 0 \end{bmatrix}, \end{gathered} \tag{11}$$

where $a = 0, \ldots, 4$; $\alpha = 1, \ldots, 4$ and Γ_a are the Dirac matrices in five dimensions, satisfying

$$\{ \Gamma_a, \Gamma_b \} = 2\eta_{ab} \text{ with } \eta_{ab} = (+,-,-,-,-) \tag{12}$$

and $\Sigma_{ab} = (i/4)[\Gamma_a, \Gamma_b]$. The ξ_α are complex numbers such that for any complex spinor λ^α

$$\lambda^\alpha T_\alpha = \lambda^\alpha \begin{bmatrix} 0 & \xi_\alpha \\ -\bar{\xi}_\alpha & 0 \end{bmatrix} = \begin{bmatrix} 0 & \lambda \\ -\bar{\lambda} & 0 \end{bmatrix} . \tag{13}$$

[In these formulas the bar is given by $\bar{\lambda} = \lambda^\dagger \Gamma_0$ with Hermitian Γ_0; see (12).]

From the definition (10), we find that the only nonvanishing d symbols are

$$\begin{aligned} d_{\otimes\otimes\otimes} &= \mathrm{str}(T_\otimes \{ T_\otimes, T_\otimes \}) = -\tfrac{15}{8} \; , \\ d_{\otimes ab} &= \mathrm{str}(T_\otimes \{ T_a, T_b \}) = \tfrac{1}{2}\eta_{ab} \; , \\ d_{\otimes abcd} &= \mathrm{str}(T_\otimes \{ T_{ab}, T_{cd} \}) = \tfrac{1}{2}(\eta_{ac}\eta_{bd} - \eta_{ad}\eta_{bc}) \; , \\ d_{abcde} &= \mathrm{str}(T_a \{ T_{bc}, T_{de} \}) = \epsilon_{abcde} \; , \\ d_{\otimes\alpha\beta} &= \mathrm{str}(T_\otimes [T_\alpha, T_\beta]) = -\tfrac{5}{4}(\bar{\xi}_\alpha \xi_\beta - \bar{\xi}_\beta \xi_\alpha) \; , \\ d_{a\alpha\beta} &= \mathrm{str}(T_a [T_\alpha, T_\beta]) = -\tfrac{1}{2}(\bar{\xi}_\alpha \Gamma_a \xi_\beta - \bar{\xi}_\beta \Gamma_a \xi_\alpha) \; , \\ d_{ab\alpha\beta} &= \mathrm{str}(T_{ab} [T_\alpha, T_\beta]) = -(\bar{\xi}_\alpha \Sigma_{ab} \xi_\beta - \bar{\xi}_\beta \Sigma_{ab} \xi_\alpha) \; . \end{aligned} \tag{14}$$

We associate to each generator T_A a connection one-form $\omega^A = (V^a, \omega^{ab}, B, \psi^\alpha)$ and a curvature two-form $R^A = (R^a, R^{ab}, R^\otimes, \rho^\alpha)$ where

$$R^A = d\omega^A - \tfrac{1}{2} f^A{}_{BC}\omega^C \omega^B \; , \tag{15}$$

$f^A{}_{BC}$ are the structure constants of SU(2,2|1), which in our conventions are defined by $[T_A, T_B\} = -i T_C f^C{}_{AB}$. Explicitly the curvatures are given by

$$\begin{aligned} R^a &= dV^a - \omega^{ab} V_b - \frac{i}{2}\bar{\psi}\Gamma^a \psi \; , \\ R^{ab} &= d\omega^{ab} - \omega^{ac}\omega_c{}^b + (V^a V^b - i\bar{\psi}\Sigma^{ab}\psi) \; , \\ \rho &= d\psi + \frac{i}{2}\omega^{ab}\Sigma_{ab}\psi + \left[\frac{i}{2} V^a \Gamma_a \psi - \frac{3i}{4} B\psi \right] , \\ R^\otimes &= dB - i\bar{\psi}\psi \; . \end{aligned} \tag{16}$$

Except for the terms in parentheses, these curvatures are covariant under the following rescalings:

$$\begin{aligned} &\omega^{ab} \to \omega^{ab}, \quad V^a \to e V^a \; , \\ &B \to eB, \quad \psi \to \sqrt{e}\,\psi \; , \\ &R^{ab} \to R^{ab}, \quad R^a \to e R^a \; , \\ &R^\otimes \to e R^\otimes, \quad \rho \to \sqrt{e}\,\rho \; . \end{aligned} \tag{17}$$

Using these connections and curvatures, we can construct the CS term of SU(2,2|1). It is defined by

$$P = d_{ABC} R^C R^B R^A = dC_5(\mathrm{CS}) \tag{18}$$

since $dP = 0$, as one can explicitly check with the definitions (14) and (16).

Under an arbitrary variation $\omega^A \to \omega^A + \delta\omega^A$, the curvatures transform into covariant derivatives of the varied connections

$$\delta R^A = D(\delta\omega^A) = d\delta\omega^A - f^A{}_{BC}\omega^C \delta\omega^B \; . \tag{19}$$

Therefore from (18), it follows that

$$\delta P = 3d_{ABC}R^C R^B \delta R^A = d[\delta C_5(\mathrm{CS})] \tag{20}$$

and using (19) and the Bianchi identities, after an integration by parts, we find

$$\delta \int C_5(\mathrm{CS}) = \int 3d_{ABC}R^C R^B \delta\omega^A . \tag{21}$$

We are interested in the invariance of $C_5(\mathrm{CS})$ under the transformation laws of five-dimensional simple supergravity. In the second-order formalism, they are, in the conventions of Ref. 6,

$$\delta V^a = -\frac{i}{2}(\bar\epsilon\Gamma^a\psi - \bar\psi\Gamma^a\epsilon) , \tag{22a}$$

$$\delta B = i(\bar\epsilon\psi - \bar\psi\epsilon) , \tag{22b}$$

$$\delta\psi = \left[d + \frac{i}{2}\hat\omega^{ab}\Sigma_{ab} \right]\epsilon + \left[\frac{i}{2}\Gamma_a V^a\epsilon - \frac{3i}{4}B\epsilon \right] + \left[\tfrac{1}{2}\Gamma^a\hat F_{ab}V^b - \frac{i}{8}\epsilon_{abcdf}\hat F^{cd}\Sigma^{ab}V^f \right]\epsilon . \tag{22c}$$

Note that these transformation laws preserve the scaling properties in (17), except for the two terms within the second set of large parentheses, if we define that ϵ has scale $\frac{1}{2}$.

In (22c) the one-form $\hat\omega^{ab} = \hat\omega^{ab}(V,\psi)$ is the usual supercovariant spin connection whose components are given by

$$\hat\omega_{\mu ab} = [-V_a{}^\nu(\partial_{[\mu}V_{\nu]b})] - (a\leftrightarrow b) + V_a{}^\rho V_b{}^\sigma(\partial_{[\rho}V_{\sigma]}{}^c)V_{c\mu} + \frac{i}{2}\bar\psi_\mu\Gamma_{[a}\psi_{b]} + \frac{i}{4}\bar\psi_a\Gamma_\mu\psi_b , \tag{23}$$

while $\hat F_{\mu\nu}$ is the supercovariant photon curl, defined by

$$R^{\otimes} = \hat F_{\mu\nu}dx^\mu \wedge dx^\nu = (\partial_\mu B_\nu - i\bar\psi_\mu\psi_\nu)dx^\mu\wedge dx^\nu . \tag{24}$$

(In this and in the following formulas, the square brackets on indices mean antisymmetrization with strength one.)

The transformations (22) consist of two parts: (1) a pure gauge part

$$\delta(\text{gauge})\omega^A = d\epsilon^A - f^A{}_{BC}\omega^C\epsilon^B , \tag{25}$$

where ϵ^A are the gauge parameters; (2) an extra part $\delta(\text{extra})\omega^A$. More precisely, looking at (22), we note that the vielbein and photon transformation laws (22a) and (22b) are pure gauge supersymmetry transformations, while the gravitino law (22c) contains, besides a pure gauge term, an extra part, proportional to the curvature $\hat F_{\mu\nu}$.

Since the choice of the spin connection as a function of the other independent fields is essentially arbitrary, we can define a new spin connection ω_{ab} as

$$\omega_{ab} = \hat\omega_{ab} - \tfrac{1}{4}\epsilon_{abcdf}\hat F^{cd}V^f . \tag{26}$$

In this way, we can absorb some of the $\hat F$-dependent terms of (22c) into the pure gauge part, and simplify the extra piece. Using (26) we have

$$\delta\psi = \left[d + \frac{i}{2}\omega^{ab}\Sigma_{ab} \right]\epsilon + \frac{i}{2}\Gamma^a V_a\epsilon - \frac{3i}{4}B\epsilon + \tfrac{1}{2}\Gamma^a\epsilon\hat F_{ab}V^b = \delta(\text{gauge})\psi + \delta(\text{extra})\psi , \tag{27}$$

where now $\delta(\text{extra})\psi = \frac{1}{2}\Gamma^a\epsilon\hat F_{ab}V^b$. This term cannot be included in the gauge part because of its different Γ-matrix structure. The choice of (26) as spin connection is also suggested by the group manifold approach, where this same expression of ω_{ab} is found by solving the torsion equation of five-dimensional simple supergravity,[6] namely,

$$R^a + \tfrac{1}{4}\epsilon^{abcdf}\hat F_{bc}V_dV_f = 0 . \tag{28}$$

We can now use this equation to determine the variation of ω^{ab} induced by (22a), (22b), and (27). In fact (28) can be viewed as a constraint on curvatures that must be preserved by supersymmetry. This implies

$$\delta R^a + \tfrac{1}{4}\epsilon^{abcdf}\delta\hat F_{bc}V_dV_f + \tfrac{1}{2}\epsilon^{abcdf}\hat F_{bc}\delta V_dV_f = 0 . \tag{29}$$

Since $\delta R^a = \delta(\text{gauge})R^a + \delta(\text{extra})R^a$, using the definition (16), we find

$$\delta(\text{gauge})R^a = \frac{i}{2}(\bar\epsilon\Gamma^a\rho - \bar\rho\Gamma^a\epsilon) , \tag{30a}$$

$$\begin{aligned}\delta(\text{extra})R^a &= -[\delta(\text{extra})\omega^{ab}]V_b - \frac{i}{2}[\delta(\text{extra})\bar\psi\Gamma^a\psi + \bar\psi\Gamma^a\delta(\text{extra})\psi] \\ &= -[\delta(\text{extra})\omega^{ab}]V_b - \frac{i}{4}\hat F^{mn}V_n(\bar\epsilon\Gamma_m\Gamma^a\psi - \bar\psi\Gamma^a\Gamma_m\epsilon) .\end{aligned} \tag{30b}$$

Inserting the expressions (30) into (29) and using the explicit form of the transformation laws, we get the equation

$$\begin{aligned}&\frac{i}{2}(\bar\epsilon\Gamma^a\rho - \bar\rho\Gamma^a\epsilon) - \frac{i}{4}\hat F^{mn}V_n(\bar\epsilon\Gamma_m\Gamma^a\psi - \bar\psi\Gamma^a\Gamma_m\epsilon) - [\delta(\text{extra})\omega^{ab}]V_b + \frac{i}{4}\epsilon^{abcdf}\hat F_{bc}(\bar\epsilon\Gamma_d\psi - \bar\psi\Gamma_d\epsilon)V_f \\ &\quad - \frac{i}{8}\epsilon^{abcdf}(\bar\epsilon\Gamma^e\psi_b - \bar\psi_b\Gamma^e\epsilon)\hat F_{ec}V_dV_f + \frac{i}{4}\epsilon^{abcdf}(\bar\epsilon\rho_{bc} - \bar\rho_{bc}\epsilon)V_dV_f = 0 .\end{aligned} \tag{31}$$

We can now solve it for $\delta(\text{extra})\omega_\mu{}^{ab}$ and find

$$\delta(\text{extra})\omega_{\mu ab} = \Bigg[i\bar{\epsilon}\Gamma_{[a}\rho_{b]\mu} + \frac{i}{2}\bar{\epsilon}\Gamma_\mu\rho_{ab} - \frac{i}{4}\epsilon_{\mu abcd}\bar{\epsilon}\rho^{cd} - \frac{i}{4}\epsilon_{abcde}\hat{F}^{de}\bar{\epsilon}\Gamma^c\psi_\mu + \frac{i}{8}\epsilon_{\mu abcd}\hat{F}^{md}\bar{\epsilon}\Gamma_m\psi^c - \frac{i}{4}\bar{\epsilon}\Gamma^c\Gamma_\mu\psi_{[a}F_{b]c} - \frac{i}{4}\hat{F}_{c[a}\bar{\epsilon}\Gamma^c\Gamma_{b]}\psi_\mu - \frac{i}{4}\hat{F}_{c\mu}\bar{\epsilon}\Gamma^c\Gamma_{[a}\psi_{b]}\Bigg] + \text{H.c.} \tag{32}$$

We observe that the $\epsilon\cdots\hat{F}^{\cdot\cdot}\bar{\epsilon}\Gamma^{\cdot}\psi^{\cdot}$ terms in Eq. (32) come from the explicit variation of the vielbein in $-\frac{1}{4}\epsilon_{abcdf}\hat{F}^{cd}V^f$, from the extra variation of the curvature $\hat{F}$ and from the torsion of the connection $\omega_\mu{}^{ab}$. We have checked that this same result is obtained by applying the chain rule to (26), i.e.,

$$\delta\omega^{ab} = \left[\frac{\delta\omega^{ab}}{\delta V}\delta V + \frac{\delta\omega^{ab}}{\delta\psi}\delta\psi + \frac{\delta\omega^{ab}}{\delta\hat{F}}\delta\hat{F} \right] = \delta(\text{gauge})\omega^{ab} + \delta(\text{extra})\omega^{ab} \tag{33}$$

and using

$$\delta(\text{gauge})\omega^{ab} = -\frac{i}{2}(\bar{\epsilon}\Sigma^{ab}\psi - \bar{\psi}\Sigma^{ab}\epsilon) .$$

The variation of $C_5(\text{CS})$ due to the pure gauge transformations in (25) always cancels, because of the Bianchi identities. Therefore the proof of the invariance of $C_5(\text{CS})$ is reduced to the extra terms

$$\delta C_5(\text{CS}) = \int 3d_{ABC}R^C R^B \delta(\text{extra})\omega^A , \tag{34}$$

where the only nonzero contributions are for $A=(ab)$ and $A=\alpha$, since only the spin connection and the gravitino have an extra part in their transformation laws.

We stress that both these extra variations are proportional to curvatures. As a consequence we have a purely algebraic problem to be solved.

We are then left with

$$\frac{1}{3}\int \delta C_5(\text{CS}) = \int [\, d_{(ab)CD}R^D R^C \delta(\text{extra})\omega^{ab} + d_{\alpha CD}R^D R^C \delta(\text{extra})\psi^\alpha] \tag{35}$$

and substituting the d_{ABC} coefficients (14) we obtain

$$\begin{aligned}\frac{1}{3}\int \delta C_5(\text{CS}) &= \int (2d_{(ab)\otimes(cd)}R^{cd}R^{\otimes} + 2d_{(ab)c(de)}R^{de}R^c + d_{(ab)\alpha\beta}\rho^\beta\rho^\alpha)\delta(\text{extra})\omega^{ab} \\ &\quad + 2(d_{\alpha\otimes\beta}\rho^\beta R^{\otimes} + d_{\alpha a\beta}\rho^\beta R^a + d_{\alpha(ab)\beta}\rho^\beta R^{ab})\delta(\text{extra})\psi^\alpha \\ &= \int 2\{(R^{\otimes}R_{ab} + \epsilon_{abcde}R^c R^{de} + \bar{\rho}\,\Sigma_{ab}\rho)\delta(\text{extra})\omega^{ab} + R^{ab}[\bar{\rho}\,\Sigma_{ab}\delta(\text{extra})\psi + \delta(\text{extra})\bar{\psi}\Sigma_{ab}\rho]\} \\ &\quad + \{R^a[\bar{\rho}\,\Gamma_a\delta(\text{extra})\psi + \delta(\text{extra})\bar{\psi}\Gamma_a\rho] + \tfrac{5}{2}R^{\otimes}[\bar{\rho}\delta(\text{extra})\psi + \delta(\text{extra})\bar{\psi}\rho]\} ,\end{aligned} \tag{36}$$

where according to (32) and (27), $\delta(\text{extra})\omega^{ab}$ contains ρ and $\hat{F}$ curvatures, while $\delta(\text{extra})\psi$ is proportional only to $\hat{F}$. Moreover, we recall that in second-order formalism we have $R^a \sim \hat{F}$ [see Eq. (28)].

From (17) we can see that the terms in the first set of parentheses of (36) have all scale e, while the others have all scale e^2. [Note that $\delta(\text{extra})\omega$ and $\delta(\text{extra})\psi$ have the same scale as ω and ψ.] This gives rise to two disconnected sectors that should separately be invariant. The terms with scale e are the same as those considered in Ref. 7: it is remarkable how this geometrical procedure allows to determine them in a much easier way.

In order to check the invariance of $C_5(\text{CS})$, we start by considering the $\bar{\rho}\,\Sigma_{ab}\rho\delta(\text{extra})\omega^{ab}$ term in (36) since it is the only term that contains at least two fermionic curvatures. It generates a $\rho\rho\rho$ term that, as demonstrated in Ref. 7, is proportional to the gravitino field equation, and some $\rho\rho\hat{F}$ terms. The former can thus be canceled by adding a $\rho\rho$ term to the gravitino transformation law. The latter read, in component notation,

$$\left[\frac{i}{2}\bar{\rho}_{\alpha\beta}\Sigma^{ab}\rho_{\gamma\delta}(-\epsilon_{abcde}\hat{F}^{de}\bar{\epsilon}\Gamma^c\psi_\mu + \tfrac{1}{2}\epsilon_{\mu abcd}\hat{F}^{md}\bar{\epsilon}\Gamma_m\psi^c - \hat{F}_{ca}\bar{\epsilon}\Gamma^c\Gamma_\mu\psi_b - \hat{F}_{ca}\bar{\epsilon}\Gamma^c\Gamma_b\psi_\mu - \hat{F}_{c\mu}\bar{\epsilon}\Gamma^c\Gamma_a\psi_b) + \text{H.c.} \right] \epsilon^{\alpha\beta\gamma\delta\mu} . \tag{37}$$

They are essentially reducible to two independent structures,

$$(\bar{\rho}\,\Gamma\Gamma\rho\bar{\epsilon}\,\Gamma\psi\hat{F} + \text{H.c.}) \text{ and } (\bar{\rho}\,\Gamma\Gamma\rho\bar{\epsilon}\,\Gamma\Gamma\psi\hat{F} + \text{H.c.}) , \tag{38}$$

that turn out to be different from zero. Since they cannot be canceled by any other term in $\delta C_5(\text{CS})$, their particular structure is enough to prevent $C_5(\text{CS})$ from being invariant under complete supersymmetry transformations.

Also the terms with scale e^2 in (36), namely,

$$\{R^a[\bar{\rho}\,\Gamma_a\delta(\text{extra})\psi + \delta(\text{extra})\bar{\psi}\Gamma_a\rho] + \tfrac{5}{2}R^{\otimes}[\bar{\rho}\delta(\text{extra})\psi + \delta(\text{extra})\bar{\psi}\rho]\} \tag{39}$$

do not cancel among themselves; using (28) and (27) they give rise to two independent structures

$$(\hat{F}\hat{F}\bar{\epsilon}\Gamma\Gamma\rho+\text{H.c.}) \text{ and } (\hat{F}\hat{F}\bar{\epsilon}\Gamma\rho+\text{H.c.}) \tag{40}$$

that are different from zero.

The Poincaré version of five-dimensional supergravity, based on the contracted superalgebra $\overline{\text{SU}(2,2\,|\,1)}$ that contains $\text{ISO}(4,1)\otimes\text{U}(1)$ as maximal bosonic subgroup, presents exactly the same features and also in this case $C_5(\text{CS})$ is not invariant. The only thing that changes in the contraction is that the terms which violate the scaling properties in the curvatures and transformation laws are dropped, but $\delta(\text{extra})$ in terms of the curvature is unchanged.

Both in the anti–de Sitter and Poincaré case, the action of five-dimensional supergravity contains terms quadratic in the curvatures that at first sight can be thought of as belonging to a CS form. Referring to the group manifold formulation[6] they read

$$h=\cdots+\tfrac{1}{4}R^{\otimes}R^{\otimes}B+\eta R^{a}R_{a}B+\cdots, \tag{41}$$

where η is a parameter of the theory that is fixed to be $\eta=\pm1$, by the requirement of the existence of nontrivial solutions for the equations of motion. The corresponding terms coming from our CS form are

$$C=d_{\otimes\otimes\otimes}R^{\otimes}R^{\otimes}B+2d_{ba\otimes}R^{\otimes}R^{a}V^{b}+d_{\otimes ab}R^{b}R^{a}B\ . \tag{42}$$

Since with an integration by parts $R^{\otimes}R^{a}V^{b}=R^{a}R^{b}B+\cdots$ we have

$$C=-\tfrac{15}{2}\left(\tfrac{1}{4}R^{\otimes}R_{\otimes}B-\tfrac{1}{5}R^{a}R_{a}B\right) \tag{43}$$

and so it is clear that even if in principle C seems to be a part of the action of five-dimensional supergravity, supersymmetry requires a ratio between the coefficients different from that of a CS term.

Although the Chern-Simons term is not invariant under the transformation rules of five-dimensional simple supergravity, it is a familiar procedure in supergravity to modify the action and transformation rules such that invariance is obtained. Let us consider the terms with scale e in (36). As we already mentioned, the terms proportional to $\rho\rho\rho$ vanish on-shell and can be canceled by modifying the gravitino transformation law. The other terms are of the form

$$FR^{\cdot\cdot}\rho,\ \ FFR^{\cdot\cdot}\psi,\ \ \rho\rho F\psi\ , \tag{44}$$

after using (28), but they do not vanish on-shell. Thus, one cannot modify the transformation rules to obtain invariance.

We have to conclude that, in spite of their clear and nice geometrical meaning, CS forms do not seem to be supersymmetric by themselves, at least in five dimensions. It may be that for superalgebras more than one d symbol exists; in that case, one would have to study those new CS terms.

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