Gravitational entropy: Beyond the black hole

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We attempt to establish the existence of gravitational entropy for a thin spherical shell of matter as a function of radius by examining the modifications to the thermodynamics of a black hole at the center of such a shell. We show that the shell has the effect of depressing the temperature of the hole, but that small exchanges of energy between the hole and its environment at the same temperature remain isentropic in the presence of the shell. The model is generalized to include de Sitter horizons too, and we find that the shell can be positioned so that a "back flow" of radiation takes place from the de Sitter horizon into the hole, thus enabling the hole-plus-shell system to be used as a device for "mining the Universe." In all cases our results are consistent with simple arguments indicating that there is no gravitational entropy associated with spherical shells which have not collapsed into black holes.

I. INTRODUCTION

The existence of a gravitational "arrow of time" has long been established.[1] Specifically, a self-gravitating system tends to grow more and more irregular with time. A uniform gas, for example, will become progressively more clumpy, as centers of gravitational condensation accrete material from their surroundings, so enhancing the density contrast. The ultimate result of this growth of inhomogeneity is the production of black holes which coalesce to form a single stationary black hole. Such a black hole is the equilibrium end state of equilibrium collapse.

This time-asymmetric tendency, for a self-gravitating system to pass from smooth to clumpy configurations should find expression through an extension of the second law of thermodynamics, in which case there must exist a gravitational quantity that plays the role of entropy. In the early 1970s, the close similarity between the laws of the thermodynamics and those of black-hole dynamics[2] was used by Bekenstein[3] to argue for the black-hole event-horizon area as a measure of the entropy of the hole. This identification was placed on a secure footing by Hawking[4] in his quantum treatment of black holes. Hawking found the entropy of the hole to be

$$S_{\text{bh}} = \tfrac{1}{4}\mathscr{A} \tag{1.1}$$

in units $\hbar = c = G = k = 1$, where $\mathscr{A}$ is the horizon area. The black-hole temperature was found to be

$$T = \frac{r_+ - M}{2\pi r_+{}^2}, \quad r_+ = M + (M^2 - Q^2)^{1/2} \tag{1.2}$$

for a Reissner-Nordström hole of mass M and charge Q. It was then possible to assert a generalized second law of thermodynamics for a closed system:

$$\Delta S_{\text{total}} \equiv \Delta S + \Delta S_{\text{bh}} \geq 0 \ , \tag{1.3}$$

where ΔS_{bh} is the change in the black-hole entropy and ΔS the change in the entropy of the hole's environment (i.e., the conventional, nongravitational entropy).

The fact that there exists a meaningful black-hole entropy confirms the conjecture that the growth of clumpiness in a self-gravitating system can be encompassed within a generalized entropic description. The black-hole case corresponds to the thermodynamic limit, at which a temperature can be ascribed to the system and quasi-equilibrium methods employed. However, as in the case of conventional thermodynamic systems, it ought to be possible to generalize the entropy concept away from quasiequilibrium states. When a star implodes to form a black hole, the conventional entropy of the stellar material disappears down the hole, while the (much greater) gravitational entropy associated with the hole's horizon appears to save the second law (1.3). This process is not instantaneous. There must exist a collapse phase during which the conventional entropy declines to zero, while the gravitational entropy rises from a low value to the high-equilibrium end-state value associated with the establishment of an event horizon. This encourages the belief that there exists a gravitational entropy for the star which rises as the star shrinks, reaching its maximum value of $\frac{1}{4}\mathscr{A}$ when the system settles down to a black-hole end state.

Penrose[5] has conjectured that gravitational entropy is in some way associated with the Weyl curvature. This is based on the tendency for cosmological models to show increases in the Weyl curvature with time, plus the fact that observational evidence suggests the Weyl curvature of the Universe was unaccountably low at the beginning. Some developments of this idea have been given by Bonnor.[6] In spite of these attempts to pin down the gravitational-entropy concept with specific mathematical expressions, the black-hole limit (1.1) is the only generally accepted case.

The simplest non-black-hole self-gravitating system is a thin spherical shell of cold matter. When the radius r of the shell tends to infinity, the gravitational entropy of the shell is presumably zero. When $r \to 2m$, where m is the mass of the shell, the system becomes a black hole, with entropy $\frac{1}{4}\mathcal{A}=4\pi m^2$. If the foregoing reasoning is correct, the gravitational entropy of the system should rise from zero to $4\pi m^2$ as r decreases from α to $2m$. One might expect S to be a monotonic decreasing function of r.

How can this conjecture be tested? The most direct way would seem to be this: If the shell is placed around a black hole, then the total gravitational entropy will be that of the black hole plus that of the shell (possibly with a correction for their mutual interaction). If the radius of the shell is now allowed to change, then any gravitational entropy associated with the shell would presumably change. As the shell can be both expanded and contracted, such a change would have to correspond to a decrease in total gravitational entropy in one direction (presumably expansion), because the horizon area remains unchanged by this maneuver in the spherically symmetric case. If, then, one were to apply the generalized second law of thermodynamics (1.3), failing to take into account the gravitational entropy of the shell, a violation of the law would be discovered. We therefore decided to carry out an investigation of the generalized second law for exchanges of energy between a black hole and a surrounding heat bath, in the presence of a shell, to see if such a violation were to occur. Any "missing entropy" could then be attributed to the shell.

If the black hole is in equilibrium with a heat bath at constant temperature, infinitesimal heat exchanges between the hole and the heat bath take place isentropically If the hole is now surrounded by a shell there is an additional degree of freedom—the shell radius. Violations of the generalized second law might be expected to occur if this radius changes. However, in the presence of the hole, the radius of the shell as such is not the only relevant parameter for representing the shell entropy. It is preferable to consider instead the distance between the shell and the horizon, because the shell gravitational entropy would go over to the known black-hole expression $4\pi M'^2$ when the shell crosses the horizon of the combined system (with mass M'), rather than when it crosses its own Schwarzschild radius.

As a result, it is simpler to regard the shell as remaining at fixed radius r, while the hole itself changes as a result of energy exchanges with the heat bath. These changes of radius will alter the distance between the shell and the horizon and, presumably, any gravitational entropy of the shell will also change.

II. SHELLS AROUND BLACK HOLES

Let us consider a charged black hole surrounded by a charged massive shell. The spacetime then consists of a Reissner-Nordström metric of mass M and charge Q inside the shell and one of mass M' and charge Q' outside the shell. The metric may be expressed as

$$ds^2=A^2\left[1-\frac{2M}{r}+\frac{Q^2}{r^2}\right]dt^2-\left[1-\frac{2M}{r}+\frac{Q^2}{r^2}\right]^{-1}dr^2$$
$$-r^2(d\theta^2+\sin^2\theta\,d\phi^2)\ ,\quad r<R \tag{2.1a}$$

and

$$ds^2=\left[1-\frac{2M'}{r}+\frac{Q'^2}{r^2}\right]dt^2-\left[1-\frac{2M'}{r}+\frac{Q'^2}{r^2}\right]^{-1}dr^2$$
$$-r^2(d\theta^2+\sin^2\theta\,d\phi^2)\ ,\quad r>R\ , \tag{2.1b}$$

where R is the shell radius and

$$A^2=(R^2-2M'R+Q'^2)(R^2-2MR+Q^2)^{-1}\ . \tag{2.2}$$

Because $M'>M$, we have $A<1$ if $Q'=Q$.

We wish to consider the effect of this shell upon the thermal radiation emitted by the black hole. We assume that there is no direct interaction between the matter in the shell and the particles created by the black hole, so the only effect of the shell is through its gravitational field. This effect is to depress the temperature of the black hole by a factor of A. This may be understood as simply the result of an additional red-shift. Because a particle produced at point x with a wave-packet frequency ω_0 in a static metric reaches infinity with a frequency of $\omega=\omega_0[g_{tt}(x)]^{1/2}$, assuming $g_{tt}=1$ at infinity as is true for the metric (2.1b), we can see that a particle created inside the shell reaches infinity with a frequency of A times the frequency it would have had at infinity in the absence of the shell. A Reissner-Nordström black hole of mass M and charge Q has a temperature $T_0=T_0(M,Q)$ and, in the absence of the shell, an observer at infinity detects an expected number of neutral particles in a normalized wave-packed mode (assumed to be sharply peaked around frequency ω) of the graybody thermal form

$$N=\frac{\Gamma}{e^{\omega/T_0}\mp 1}\ , \tag{2.3}$$

where Γ is the transmission probability for that mode. In the presence of the shell, a mode which has the same frequency as before inside the shell now has frequency $\omega'=A\omega$ as seen at infinity, and the gravitational field of the shell changes the transmission probability from Γ to Γ', so

$$N'=\frac{\Gamma'}{\Gamma}N=\frac{\Gamma'}{e^{\omega'/(AT_0)}\mp 1}\ , \tag{2.4}$$

which is a graybody thermal spectrum of temperature

$$T=AT_0\ . \tag{2.5}$$

From the definition (2.2) of A it will be noted that the external temperature of the hole can be expressed to an arbitrarily small value by letting $R \to 2M'$. Thus the hole's radiance can effectively be shut off by positioning a shell very close to the horizon. Of course, in this position the shell would be subject to enormous stresses if it were to remain static, and we do not suggest that this positioning can be done with real matter. However, it is of interest to note that it is often supposed that the particles emanating from a black hole are created in the region just outside the horizon, in which case it might seem as though the presence of the shell would make little difference when it is inside this "creation zone." Clearly this simple picture is inadequate.

It is interesting to note that, for a shell of radius $R < R_c$, where R_c is a critical radius, the temperature of the black hole + shell is *less* than that of the black hole which would subsequently form if the shell were allowed to free fall into the hole from its fixed position. If $m \ll M$ we may determine R_c for the uncharged case as follows: Set

$$\frac{A}{8\pi M} = \frac{1}{8\pi M'} \ . \tag{2.6}$$

This gives

$$\frac{R_c - 2M'}{R_c - 2M} = \frac{M^2}{M'^2}$$

so

$$R_c = 2(M'^2 + M'M + M^2)/(M' + M) \ . \tag{2.7}$$

If $m \ll M$ then $M' \simeq M$, so $R \simeq 3M$. Thus if a shell is held fixed between the horizon and $R \simeq 3M$, and then allowed to fall into the hole, the temperature and magnitude of the outgoing flux will *rise* as a result of the infall.

A more detailed derivation of Eq. (2.5) is the following. Define the coordinates r^* and r'^* by

$$\frac{dr^*}{dr} = \left[1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right]^{-1}, \quad r < R \tag{2.8}$$

and

$$\frac{dr'^*}{dr} = \left[1 - \frac{2M'}{r} + \frac{Q'}{r^2}\right]^{-1}, \quad r > R \ . \tag{2.9}$$

Then null coordinates outside of the shell are

$$u = t - r'^* \quad \text{and} \quad v = t + r'^* \ , \tag{2.10}$$

and those inside the shell are

$$U = At - r^* \quad \text{and} \quad V = At + r^* \ . \tag{2.11}$$

An outgoing null ray is labeled by the coordinate U inside the shell and u outside the shell, and the values of these coordinates are related by

$$u = A^{-1}U + u_0 \ , \tag{2.12}$$

where u_0 is a constant. Hawking[4] showed that null rays with v = const on $\mathscr{I}^-$ which pass through a collapsing body just before the horizon forms reach $\mathscr{I}^+$ as rays with U= const, where

$$U = -\kappa^{-1}\ln\alpha(v_0 - v) \ . \tag{2.13}$$

Here α and v_0 are constants and κ is the black-hole surface gravity. This relation leads to a thermal flux of particles at a temperature $T_0 = \kappa/2\pi$. When a shell is present, the relation between u and v now becomes

$$u = -(A\kappa)^{-1}\ln\alpha(v_0 - v) + u_0 \tag{2.14}$$

and hence the temperature is just $T = AT_0$. Here κ is the surface gravity of a black hole of mass M and charge Q. In Sec. III, it will be shown that the surface gravity for the spacetime containing a shell is $\kappa' = A\kappa$, so $T = \kappa'/2\pi$.

III. SHELLS IN BLACK-HOLE—de SITTER SPACETIMES

We now discuss the matching of the spacetime metric on opposite sides of the shell. Take the metric to be a Reissner-Nordström de Sitter metric of mass M, charge Q, and positive cosmological constant Λ within the shell and one of mass M', charge Q' and the same cosmological constant Λ outside of the shell:

$$ds^2 = A^2\left[1 - \frac{2M}{r} + \frac{Q^2}{r^2} - \tfrac{1}{3}\Lambda r^2\right]dt^2 - \left[1 - \frac{2M}{r} + \frac{Q^2}{r^2} - \tfrac{1}{3}\Lambda r^2\right]^{-1} dr^2 - r^2 d\Omega^2 \ , \quad r < R \ , \tag{3.1a}$$

$$ds^2 = \left[1 - \frac{2M'}{r} + \frac{Q'^2}{r^2} - \tfrac{1}{3}\Lambda r^2\right]dt^2 - \left[1 - \frac{2M'}{r} + \frac{Q'^2}{r^2} - \tfrac{1}{3}\Lambda r^2\right]^{-1} dr^2 - r^2 d\Omega^2 \ , \quad r > R \ , \tag{3.1b}$$

where as before R is the shell radius and

$$A^2 = (R^2 - 2M'R + Q'^2 - \tfrac{1}{3}\Lambda R^4)^{-1} \times (R^2 - 2MR + Q^2 - \tfrac{1}{3}\Lambda R^4) \ . \tag{3.2}$$

The electromagnetic junction conditions at the shell require it to have charge $q = Q' - Q$. The gravitational junction conditions at the shell are more complicated[7] and require that (1) the metric of the three-dimensional hypersurface defined by the shell world history be continuous and that (2) the extrinsic curvature of this hypersurface have a discontinuity proportional to the stress energy in the surface. Condition (1) is satisfied by the above form of the metric; the constant A is chosen so that the two forms of the metric in Eq. (3.1) agree on the hypersurface $r = R$. The precise form of condition (2) is

$$\Delta[K^i{}_j - \delta^i{}_j K] = -8\pi S^i{}_j \ , \tag{3.3}$$

where S_{ij} is the surface stress-energy term, $K_{\mu\nu}$ is the extrinsic curvature, and Δ denotes the differences between the quantity in brackets on the upper ($r > R$) and the lower ($r < R$) sides of the hypersurface. The extrinsic curvature is defined by

$$K_{\mu\nu} = -n_{\mu;\nu} \ , \tag{3.4}$$

where n^μ is the unit-normal vector to the hypersurface and is given by

$$n^\mu=\delta^\mu_r(-g_{rr})^{-1/2} . \tag{3.5}$$

Using the metric of Eq. (3.1a) we find that

$$K^t{}_t-K=\frac{2}{R}\left[1-\frac{2M}{R}+\frac{Q^2}{R^2}-\tfrac{1}{3}\Lambda R^2\right]^{1/2} \tag{3.6}$$

on the lower side of the shell and has the same form with M and Q replaced by M' and Q' on the upper side.

Let $u^\mu=\delta^\mu_t(g_{tt})^{-1/2}$ be the four-velocity of an observer at rest at $r=R$. Then

$$S^t{}_t=S_{\mu\nu}u^\mu u^\nu=\frac{m}{4\pi R^2} , \tag{3.7}$$

where m is the mass of the shell measured by this observer (i.e., the proper volume integral of the local energy density). The junction condition Eq. (3.3) for $\mu=\nu=t$ then yields

$$m=(R^2-2MR+Q^2-\tfrac{1}{3}\Lambda R^4)^{1/2} -(R^2-2M'R+Q'^2-\tfrac{1}{3}\Lambda R^4)^{1/2} , \tag{3.8}$$

or, equivalently,

$$M'=M+m\left[1-\frac{2M}{R}+\frac{Q^2}{R^2}-\tfrac{1}{3}\Lambda R^2\right]^{1/2} -\frac{m^2}{2R}+\frac{Q'^2-Q^2}{2R} . \tag{3.9}$$

An alternative method to obtain (3.8) is to use the standard $G^t{}_t=-8\pi T^t{}_t$ equation to integrate

$$dm=4\pi r^2T^t{}_t(-g_{rr})^{1/2}dr =\tfrac{1}{2}(-g_{rr})^{1/2}d[r(1+g^{rr})]$$

across the thin shell at $r=R$. This condition relates the mass parameters M' and M of the exterior and interior metric to the locally measured mass m of the matter in the shell. In an asymmetrically flat black-hole spacetime ($\Lambda=0$) we could envision building the shell by slowly lowering matter from infinity; in this case $M'-M$ rather than m is the mass measured at infinity of the net matter and energy required to build the shell. The difference, $M'-M-m$, is the gravitational and electrostatic potential energy of the shell. If M and M' vary with m, R, Q, Q', and Λ held fixed, then we find

$$\frac{dM'}{dM}=A . \tag{3.10}$$

This relation will be needed in the discussion in Sec. V. The remaining, spatial components of the junction condition, Eq. (3.3), can be interpreted as determining the stresses which are required to hold up the shell.

The Reissner–Nordström–de Sitter spacetime contains two horizons of interest to the present discussion. The cosmological horizon occurs at $r=r_2$, where r_2 is the largest root of the equation:

$$r^2-2M'r+Q'^2-\tfrac{1}{3}\Lambda r^4=0 . \tag{3.11}$$

The (outer) black-hole horizon occurs at $r=r_1$, at the second largest root of

$$r^2-2Mr+Q^2-\tfrac{1}{3}\Lambda r^4=0 . \tag{3.12}$$

If $Q\neq 0$, the next root is the radius of the inner black-hole horizon, with which we will not be concerned. The smallest root is always negative (for $\Lambda>0$, which we are assuming) and hence has no physical significance.

If $\Lambda\to 0$, then $r_2\to\infty$ and

$$r_1\to r_+=M+(M^2-Q^2)^{1/2} .$$

The shell must be located between the black hole and the cosmological horizon, so

$$r_1<R<r_2 . \tag{3.13}$$

The surface gravity κ for both horizons may be found from the relation[8]

$$\kappa^2=-\tfrac{1}{2}l_{\mu;\nu}l^{\mu;\nu} , \tag{3.14}$$

where l^μ is the Killing vector which is timelike in the region between the horizon and which is normalized so that it would have $l_\mu l^\mu=1$, where $g_{tt}=1$. Explicitly,

$$l^\mu=\delta^\mu_t . \tag{3.15}$$

For a metric of the form

$$ds^2=g_{tt}dt^2-g_{rr}dr^2-r^2d\Omega^2 , \tag{3.16}$$

where g_{tt} and g_{rr} are functions of r only, Eqs. (3.14) and (3.15) yield

$$\kappa^2=-\tfrac{1}{4}g^{tt}g^{rr}(g_{tt,r})^2 . \tag{3.17}$$

Using the form of the metric in Eq. (3.1) we find that the surface gravity of the black-hole horizon is

$$\kappa_1=A(Mr_1-Q^2-\tfrac{1}{3}\Lambda r_1{}^4)r_1{}^{-3} \tag{3.18}$$

and that of the cosmological horizon is

$$\kappa_2=(\tfrac{1}{3}\Lambda r_2{}^4+Q'^2-M'r_2)r_2{}^{-3} . \tag{3.19}$$

In both cases we have chosen the sign of the square root which makes $\kappa>0$.

In the absence of the shell, $A=1$, and κ_1 and κ_2 are equal to the expression given by Gibbons and Hawking.[9] If we identify $\kappa/2\pi$ as the temperature of a horizon, then the black-hole temperature is

$$T_1=\kappa_1/2\pi=AT_0 , \tag{3.20}$$

where T_0 is the temperature of the black-hole horizon without the shell, and the cosmological horizon temperature is

$$T_2=\kappa_2/2\pi . \tag{3.21}$$

Again we see that the effect of the shell is to suppress the black-hole temperature by a factor of A. Without the shell, the black-hole temperature is always greater than the temperature of the Universe:

$$T_0>T_2 \text{ if } M=M',\ Q=Q' . \tag{3.22}$$

However, with the shell the factor A can be made arbitrarily small and we can have

$$T_1 \le T_2 \ . \tag{3.23}$$

In this case we can have the black hole in thermal equilibrium with the Universe, or we can arrange that the Universe is hotter than the black hole. In the latter case, the black hole gains mass as a result of an inward flux of thermal radiation. Thus the hole-plus-shell system can in principle be used as a device for "mining the Universe."

IV. A TWO-DIMENSIONAL MODEL

In this section a two-dimensional spacetime model which illuminates the thermal properties of black-hole—de Sitter space will be investigated. In black-hole spacetime in two dimensions, Christensen and Fulling[10] have shown that the Hawking effect for a massless, conformally invariant field may be derived from (i) the trace anomaly, (ii) the conservation law for the energy-momentum tensor $T_{\mu\nu}$, and (iii) the finiteness of $T_{\mu\nu}$ on the black-hole horizon. Here we wish to adapt their arguments to the case of a two-dimensional Reissner—Nordström—de Sitter space containing a shell. Let the metric be

$$ds^2=\alpha(r)B(r)dt^2-B^{-1}(r)dr^2 \ . \tag{4.1}$$

When the space contains a thin shell at $r=R$, we would have

$$B(r)=\begin{cases} 1-\dfrac{2M}{r}+\dfrac{Q^2}{r^2}-\frac{1}{3}\Lambda r^2, & r<R \ , \\ 1-\dfrac{2M'}{r}+\dfrac{Q'^2}{r^2}-\frac{1}{3}\Lambda r^2, & r>R \ , \end{cases} \tag{4.2}$$

and

$$\alpha(r)=\begin{cases} A^2, & r<R \ , \\ 1, & r>R \ . \end{cases} \tag{4.3}$$

However, our argument will only require that α and B approach the above limiting forms near the horizon. Hence we may take these to be smooth functions associated with a more general radial distribution of matter. Let $T^{\mu\nu}$ be the expectation value of the energy-momentum tensor for a massless scalar field. We require $T^{\mu\nu}$ to be conserved ($T^{\mu\nu}{}_{;\nu}=0$) and to be time independent. This leads to the relations

$$\alpha^{1/2}T_t{}^{\nu}{}_{;\nu}=(\alpha^{1/2}T_t{}^r)_{,r}=0 \tag{4.4}$$

and

$$(g_{tt}T^r{}_r)_{,r}=\tfrac{1}{2}g_{tt,r}T \ , \tag{4.5}$$

where $T=T^{\mu}{}_{\mu}$. The first of these relations tells us that the flux of energy at infinity, $\alpha^{1/2}T_t{}^r$, is constant. In particular, for a shell

$$T^r{}_t=\begin{cases} A^{-1}c, & r<R \ , \\ c, & r>R \ , \end{cases} \tag{4.6}$$

where c is a constant. If we integrate Eq. (4.5), we find

$$g_{tt}T^r{}_r=\tfrac{1}{2}\int_{r_1}^{r} g_{tt,r}T\,dr+c_1 \ , \tag{4.7}$$

where c_1 is another constant.

The constants c and c_1 are fixed by the requirement that $T_{\mu\nu}$ be finite on the horizon in well-behaved coordinate systems, that is, that physically allowable observers should not see infinite stresses at the horizon. Define the r^* coordinate by

$$\frac{dr^*}{dr}=\alpha^{-1/2}B^{-1} \ , \tag{4.8}$$

and null coordinates $u=t-r^*$ and $v=t+r^*$. Then

$$\begin{aligned} ds^2&=\alpha(r)B(r)(dt^2-dr^{*2}) \\ &=\alpha(r)B(r)du\,dv \ . \end{aligned} \tag{4.9}$$

Because $B(r_1)=B(r_2)=0$, the (t,r), (t,r^*), and (u,v) coordinates are not well behaved on the horizon. However, we can define Kruskal-type coordinates which are well behaved on each horizon. Near the horizon, $B(r)$ has the asymptotic forms

$$B(r)\sim\begin{cases} B_1(r-r_1), & r\to r_1 \ , \\ B_2(r_2-r), & r\to r_2 \ , \end{cases} \tag{4.10}$$

where B_1 and B_2 are positive constants.

Thus

$$r^*\sim\begin{cases} A^{-1}B_1{}^{-1}\ln|r-r_1|, & r\to r_1 \ , \\ -B_2{}^{-1}\ln|r_2-r|, & r\to r_2 \ . \end{cases} \tag{4.11}$$

Let

$$\begin{aligned} \bar{u}_1&=-2A^{-1}B_1{}^{-1}e^{-uAB_1/2} \ , \\ \bar{v}_1&=2A^{-1}B_1{}^{-1}e^{vAB_1/2} \ , \\ \bar{u}_2&=2B_2{}^{-1}e^{uB_2/2} \ , \\ \bar{v}_2&=-2B_2{}^{-1}e^{-vB_2/2} \ . \end{aligned} \tag{4.12}$$

Because

$$d\bar{u}_1 d\bar{v}_1=e^{AB_1r^*}du\,dv\sim(r-r_1)du\,dv, \quad r\to r_1$$

and (4.13)

$$d\bar{u}_2 d\bar{v}_2=e^{-B_2r^*}du\,dv\sim(r_2-r)du\,dv, \quad r\to r_2$$

we see that the $(\bar{u}_1,\bar{v}_1)$ coordinates are well behaved on the black-hole horizon ($r=r_1$) and that the $(\bar{u}_2,\bar{v}_2)$ coordinates are well behaved on the cosmological horizon ($r=r_2$). On the black-hole horizon $u\to\infty$ and on the cosmological horizon $v\to\infty$. The energy-momentum tensor in the $(\bar{u}_1,\bar{v}_1)$ coordinates must be finite at $r=r_1$ and that in the $(\bar{u}_2,\bar{v}_2)$ coordinates must be finite at $r=r_2$. Because of the relation

$$T_{\bar{u}_1\bar{u}_1}=e^{AB_1u}T_{uu} \quad \text{and} \quad T_{\bar{v}_2\bar{v}_2}=e^{B_2v}T_{vv} \ , \tag{4.14}$$

we must have that $T_{uu}=0$ at $r=r_1$ and $T_{vv}=0$ at $r=r_2$. In terms of the (t,r) coordinates, T_{uu} is

$$\begin{aligned} T_{uu}&=\tfrac{1}{4}(T_{tt}-2T_{tr^*}+T_{r^*r^*}) \\ &=\tfrac{1}{4}(T_{tt}-2\alpha^{1/2}BT_{tr}+\alpha B^2T_{rr}) \ . \end{aligned} \tag{4.15}$$

From Eqs. (4.6) and (4.7) and the relation $T^t{}_t = T - T^r{}_r$, we find that $T_{uu}=0$ at $r=r_1$ implies that

$$c_1 = c \ . \tag{4.16}$$

Similarly, the requirement that $T_{vv}=0$ at $r=r_2$ leads to the result that

$$c = \tfrac{1}{4}\int_{r_1}^{r_2} g_{tt,r} T \, dr \ . \tag{4.17}$$

For a massless scalar field, T is determined entirely by the conformal anomaly in two dimensions:[10]

$$\begin{aligned} T &= -\frac{1}{24\pi}R \\ &= -\frac{1}{48\pi\alpha^2}[2\alpha^2 B'' + 3\alpha\alpha' B' + (2\alpha\alpha'' - \alpha'^2)B] \ , \end{aligned} \tag{4.18}$$

where R is the scalar curvature. Noting that

$$g_{tt,r}T = -\frac{1}{24\pi}\frac{d}{dr}\left[\tfrac{1}{2}\alpha B'^2 + \frac{1}{2\alpha}\alpha'^2 B'^2 + \alpha' B B'\right] , \tag{4.19}$$

we find

$$c = \frac{1}{192\pi}\{A^2[B'(r_1)]^2 - [B'(r_2)]^2\} \ . \tag{4.20}$$

However, from Eq. (4.6), c is just the outgoing energy flux across the cosmological horizon. With κ_1 and κ_2 given by Eqs. (3.18) and (3.19), or equivalently by

$$\kappa_1 = \tfrac{1}{2}A\,|B'(r_1)| \ , \quad \kappa_2 = \tfrac{1}{2}\,|B_2'(r_2)| \ , \tag{4.21}$$

we have for the outgoing flux

$$c = \frac{1}{48\pi}(\kappa_1{}^2 - \kappa_2{}^2) \ . \tag{4.22}$$

This result is in agreement with the discussion of Sec. III; the black hole surrounded by a shell is at a temperature of $T_1 = \kappa_1/2\pi$ and the Universe is at a temperature of $T_2 = \kappa_2/2\pi$.

V. THE GENERALIZED SECOND LAW

In this section the second law of thermodynamics and the expression for the entropy of a black hole (or black-hole–de Sitter spacetime) with a shell will be treated. As discussed above, the principal motivation for this investigation was to search for an additional contribution to the entropy which can be identified as the gravitational entropy of the shell. Let us first consider a charged black hole surrounded by a shell in equilibrium with a heat bath in a finite volume V of asymptotically flat spacetime ($\Lambda=0$). If T_r is the temperature of the heat bath (which is assumed to be dominated by massless radiation with radiation constant a), the total energy of the system is approximately

$$E = M' + aVT_r{}^4 \ . \tag{5.1}$$

Here we assume that the black hole and the heat bath are enclosed within a cavity whose dimensions are very large compared to the size of the black hole, but very small compared to the size where the self-gravity of the radiation becomes significant. Thus the expression for the energy and entropy of the heat bath are taken to be those in flat spacetime. There are some small corrections to these quantities produced by the spacetime curvature which are being ignored in the present analysis.

Assume that the total entropy of the system is the sum of that of the black hole, one-quarter of the horizon area, and of that of the heat bath, so

$$S = \pi r_+{}^2 + \tfrac{4}{3}aVT_r{}^3 \ , \tag{5.2}$$

where

$$r_+ = M + (M^2 - Q^2)^{1/2} \tag{5.3}$$

is the horizon radius. If a small amount of energy is exchanged between the black hole and the heat bath, the total energy remains constant so

$$dM' = -4aVT_r{}^3 dT_r \ . \tag{5.4}$$

The entropy change is

$$dS = \left[2\pi r\left[\frac{dr_+}{dM}\right]\left[\frac{dM}{dM'}\right] - T_r{}^{-1}\right]dM' \ , \tag{5.5}$$

where the charge Q is held constant. The temperature of the black hole is

$$T = AT_0 \ , \tag{5.6}$$

where

$$T_0 = \frac{Mr_+ - Q^2}{2\pi r_+{}^3} \tag{5.7}$$

is the temperature of a charged black hole without a shell. If we use these relations and the fact that $dM/dM' = A^{-1}$, where A is as given in Eq. (2.2), we find that

$$dS = \left[\frac{1}{T} - \frac{1}{T_r}\right]dM' \ . \tag{5.8}$$

Thus $dS=0$ if $T=T_r$; otherwise $dS>0$ because if $T_r > T$, then $dM'>0$ (energy flows into the black hole), and if $T_r < T$, then $dM'<0$. This is exactly the result which is to be expected if S is indeed the correct total entropy. The entropy change is zero for a reversible transformation such as a transfer of energy between two systems at equal temperature, and otherwise the entropy increases.

A similar result applies in the case of black-hole–de Sitter spacetimes containing a shell. Assume that the total entropy is one-quarter of the sum of the areas of the black hole and cosmological horizons (i.e., ignore any matter entropy between the horizons):

$$S = \pi(r_1{}^2 + r_2{}^2) \ . \tag{5.9}$$

If we differentiate Eq. (3.11) with respect to M with Q, Q', and Λ held fixed and use the expression Eq. (3.19) for κ_2, we find

$$dM' = -r_2\kappa_2 dr_2 \ . \tag{5.10}$$

Similarly from Eqs. (3.12) and (3.18) we have

$$dM = A^{-1}\kappa_1 r_1 dr_1 \ . \tag{5.11}$$

Using these relations and Eq. (5.9) we find that

$$dS = \left[\frac{1}{T_1} - \frac{1}{T_2}\right] dM' \ . \tag{5.12}$$

Thus again in this case $dS = 0$ only if $T_1 = T_2$ and $dS > 0$ otherwise. The black-hole—de Sitter spacetime with a shell behaves in the same way as does the black hole surrounded by a shell in contact with a heat bath, where the role of the heat bath is now played by the cosmological horizon.

Finally we note that attempts to violate this second law by the "box-lowering" techniques of Bekenstein,[3] and Unruh and Wald[11] will also fail in the presence of a shell. The latter authors found that buoyancy effects caused by the Hawking radiation would save the second law providing the general relation $dM = T\,dS$ remains true. In the presence of the shell we have

$$dM' = \frac{dM'}{dM} dM = A\,dM = AT_0 dS \ , \tag{5.13}$$

where $dS = d(4\pi M^2)$. Using (3.20) we conclude

$$dM' = T\,dS \tag{5.14}$$

as required by Unruh and Wald.

VI. CONCLUSIONS

In both of the situations analyzed above, the second law takes precisely the form that one would expect with the entropy given by Eqs. (5.2) and (5.9), where the only gravitational contribution to the entropy is one-quarter of the area of all event horizons. If there were any additional contributions to the entropy which depend upon M', then we would not have $dS = 0$ when $T_1 = T_2$. Then the transfer of heat between two bodies at equal temperature would either violate the second law (if $dS < 0$) or be an irreversible process (if $dS > 0$). Either outcome would conflict with conventional notions of thermodynamics. Consequently, our results must be interpreted as evidence against a gravitational entropy which is not associated with an event horizon.

The foregoing is also consistent with very simple arguments showing that there is no gravitational entropy associated with any cold spherical shells not inside an event horizon. No matter what the radius (if $R > M'$) or what the motion of such a shell is, its gravitational field is spherically symmetric and therefore carries off no irreversible gravitational radiation if conservative forces in the shell are permitted to expand the shell to radial infinity. Thus, as long as one ignores dissipative effects in any matter present or nonequilibrium fluxes of matter into or out of horizons, the motion of a spherical shell is reversible. Assuming that the shell and its gravitational field have zero entropy at radial infinity, with purely conservative forces they will have zero entropy at all other radii, no matter what the shell motion, as long as it does not enter an event horizon. One could design a shell which is capable of raising and lowering itself in the gravitational field of a black hole by changing its internal stresses without the assistance of any external agent. The black hole and shell then form a truly isolated system which is capable of reversibly changing itself from a configuration in which the shell radius is arbitrarily large to one in which the shell is located anywhere outside of the event horizon.

By considering an infinite set of concentric nondissipative shells, possibly with conservative radial forces between them, one can extend this argument easily to see that no spherical gravitational field other than a black hole has any entropy. The argument also applies to nonspherical stationary fields without horizons if one imagines allowing the matter source to spread out very slowly and adiabatically with the appropriate conservative forces to make the gravitational field become arbitrarily weak (and hence presumably have zero entropy) in a reversible way. One might object that matter realistically could not be expected to have precisely the appropriate conservative forces in all cases, but as long as these imagined forces do not violate the laws of gravity and thermodynamics, they are sufficient for these arguments about gravitational entropy.

These conclusions are the same as those Gibbons and Hawking[9] found for stationary configurations with perfect-fluid sources: a definite classical gravitational field without horizons has no entropy. Because of the wide variety of spherical and/or stationary metrics, for which this fact is easily seen to be true by the arguments above, there would appear to be great difficulty in constructing any functional of the Weyl tensor (or indeed virtually any other measure of the curvature) that would give the nonzero gravitational entropy of black holes and yet zero for the large class of metrics known to have no gravitational entropy.

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