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Quantum propagation near black holes

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This paper is an investigation of quantum wave propagation near the event horizon of an astrophysical black hole. Solutions of the wave equations for Dirac and electromagnetic quanta are determined near the horizon with more accuracy than can be found in previously existing literature. For these black holes, the study reveals that the separation constant squared of the Dirac equation is approximately Carter's fourth constant of motion in positive-definite form. This is also the value of the separation constant of the electromagnetic equation. An explicit calculation of causal propagators reveals that there are no global effects that alter the local structure of quantum propagation. A framework for computing quantum-electrodynamical scattering on the Kerr background is outlined using the methods that are developed in this article.

I. INTRODUCTION

In this paper analytic solutions of the electromagnetic wave and the Dirac equations are determined in the vicinity of the event horizon of an astrophysical black hole to a higher degree of accuracy than those which can be found in the previously existing literature. These solutions reveal an insight into the dynamics of wave propagation near the horizon that is intimately connected with the global constants of motion.

Presumably, the physical motivation for studying the wave equations near black holes is the potential astrophysical applications. Previously, the attitude that has been taken toward these equations has been stimulated by the problem of scattering waves off of the gravitational potential. In this paper the inclination is toward doing quantum-electrodynamical scattering on the background of the Kerr spacetime. The most important results that have been obtained from the previous point of view are superradiant scattering and Hawking radiation (for a good discussion of these phenomena see Birrell and Davies, Ref. 30). Unfortunately, Hawking radiation is not significant for astrophysical black holes as they are thermodynamically cold (see Ref. 31). Superradiant scattering is probably an unrealistic way of extracting energy from a black hole as well, the reason being that black holes are probably not isolated but are surrounded by matter and waves striking the hole from infinity are therefore few and far between (see Ref. 32).

Perhaps, the most likely way of extracting energy from a black hole is by exerting large-scale torques as described by Blanford and Znajek (see Ref. 33). Any energy liberated near the hole is most likely swallowed by the hole. The magnetosphere model of Blanford and Znajek circumvents this difficulty by extracting energy through a macroscopic process. The idea of high-energy plasma and strong electromagnetic fields surrounding a black hole suggests the relevance of computing quantum electrodynamics on the background of the hole. Even though much of the radiation produced in such processes is unable to escape the gravitational pull of the hole, the high-energy quantum processes near the hole would undoubtedly affect the plasma flow near the horizon. This is important since the flow near the horizon is a boundary condition on the global plasma flow. As such, it can affect the power extraction from the torqued magnetosphere. In these magnetospheric models, one has a natural environment for high-energy QED. There are photon number densities of $10^{17}/\mathrm{cm}^3$ or higher, magnetic fields of the order 10^5 G (and strong electric fields as well) along with an abundance of ultrarelativistic particles. The main calculations of interest would be Compton scattering and pair production by photons scattering off of the strong gravitomagnetic fields.

Consequently, the goal of this paper is to lay the groundwork for computing QED scattering processes on the background of the Kerr geometry. This will be done by analyzing the radial wave functions of the separated solutions. The previously existing solutions are inadequate for this purpose, the reason being that near the horizon all of the particle trajectories that are described by these solutions lie on the ingoing principal null congruence [for a definition of principal null congruences see the work of Misner, Thorne, and Wheeler (MTW)[34]]. In order to perform a scattering calculation, one needs to distinguish the trajectories corresponding to different momenta. This is accomplished in the next level of approximating the solutions which introduce a quantity that differentiates the trajectories near the hole. This quantity is Carter's fourth constant of motion (for a discussion, see MTW, Ref. 35). It is a globally conserved quantity that is absent in the zeroth-order solutions mentioned above.

There are two approaches to studying these radial solutions. First, one can study the analyticity properties of the radial functions as Candelas has done for the Schwarzschild geometry (see Ref. 53). The second method is to improve on the analytic solutions by computing to higher order. This is the approach that is taken in this paper.

An elegant machinery exists in the literature on the

Dirac equation in curved space-time and in the Kerr space-time in particular. However, explicit calculations are performed only in an asymptotic sense, since the mathematical expressions are actually extraordinarily complicated. Thus, qualitative arguments are implemented by physicists whenever possible. One of the most discouraging obstacles preventing physicists from performing calculations is that the separation of the equations depends on a separation constant with an unknown physical character. All solutions must depend on the value of this constant. Previous descriptions of this constant involve a numerical expansion that is based on a physically ambiguous quantum number l (Ref. 1). In the following, the square of this constant is revealed as Carter's fourth constant of motion in positive-definite form. To obtain this result, the elegance of the theory must be abandoned in favor of long brute-force calculations.

During the course of analyzing these solutions, a local momentum-space structure is developed. Since scattering calculations are most conveniently performed in a momentum space, this is a necessary development if the solutions that are found are used in calculations. There is no invariant meaning to a dual momentum space if the space-time is curved (see Ref. 36). However, there is a well-defined dual momentum space to the local coordinate patch of each locally Lorentz observer. In Secs. IV and V the transformations from these local spaces to the stationary frames at asymptotic infinity are well defined. The rudiments of scattering are constructed in the local momentum space, the Feynman propagators. It is shown in Sec. VII and Appendix C that the Feynman propagator that is computed in the frame at asymptotic infinity is equivalent to the propagator computed locally. The Feynman propagators that are found in Sec. VII are shown to reduce to flat-space propagators in a local Lorentz frame. This has been conjectured by many physicists based on Hadamard's method of inverting the local structure of the metric near a point. A particularly good treatment of these types of calculations is given by Adler, Lieberman, and Ng. They qualify their results by noting that global effects, such as the Hawking effect, do not show up in these types of calculations.[2] Without doing an explicit calculation, no one can be sure that global effects and boundary conditions do not produce other such processes that might alter the Green's-function flat-space character. This null result is not at all obvious upon initial inspection of the complicated dependence of the solutions and propagators, as viewed in the stationary frames at asymptotic infinity, on the separation constant k.

These problems are not addressed in all generality. This paper is aimed at QED calculations around astrophysical black holes, such as those that are believed to exist in galactic nuclei, quasars, and binary systems. These are characterized by large masses M (up to 10^8-10^9 solar masses for quasars) and large angular momentum $M^2=a^2$, where a is the angular momentum per unit mass of the black hole. The wave equations are solved for the cases $a^2<M^2$, $M^2 \gg M^2-a^2$, and the maximal-angular-momentum case $a^2=M^2$.

The quantum fields will be analyzed in four stages. First, the equations of motion are solved to good approximation near the horizon. The solutions are rewritten in terms of a local Lorentz basis and local momenta are found. In this basis, the solution looks like a plane wave.

Second, the local mometum is squared to form a Lorentz invariant. For the Dirac equation, this is set equal to the electron mass squared. The result of this is that the separation constant squared k^2 must be equal to Carter's fourth constant of motion in positive-definite form $\mathscr{K}$.

The local momenta can be used to find a local group velocity of the waves. By the transformation of coordinates from this local basis to the stationary frames at infinity, one can express the group velocity in Boyer-Lindquist coordinates. The group velocity of a quantum wave should correspond to the velocity of a classical trajectory. The classical trajectories in the Kerr space-time are given by the solutions to Carter's equations of motion. By expanding Carter's equations of motion about the horizon (i.e., choose $\Delta=r^2-2Mr+a^2$ as a small parameter), the two expressions agree exactly to $O(\Delta^2)$, if k^2 is Carter's fourth constant of motion.

Finally, causal propagators are formed. This calculation requires the same value of k^2. The analysis above is repeated for photons. The result is that the relevant separation constant k_+ is Carter's fourth constant.

II. THE DIRAC EQUATION

The massive Dirac equation was separated in the 1970s by Chandrasekhar in the Newman-Penrose spinor formalism.[3] Field quantization and complete sets of solutions were studied by Iyer and Kumar in the four-component formalism.[4] They analyzed both the case of an eternal black hole and one with a dynamical past.[5] The distinction is irrelevant for the purposes of this paper.

As with solutions of the scalar wave equation, the solution of the "radial" component of the waves at the event horizon is accomplished by a change of variables such that the equation looks like that of the harmonic oscillator.[6] However, this transformed equation involves the second-order differentiation of a function of one variable with respect to a different variable. As such, the solution cannot be extended away from the horizon without doing some overly complicated mathematics. This asymptotic form of the solution is independent of one of the quantum numbers of the field, the separation constant that was mentioned in the Introduction.

In the following, the radial equation is solved to the next level of approximation. To lowest order the waves thread the ingoing principal null congruences. The calculations in this section will deal with the deviation of the group velocity of the waves from the principal null trajectories. This involves the separation constant k. The relationship between the null congruences and the group velocities of the waves is elucidated in Sec. V.

The free-particle solutions of the Dirac equation can be separated in terms of four functions:[7]

$$\psi(t,r,\theta,\phi)=e^{-iwt}e^{-im\phi}\left[\frac{S^-(\theta)R_1(r)}{\sqrt{2}\bar{\rho}^*},\frac{S^+(\theta)R_2(r)}{\sqrt{\Delta}},\frac{-S^-(\theta)R_2(r)}{\sqrt{\Delta}},\frac{-S^+(\theta)R_1(r)}{\sqrt{2}\bar{\rho}}\right], \quad (2.1)$$

where $\bar{\rho}=r+ia\cos(\theta)$.

The angular functions $S_k^{\pm}(\theta)$ satisfy a complicated second-order linear differential equation:[8]

$$\left[\frac{1}{\sin(\theta)}\frac{d}{d\theta}\left[\sin(\theta)\frac{d}{d\theta}\right]+\frac{am_e\sin(\theta)}{k+am_e\cos(\theta)}\frac{d}{d\theta}+[\tfrac{1}{2}+aw\cos(\theta)]^2-\tfrac{3}{4}+2awm-a^2w^2-\frac{[m-\frac{1}{2}\cos(\theta)]^2}{\sin^2(\theta)}\right.$$
$$\left.+\frac{am_e[\frac{1}{2}\cos(\theta)+aw\sin^2(\theta)-m]}{k+m_ea\cos(\theta)}-a^2m_e{}^2\cos^2(\theta)+k^2\right]S^-(\theta)=0\,. \quad (2.2)$$

The equation for $S^+(\theta)$ is obtained by the substitution $\theta\to\pi-\theta$ in (2.2). The solutions to these equations are not found in this paper. According to the Sturm-Liouville theory, these orthogonal functions are complete.[9] This fact will be used in the construction of the propagators of the field.

Chandrasekhar describes the radial functions in terms of two coupled first-order linear differential equations:[10]

$$\frac{d}{dr}R_1-(i/\Delta)[w(r^2+a^2)+ma]R_1=[(k-im_er)/\Delta^{1/2}]R_2\,,$$

$$\frac{d}{dr}R_2+(i/\Delta)[w(r^2+a^2)+ma]R_2=[(k+im_er)/\Delta^{1/2}]R_1\,.$$

These can be decoupled by differentiation to yield second-order linear differential equations with variable coefficients:

$$\frac{d^2}{dr^2}(R_1)-[1/\Delta(k^2+m_e{}^2r^2)][m_e{}^2r(a^2-Mr)-k^2(r-M)-ikm_e\Delta]\frac{d}{dr}R_1+V_1(r)R_1=0\,, \quad (2.3)$$

where

$$V_1(r)=\Delta^{-2}[1/(k^2+m_e{}^2r^2)]\{(k^2+m_e{}^2r^2)[w(r^2+a^2)+ma^2+i[w(r^2+a^2)+ma][(r-M)(k^2+m_e{}^2r^2)]\}$$
$$+\Delta^{-1}[1/(k^2+m_e{}^2r^2)](km_e[w(r^2+a^2)+ma]-(k^2+m_e{}^2r^2)^2i\{m_e{}^2r[w(r^2+a^2)+ma]-2(k^2+m_e{}^2r^2)rw\})\,. \quad (2.4)$$

The parameters in these equations can be related to physical quantities. The mass of the fermion is m_e. The energy of the fermion, as measured in the stationary frames at asymptotic infinity, is w. The conventions of this paper identify the quantity m with the negative of the conserved component of the angular momentum of the fermion along the angular momentum vector of the black hole. It should be noted that $R_2(r)$ satisfies the complex conjugate of Eq. (2.4).

When $a^2<M^2$, the radial equations have regular singular points at the event horizons, $r=r_\pm=M\pm(M^2-a^2)^{1/2}$. When $a^2=M^2$, $r_+=r_-=a$, there is one singular point, and it is irregular, at $r=a$. The latter case has no general method of solution and will be considered in the next section. The case $a^2<M^2$ will be addressed presently. For these parameter values, there exist power-series solutions about the event horizon $r=r_+$ with a radius of convergence equal to $|r_+-r_-|$.

The lowest-order solution has been found by Iyer and Kumar to be (see Ref. 37)

$$R_2(r-r_4)=\exp\left[-i\int_{r_+}^{r}\frac{w(r^2+a^2)+ma}{\Delta(r)}dr\right] \quad (2.5)$$

using the previously mentioned change-of-variables technique. In order to get more details than this one must investigate the power-series solutions. It is not easy with the power-series solution to even extract the lowest-order term, (2.5) (that is why the change of variables in used to find it) but once it is found a little more effort gives the next level of corrections. The following calculations are an attempt to extract as much information on the oscillatory nature of the solutions as is possible. This is what determines the momentum of the wave and is therefore most relevant for studying scattering on the Kerr background.

The following calculations are quite long and involved. This is primarily due to the expansions of the various coefficients of the differential equations that are tabulated in Appendix A. The problem is further complicated by the existence of many large parameters. Thus, in preliminary attempts to solve the recursion formulas one has expressions that do not appear to converge anywhere. These large terms reflect the high-energy oscillatory nature of the particles as they approach the hole. The confusion that is caused by these large terms is remedied only if their contribution is factored out of the recursion relation to all orders. It is during this factorization that most of the physical information is extracted. The factorization must be done with great precision, keeping track of even the smallest terms or the results are not readily understandable.

A series of steps are indicated in the following that greatly simplify the amount of labor. However, in the end one still has numerous terms to manipulate. The procedure that is followed manages basically to separate out the most dominant terms (factorization of the recursion) before the final manipulation is done. The cumbersome last step involves the calculation of a plethora of small term corrections. These can be computed in a straightforward manner. It is not instructive to cover a page with 30 or so small terms. Three or four of the largest or potentially most confusing terms are recorded in the solution of the nonoscillatory behavior. The main goal of the calculations is to factor out the dominant oscillatory behavior then find the lowest-order correction to the nonoscillatory amplitude.

There are two independent solutions for each equation:

$$R_1^+(r-r_+)=|r-r_+|^{\lambda_+}\sum_{n=0}^{\infty}a_n(r-r_+)^n\,, \quad (2.6a)$$

$$R_1^-(r-r_+)=|r-r_+|^{\lambda_-}\sum_{n=0}^{\infty}\tilde{a}_n(r-r_+)^n\,. \quad (2.6b)$$

The preliminary step is to determine the values of λ by the method of Frobenius. This requires the solution of the indicial equation. The variable coefficients and the potential are expanded in power series about the event horizon in Appendix A. Utilizing the expansions of Appendix A, one finds

$$\lambda_1^{\pm}=\tfrac{1}{4}[1\mp1\pm2iw_+/(M^2-a^2)^{1/2}]\,, \quad (2.7a)$$

$$\lambda_2^{\pm}=\tfrac{1}{4}[1\pm1\pm2iw_+/(M^2-a^2)^{1/2}]\,, \quad (2.7b)$$

where $w_+=w(r_+{}^2+a^2)+ma$.

The physical states near the horizon correspond to waves that appear to be ingoing to all local observers.[11] Outgoing states can be interpreted as arising from particle creation in the gravitational potential. The probability amplitude for such a process would manifest itself as an indeterminancy of particle number for the quantum field on a coordinate patch containing both the event horizon and the point of observation. This is proportional to the ratio of the Compton wavelength of the created quanta to the radius of curvature of the background space-time.[12] For an astrophysical black hole, this ratio is small as the curvature near the event horizon is negligible on a quantum scale. Consequently, the relevant solutions of the indicial equations are

$$\lambda_1=\tfrac{1}{2}-iw_+/2(M^2-a^2)^{1/2}\,, \quad (2.8a)$$

$$\lambda_2=-iw_+/2(M^2-a^2)^{1/2}\,. \quad (2.8b)$$

In terms of the spinor defined by (2.1), the first and fourth components are $O(\Delta)$ of the second and third components. This is merely a consequence of the behavior near the horizon of the particular tetrad that was used in the separation of the Dirac equation (see Ref. 38). Solutions R_2 will be solved for below as their amplitudes contribute $1/\Delta$ times as much as the R_1 solution to the spinor. It should be recalled that the oscillatory nature of the solutions is basically found to all orders. The imaginary exponential behavior of R_1 can be solved for in the same way as is done for R_2. To the level of accuracy obtained in this paper, it is the same as for R_2. R_1 is not solved for in this paper. The calculation is analogous to that for R_2 except that one has to take the complex conjugate of the coefficient B and the potential V in Appendix A as well as use the different value of λ in the recursion relation. As stated earlier, most of the relevant information for scattering is in the oscillatory behavior of the solutions. The nonoscillatory corrections are computed mainly to see if the solutions appear to converge and as a check that most of the oscillatory behavior was factored out. Even though there will be no further reference to these smaller solutions, it should be remembered that the first and fourth components of the spinor display the same oscillatory nature as the second and third components in the vicinity of the hole but with a smaller amplitude inside the radius of convergence of the power series.

$R_2(r-r_+)$ will be solved to $O((r-r_+)^2)$ in the following using Appendixes A and B. The recursion formula that determines a_1 is given in Appendix B as

$$a_1[\lambda(\lambda+1)A_2+(\lambda+1)\mathrm{Re}(B_1)+V_0]$$
$$=-a_0[\lambda(\lambda-1)A_3+\lambda B_2-V_1]\,. \quad (2.9)$$

Certain terms cancel identically as a result of the indicial relation, leaving

$$a_1[2\lambda A_2+\mathrm{Re}(B_1)]=-a_0[\lambda(\lambda-1)A_3+\lambda B_2+V_1]\,. \quad (2.10)$$

This recursion relation involves large numbers and can be factorized to give a new recursion. The coefficients that solve the new recursion are small. The convergence of this sequence of coefficients to a finite sum is much more obvious. It is interesting at this point to look at the size of the numbers involved. In particular, for an electron near an astrophysical black hole, an x-ray quasar for example, $m_e{}^2r_+{}^2\sim10^{50}$. By definition, a physical state at infinity has a value of $w^2r_+{}^2$ even larger than this.

Noting that the λ^2 term in front of the coefficient a_n in Appendix B cancels out due to the indicial relation, as in (2.10), one can derive the approximate recursion relation

$$a_nA_2(n^2-\tfrac{1}{2}n+2n\lambda)\approx-a_{n-1}[(n+\lambda-1)(n+\lambda-2)A_3+(n-1+\lambda)\mathrm{Re}(B_2)+i\,\mathrm{Im}(V_1)+\mathrm{Re}(V_1)$$
$$+i(n-1+\lambda)\mathrm{Im}(B_2)]-a_{n-2}\lambda^2a_4-a_{n-3}\lambda^2a_5-\cdots\,. \quad (2.11)$$

The term on the left (rewritten using $B_1=\tfrac{1}{2}A_2$ from Appendix A) and the first term on the right are exact. The other terms on the right have been approximated. A factor of λ^2 is retained as the dominant contribution in approximating the higher-order terms of the recursion. This relation when combined with the remarks on the large parameters in the previous paragraph imply that each $|a_n|$ is larger than the terms that preceded it by about the order of magnitude of m_ea.

Consequently one has the amazing simplification that the dominant behavior of the solution is essentially given by a two-term recursion. To high accuracy [see the remarks that follow (2.17)], one approximates further by just dropping these higher-order terms from the recursion:

$$a_n(n^2-\tfrac{1}{2}n+2n\lambda)A_2 \approx -a_{n-1}\{[n^2+2-3n+\lambda^2+(2n-3)\lambda]A_3+[(n-1)+\lambda]\mathrm{Re}(B_2)+i\,\mathrm{Im}(V_1)+i(n-1+\lambda)\mathrm{Im}(B_2)+\mathrm{Re}(V_1)\}\ . \tag{2.12}$$

At this point it is instructive to factor out the oscillatory behavior from the radial wave function by writing the summation in (2.6) as

$$\sum_{n=0}^{\infty} a_n(r-r_+)^n=\sum_{n=0}^{\infty} a_n^0(r-r_+)^n \sum_{n=0}^{\infty} \alpha_n(r-r_+)^n\ , \tag{2.13}$$

where a_n^0 are the coefficients of an imaginary exponential series. These coefficients are characterized by the fact that the ratio a_n^0/a_{n-1}^0 is proportional to $1/n$ and is imaginary. Furthermore, as a consequence of the high-energy frequencies of the particles as they approach the horizon, $|a_n^0|$ is at least the order of magnitude of $m_e a$ larger than $|\alpha_n|$. This can be seen from the individual terms in the recursion of Appendix B. The $|a_n^0|$ vary as $(wa)^n/(r_+-r_-)^n$ and the $|\alpha_n|$ vary as $(wa)^k r_+^{-m}/(r_+-r_-)^{n-m}$ where k and m are positive integers less than n. As stated before, $|a_n^0|$ is about the order of magnitude of $m_e a$ larger than $|a_{n-1}^0|$. By definition, $a_n=\sum_{k=0}^{n} a_{n-k}^0\alpha_k$ and the previous statements imply that $|a_{n-k}^0\alpha_k| \ll |a_{n-k}^0 a_k^0| \sim |a_n^0|$. Noticing this, one can write $a_n \approx a_n^0$ since the contributions of the cross terms $a_{n-k}^0\alpha_k$ are at least the order of magnitude of $m_e a$ smaller than $|a_n^0|$ due to the high-frequency oscillations. Hence, it is natural to express the coefficients of the power series as

$$a_n=a_n^0+\alpha_n'\ , \tag{2.14}$$

where α_n' contains all of the cross terms of the type mentioned above including α_n. In the special case of $n=1$, to be computed in (2.17), $\alpha_1'=\alpha_1$.

The coefficients of the imaginary exponential will be solved for first. Noting the characteristic ratio mentioned in the previous paragraph, only certain terms in (2.12) have the relevant form. In light of the remarks that preceded (2.14), one has the following effective recursion relation obtained from (2.12) with virtually no approximation [see the remarks following (2.17)]:

$$a_n^0 \approx -a_{n-1}^0[(2+\lambda^2)A_3/2n\lambda A_2 -\mathrm{Re}(B_2)/2n\lambda A_2+\mathrm{Re}(V_1)/2n\lambda A_2 +i\lambda\,\mathrm{Im}(B_2)/2n\lambda A_2]\ , \tag{2.15}$$

where the fact that $|\lambda|$ is many orders of magnitude larger than the other terms in the coefficient of a_n^0 was used to obtain the approximate coefficient $2n\lambda$.

After a long calculation that includes the identical cancellation of five terms one obtains

$$a_n^0 \approx (a_{n-1}^0/n)[iw_+/(r_+-r_-)^2-2iwr_+/(r_+-r_-)-5i/8w_+-2i\Gamma/w_+ +i\Lambda/2w_+ +ikm_e w(r_+{}^2+a^2)/4w_+\Lambda+i\Gamma w_+/(r_+-r_-)^2]\ , \tag{2.16}$$

where the parameters $\Lambda \equiv k^2+m_e{}^2 r_+{}^2$ and $\Gamma \equiv m_e{}^2 r_+{}^2/\Lambda$ have been introduced. It should be noted that all of the approximations that were used in deriving (2.16) from (2.3) involved dropping terms that were on the order of magnitude of $m_e a$ smaller than those that were retained.

Using the value of a_1^0 in (2.16) and the defining relation (2.14), one can insert these into the recursion (2.10) and solve to get α_1:

$$\alpha_1=5\Gamma/[4(r_+-r_-)]+O(1/r_+)\ . \tag{2.17}$$

The small terms in (2.16) should not be taken that seriously, namely, those of order i/w_+, since these are at least the order of magnitude of $m_e a$ smaller than the next largest terms. Other contributions of this magnitude can come from the $-\frac{1}{2}n$ correction to the coefficient of a_n^0 as well as from the next highest term, a_{n-2}, in the recursion [this is the order of the largest oscillatory term that results from considering the effect of a_{n-2} in the recursion (2.11)].

To analyze the oscillatory part of the radial solution (2.16), it is useful to introduce the following simplifying approximations that are valid for r near r_+:

$$(r^2+a^2)/\Delta(r)=(r_+{}^2+a^2)/(r_+-r_-)(r-r_+)+2r_+/(r_+-r_-)-(r_+{}^2+a^2)/(r_+-r_-)^2+O(r-r_+)\ , \tag{2.18a}$$

$$1/\Delta(r)=1/(r_+-r_-)(r-r_+)-1/(r_+-r_-)^2+O(r-r_+)\ . \tag{2.18b}$$

With the aid of (2.18), the first two terms in (2.16) combine with the overall factor $(r-r_+)^\lambda$ from the indicial condition to give (2.5). This is the solution that has previously been found by Chandrasekhar mentioned earlier. All of the remaining terms in (2.16) and (2.17) are an improvement of this solution.

For small-point separations the integral in (2.5) can be linearly approximated. If this is the case then one of the terms in (2.16) which can be written as

$$[i\Gamma w_+/(r_+-r_-)^2]^n(r-r_+)^n=(-2i\Gamma/w_+)^n(r-r_+)^n[(-i)^2 w_+{}^2(r-r_+)^2/2\Delta^2] \tag{2.19}$$

with $\Delta \approx (r_+ - r_-)(r - r_+)$ looks suggestively like a cross term in the expansion of the exponential

$$\exp\left[-i\,2\Gamma(r-r_+)/w_+ - i\int_{r_+}^{r}\frac{w(r^2+a^2)+ma}{\Delta(r)}dr\right] .$$

It appears as a first-order term since (2.5) actually contains a "zeroth"-order term from the logarithmic contribution that is present from the indicial factor. This turns out to be a valid explanation of the term since when $a^2=M^2$, there is no corresponding term in the solution that is found is Sec. III. Furthermore, if (2.19) were not a cross term but a contribution to the oscillatory nature of the wave in its own right then one has a physical contradiction. This is illustrated by the discussion of Sec. V which equates the group velocity of the particle waves with the velocity that is derived from the equations of classical motion in the Kerr space-time. The contribution of (2.19) to the group velocity would produce an effect that goes like $1/(r_+ - r_-)^2$. No such term exists in the equations of classical motion when they are analyzed near the event horizon. This lends support to the cross-term explanation of (2.19). The cross-term argument is meant only to be heuristic since there are other oscillatory terms of order i/w_+ that have not been included as remarked in the paragraph following (2.17). The term $2i\Gamma/w_+$ is unique among these as it and the first-order term in (2.19) are both derived from the same term (namely, the first) on the right-hand side of (2.15).

Motivated by these remarks, the term $i\Gamma w_+/(r_+ - r_-)^2$ is dropped from the exponential in (2.16) in order that it not be counted twice. The final solution can be written as

$$R_2(r-r_+) \approx \{1+[5\Gamma/8(M^2-a^2)^{1/2}+O(1/r_+)](r-r_+)\}$$
$$\times\exp\left[-i\int_{r_+}^{r}\left[\frac{w(r^2+a^2)+ma}{\Delta(r)} - \frac{(k^2+m_e^2 r_+^2)-(\frac{5}{4}+4\Gamma)+\frac{1}{2}k\Gamma w(r_+^2+a^2)/m_e r_+^2}{2w_+}\right]dr + O(i/w_+)\right] . \quad (2.20)$$

It should be noticed that the first term in the expression for α_1 (the amplitude to first order) is a reflection of the fact that the power-series solution has a radius of convergence $|r_+ - r_-|$. The corrections to the unit amplitude are always small and well behaved inside of the radius of convergence.

Another item of note is that the contributions to the exponential given by $\frac{5}{4}$ and 4Γ are of order unity and are obviously negligible corrections compared to the large numbers in the other terms in the numerator above $2w_+$. The ratio of the other two terms in the numerator is given by $km_e w(r_+^2+a^2)/(k^2+m_e^2+(k^2+m_e^2 r_+^2)^2$. It can be shown that this ratio is small as well. For instance, it is maximized with respect to k (by taking a derivative and setting it equal to zero) when $k^2=\frac{1}{3}m_e^2 r_+^2$. For this value, the ratio is of the order unity only for electron energies of 10^{23} MeV as observed from infinity. Since this paper is aimed at realistic scattering problems, these kinds of electron energies will not be considered. To compactify the expressions, these corrections will be dropped in most of the following calculations. It will turn out in Sec. IV that this value for k^2 is unattainable. The minimum value of k^2 that is consistent with the results of Sec. IV is $m_e^2 a^2\cos^2\theta$ [see Eq. (4.10)]. Thus, the value of k^2 that maximizes the ratio cannot be achieved for many values of θ. In the strictest sense the ratio is maximized by taking the minimum possible value of k^2. For the true maximum of the ratio a similarly large energy at infinity is required to bring the ratio close to one. The minimum value of k^2 requires that an equally astronomical value of angular momentum about the symmetry axis for the particle trajectories. Even though these are highly improbable trajectories, it is interesting to observe that the mathematical description of particle states can change if there are extreme ultrarelativistic energies and an enormous value of orbital angular momentum that is chosen in just the right way. For the sake of this paper, the oscillatory nature of the radial wave functions can be written with high accuracy [there is an inherent error on the order of $(wm/a)/10^{46}$ MeV^2] and more compactly than (2.20) by the exponential

$$\exp(R) = -iw\int_{r_+}^{r} dr(r^2+a^2)/\Delta(r)$$
$$-ima\int_{r_+}^{r} dr/\Delta(r)$$
$$+i(k_2+m_e^2 r_+^2)(r-r_+)/2w_+ . \quad (2.21)$$

It is shown in Sec. IV that the quantity in the integrand of the exponent of (2.20) and (2.21) when combined with the separation factors $e^{-iwt}e^{im\phi}$ is a locally measured momentum, p^μ. Thus the solution can be written in the form

$$R_2(r-r_+) \sim \exp\left[-i\int p^\mu dx_\mu\right] . \quad (2.22)$$

This is reminiscent of the WKB approximation that is often used for solving the Schrödinger equation (see Ref. 39). This is usually a valid approximation if the momentum varies on distance scales that are very large as compared to the wavelength of oscillation, which is clearly the case here.

In summary, a power-series solution of the radial equation was studied by means of its recursion relation in Sec. II. The main step was the factorization of the large oscillatory terms from the recursion. The main result (2.20) was obtained by neglecting only those terms that were many magnitudes smaller than those that were retained. Hence, one expects (2.20) to reflect the oscillatory nature of the particle waves near the horizon with a high degree of accuracy. Using the general recursion of Appendix B along with the expansions in Appendix A, the factorization method that was developed in this section can be implemented to reduce the amount of labor in calculating higher-order terms.

III. MAXIMUM BLACK-HOLE ANGULAR MOMENTUM

When a black hole is spinning with its maximum value of angular momentum, $a^2=M^2$, the radial component of the Dirac equation (2.3) becomes the intractable case possessing an irregular singular point at $r=a$. There is a confluence of the singular points $r=r_+$ and $r=r_-$ at $r=a$.

$$(r-a)^4\frac{d^2}{dr^2}(R_2)+[(r-a)^2/[(k^2+m_e{}^2a^2)] \times[(m_e{}^2ra+k^2)(r-a)+ikm_e(r-a)^2]\frac{d}{dr}(R_2) +(r-a)^4V^*(r)R_2=0\ , \quad (3.1)$$

where $V^*(r)$ is the complex conjugate of (2.4) with $r_+=r_-=a$.

Normally, this equation would be difficult to solve near the singular point. There are no general methods, such as Frobenius' method for regular singular points, for the irregular case. However, the results of the last section combined with the limiting form of (3.1) at the horizon can be used to motivate a good guess.

As r approaches a, the equation reduces to

$$(r-a)^4\frac{d^2}{dr^2}(R_2)+(r-a)^3\frac{d}{dr}(R_2) +(2wa^2+ma)^2R_2=0\ . \quad (3.2)$$

If one makes the substitution

$$z=(2wa^2+ma)/(r-a) \quad (3.3)$$

then (3.2) becomes

$$\frac{d^2}{dz^2}(R_2)+(1/z)\frac{d}{dz}(R_2)+R_2=0\ . \quad (3.4)$$

This is Bessel's equation. Hence, the solution will be a Hankel function that corresponds to an ingoing group velocity as noted in the remarks preceding (2.8). Contrary to what one would expect on preliminary inspection, it is a Hankel function of the first kind (positive exponent). The reason will be clear when local momenta are found in the next section. The solution to (3.4) is

$$R_2(r-a)=H_0^{(1)}(z) =H_0^{(1)}((2wa^2+ma)/(r-a))\ . \quad (3.5)$$

This gives a big clue as to how one should solve the equation away from the horizon. It should be noted that (3.5) vanishes at the horizon.

Taking a cue from expression (2.21), the solution to guess is of the form

$$R_2(r-a)=H_0^{(1)}((2wa^2+ma)/(r-a)+\tfrac{1}{2}(k^2+m_e{}^2a^2)(r-a)/(2wa^2+ma))\sum_{n=0}^{\infty}a_n(r-a)^{n+\lambda}\ . \quad (3.6)$$

The relationship between (2.21) and (3.6) becomes apparent when one looks at the asymptotic form of the Hankel function:[13]

$$H_\nu^{(1)}\underset{r\to a}{\sim}([1+i(4\nu^2-1)(r-a)/8w_+]\{4w_+(r-a)/\pi[2w_+{}^2+(k^2+m_e{}^2a^2)(r-a)^2]\}^{1/2}) \times\exp\{i[w_+/(r-a)+(k^2+m_e{}^2a^2)(r-a)/2w_+-\tfrac{1}{2}\nu\pi-\tfrac{1}{4}\pi]\}\ . \quad (3.7)$$

In order to solve for a_n and λ_1 one must expand the coefficients and potential of (3.1) in power series about $r=a$. This can be done by using the expansions in Appendix A with $r_+=r_-=a$. Upon substitution of (3.6) into (3.1), one finds that the terms of order $(r-a)^\lambda$ all cancel by virtue of the limiting form of Eq. (3.4), and the solution (3.5). The vanishing of the coefficients of the terms of the order $(r-a)^{\lambda+1}$ determines λ. Without any approximation, one finds

$$\lambda=-2aiw-\tfrac{1}{2}\ . \quad (3.8)$$

Equation (3.7) aids in the calculations leading to (3.8), since as

$$r\to a:\ H_1\to-iH_0 \text{ and } H_2\to-H_0\ . \quad (3.9)$$

The following identity was also used:[14]

$$\{2w_+(r-a)/[2w_+{}^2+(k^2+m_e{}^2a^2)(r-a)^2]\}H_1=\tfrac{1}{2}(H_0+H_2)\ . \quad (3.10)$$

Using (3.7), (3.8), and (3.6), one can form the combined oscillatory term near the horizon:

$$\exp\{i[(2wa^2+ma)/(r-a)+(k^2+m_e{}^2a^2)(r-a)/2w_+-2wa\ln(r-a)]\}\ . \quad (3.11)$$

The factor of $(r-a)^{-1/2}$ from λ cancels the $(r-a)^{1/2}$ in the asymptotic expression (3.7). Thus, one has the same dependence on $(r-a)$ as in (2.8b), i.e., $R_2(r-a)\sim\Delta^0=1$.

The vanishing of terms to order $(r-a)^{\lambda+2}$ yields

$$a_1=-iw-i/8w_+ +ikm_e wa^2/2w_+\Lambda+O(\Gamma/a)\ . \tag{3.12}$$

A rough computation that estimates the higher order a_n shows that the term $(-iw)^n/n!$ is always present. Thus, this can be factored out to give

$$R_2(r-a)\sim\exp\{i[w(r-a)-w_+/(r-a)-(k^2+m_e{}^2a^2)(r-a)/2w_+ +2wa\ln(r-a)]\} \tag{3.13}$$

or, equivalently,

$$R_2(r-a)\sim\exp\left[-iw\int_{r_+}^{r}dr(r^2+a^2)/\Delta(r)-ima\int_{r_+}^{r}dr/\Delta(r)+i(k^2+m_e{}^2a^2)(r-a)/2w_+\right]\ . \tag{3.14}$$

In this form, the solution when $a^2=M^2$ is the same as the one found for $a^2<M^2$ in (2.21). Even the correction $ikm_e w(r_+{}^2+a^2)/(4w_+\Lambda)$ of the exponential in (2.20) is present to first order in (3.12). Presumably, if one were to take the entire exponential of (2.20) and put it inside of the argument of the Hankel function in (3.6) a more accurate answer would be obtained. However, the algebra involved in substituting such a term into the differential equation would be intractable.

IV. LOCAL LORENTZ DESCRIPTION

The solutions that were derived in the previous sections will be recast in terms of the coordinates of local Lorentz frames that are instantaneously at rest with respect to a zero angular momentum frame (ZAMF), near the event horizon. The ZAMF's are accelerating (noninertial) frames. Thus, the two frames will coincide only for an instant.[15] These are useful frames to pick since the ZAMF's at $r=r_+$ corotate with the horizon. ZAMF's are defined at a constant value of the coordinate r. Therefore, the local Lorentz frame has no initial "radial" velocity. Determining the trajectories of particles in this frame becomes basically a one-dimensional (radial) problem near the horizon. As stated earler, the particle waves nearly thread the principal null congruences. So, for these initially corotating observers particles will appear to be moving inward along the radial direction with essentially the speed of light.

The Kerr metric can be expressed as the following line element in Boyer-Lindquist coordinates:

$$ds^2=-(1-2Mr/\rho^2)dt^2+\rho^2d\theta^2+(\Delta^2/\rho^2)dr^2+[(r^2+a^2)+(2Mra^2/\rho^2)\sin^2\theta]\sin^2\theta\,d\phi^2-(4Mra/\rho^2)\sin^2\theta\,d\phi\,dt\ , \tag{4.1}$$

where $\rho^2=r^2+a^2\cos^2\theta$.

Denote the orthogonal stationary frame at infinity by the usual sperical coordinate basis $\tilde{e}_i$. Let e_j denote a leg of the tetrad that is carried by a ZAMF. The coordinate transformation between the two frames is

$$\begin{bmatrix}e_0\\ e_\phi\end{bmatrix}=\begin{bmatrix}|g_{tt}-\Omega^2g_{\phi\phi}|^{1/2} & \Omega|g_{tt}-\Omega^2g_{\phi\phi}|^{-1/2}\\ 0 & g_{\phi\phi}{}^{-1/2}\end{bmatrix}\begin{bmatrix}\tilde{e}_t\\ \tilde{e}_\phi\end{bmatrix}\ . \tag{4.2}$$

$e_r=(\Delta^{1/2}/\rho)\tilde{e}_r$ and $e_\theta=(1/\rho)\tilde{e}_\theta$, where $\Omega\equiv-g_{\phi t}/g_{\phi\phi}$.

The basis covectors transform as

$$\begin{bmatrix}w^0\\ w^\phi\end{bmatrix}=\begin{bmatrix}|g_{tt}-\Omega^2g_{\phi\phi}|^{1/2} & 0\\ -\Omega g_{\phi\phi}{}^{1/2} & g_{\phi\phi}{}^{1/2}\end{bmatrix}\begin{bmatrix}dt\\ d\phi\end{bmatrix}\ . \tag{4.3}$$

Let X_i denote the basis vectors of the local Lorentz frame that is instantaneously at rest with respect to the ZAMF. One can define local momentum operators in the dual space to the local Lorentz frame that is spanned by the X_i's:

$$\mathcal{P}_0=i\,|g_{tt}-\Omega^2g_{\phi\phi}|^{-1/2}(\partial/\partial t+\Omega\partial/\partial\phi)\ , \tag{4.4a}$$

$$\mathcal{P}_r=-i(\Delta^{1/2}/\rho)\partial/\partial r\ , \tag{4.4b}$$

$$\mathcal{P}_\phi=-ig_{\phi\phi}{}^{-1/2}\partial/\partial\phi\ , \tag{4.4c}$$

$$\mathcal{P}_\theta=-(i/\rho)\partial/\partial\theta\ . \tag{4.4d}$$

The locally measured energy and momentum are given by the eigenvalue equations

$$\mathcal{P}_0\psi=p_0\psi\ \text{and}\ \boldsymbol{\mathcal{P}}\psi=\mathbf{p}\psi\ . \tag{4.5}$$

Some useful approximations for computing the local four-momentum near the horizon are

$$\Omega\approx[a/(r^2+a^2)][1-\Delta\rho^2/(r^2+a^2)^2]\ , \tag{4.6a}$$

$$|g_{tt}-\Omega^2g_{\phi\phi}|^{-1/2}\approx[(r^2+a^2)/(\rho\Delta^{1/2})][1-\Delta a^2\sin^2\theta/2(r^2+a^2)^2]\ , \tag{4.6b}$$

$$g_{\phi\phi}{}^{1/2}\approx[(r^2+a^2)/\rho][1-\Delta a^2\sin^2\theta/2(r^2+a^2)^2]\sin\theta\ . \tag{4.6c}$$

By factoring out the oscillatory behavior of the expressions for the Dirac spinors, one can compute the locally measured momentum p_μ. For the case $a^2=M^2$, using (3.14), and for $a^2<M^2$, using (2.21), Eqs. (4.4) and (4.6) can be applied to yield the same momentum vector:

$$p_0 \approx [w(r^2+a^2)/\rho\Delta^{1/2}][1-\Delta a^2\sin^2\theta/2(r^2+a^2)^2]+(ma/\rho\Delta^{1/2})[1-\Delta(\rho^2+\tfrac{1}{2}a^2\sin^2\theta)/(r^2+a^2)^2] , \quad (4.7a)$$

$$p_r \approx -(1/\rho\Delta^{1/2})\{w(r^2+a^2)+ma-\Delta(k^2+m_e{}^2r_+{}^2)/2[w(r_+{}^2+a^2)+ma]\} , \quad (4.7b)$$

$$p_\phi \approx m\rho/[(r^2+a^2)\sin\theta] . \quad (4.7c)$$

It should be noted that the terms $-i(5/8+2\Gamma)/w_+$ and $ikm_e w(r_+{}^2+a^2)/4w_+\Lambda$ were treated as being negligibly small in the calculation of the local momentum from (2.20) for the reasons that were mentioned in the discussion leading to (2.21).

For small space-time point separation, the integrals in the exponentials (3.14) and (2.21) can be linearly approximated. Consequently, for small point separations, a factor $e^{ip\cdot x}$ can be extracted from the spinor. Thus, the spinor wave functions look like plane waves to these local Lorentz observers in the vicinity of their point of observation.

This suggests that information concerning k, the separation constant, can be ascertained by computing the Lorentz invariant

$$p_\mu p^\mu = p_0{}^2-p_r{}^2-p_\phi{}^2-p_\theta{}^2=m_e{}^2 .$$

The calculation is made easier by noting

$$p_r \approx [1+\Delta a^2\sin^2\theta/2(r^2+a^2)^2]p_0+\rho ma\Delta^{1/2}/(r^2+a^2)^2 - \tfrac{1}{2}(k^2+m_e{}^2r_+{}^2)/p_0\rho^2 , \quad (4.8a)$$

$$p_\theta \approx (1/\rho)\tilde{p}_\theta , \quad (4.8b)$$

where $\tilde{p}_\theta$ is the momentum conjugate to the coordinate θ as measured in the stationary frames at asymptotic infinity.

Computing the local Lorentz invariant, one finds

$$k^2=\mathcal{K}+O(r-r_+) , \quad (4.9)$$

where

$$\mathcal{K}=\tilde{p}_\theta{}^2+(wa\sin\theta+m/\sin\theta)^2+m_e{}^2a^2\cos^2\theta \quad (4.10)$$

is Carter's fourth constant of motion in positive-definite form.

Equation (4.10) removes the physical ambiguity in the radial solutions. Carter's fourth constant of motion is a well-defined physical quantity. Consequently, the radial solutions that were found in Secs. II and III can be completely defined in terms of physically measurable quantities.

The developments of this section give a definition of a local dual momentum space at a point of space-time. The definition of a local momentum space that was presented here is not unique. It clearly depends on the local Lorentz space to which it is dual. The choice of this particular momentum space is most natural due to the special role of the ZAMF's near the horizon as stated in the introduction to this section. Equations (4.4) and (4.6) provide a way of transforming the local momentum operators and their eigenvalues to the stationary frames at asymptotic infinity. As alluded to in the Introduction this is a crucial construction to performing calculations on a curved background.

V. THE GROUP VELOCITY OF THE WAVES

The group velocity of the wave functions that were found in Secs. II and III are calculated in this section. The velocities of classical freely falling particle trajectories should be equivalent to the group velocities of the solutions to the wave equations. Carter has derived the equations for the classical trajectories in the Kerr space-time (see Ref. 40). Comparison of these two velocities requires that $k^2=\mathcal{K}$ as was found in Sec. IV. The equivalence of these two velocities is particularly useful for transforming the characterization of the asymptotic scattering states. It provides a simple well-defined way of going back and forth between the local coordinate description of the states and the corresponding description in terms of the stationary frames at asymptotic infinity. As was stated earlier, this is a construction of practical importance if QED scatterings are to be calculated on the Kerr background.

One can define the group velocity dp_0/dp_i in the local Lorentz basis defined in Sec. IV. Using the invariance of $p^\mu p_\mu$, the group velocity can be written as

$$dX^r/dX^0=p^r/p^0 , \quad (5.1a)$$

$$dX^\phi/dX^0=p^\phi/p^0 . \quad (5.1b)$$

Using the coordinate transformations (4.3) and the approximations (4.6)

$$dX^r/dX^0 \approx [(r^2+a^2)/\Delta][1-\Delta a^2\sin^2\theta/2(r^2+a^2)^2](dr/dt) , \quad (5.2a)$$

$$dX^\phi/dX^0 \approx [(r^2+a^2)^2/\rho^2\Delta^{1/2}][1-\Delta a^2\sin^2\theta/(r^2+a^2)^2][-\Omega+(d\phi/dt)]\sin\theta . \quad (5.2b)$$

Using (5.1) combined with (5.2), one can find dr/dt and $d\phi/dt$ for the group velocity of the waves. Expanding in the small parameter Δ, one has, to $O(\Delta^2)$,

$$dr/dt \approx -[\Delta/(r^2+a^2)][1+\Delta(wa^2\sin^2\theta+ma)/P(r^2+a^2)-\Delta(k^2+m_e{}^2r_+{}^2)/2P^2] , \quad (5.3a)$$

$$d\phi/dt \approx [a/(r^2+a^2)]\{1-\Delta\rho^2[w+m/(a\sin\theta)]/P(r^2+a^2)\} , \quad (5.3b)$$

where $P = w(r^2+a^2)+ma$.

If one were to keep only the first term in (5.3a) and (5.3b) and ignore the corrections of $O(\Delta)$ then the expressions reduce to the group velocity corresponding to a principal null congruence. This was mentioned earlier. The deviation of dr/dt from this trajectory depends on the separation constant squared.

It is instructive to compare (5.3) with Carter's equations that describe classical motion about a black hole (see Ref. 41):

$$\rho^2(dt/d\lambda) = -a(aw\sin^2\theta+m)+(r^2+a^2)/\Delta P \ , \tag{5.4a}$$

$$\rho^2(dr/d\lambda) = -(R)^{1/2} \ , \tag{5.4b}$$

$$\rho^2(d\phi/d\lambda) = -(aw + m/a\sin^2\theta)+P \ , \tag{5.4c}$$

where $R = P^2 - \Delta(m_e{}^2 r^2 + \mathscr{K})$ and λ is a suitably normalized affine parameter. A straightforward calculation shows that an approximation of (5.4) to $O(\Delta^2)$ retrieves (5.3) if and only if $k^2 = \mathscr{K}$.

Since the expressions for the local momentum (4.7) depend on global constants of motion, the comparison with Carter's equations in Boyer-Lindquist coordinates is quite natural. Recalling Eq. (2.22), the solutions of the wave equations can be written in local coordinates like $\exp(-i\int p^\mu dx_\mu)$, where p^μ is the locally measured momentum. The general form of an "in" or "out" scattering state on a large coordinate patch near the event horizon is therefore of this form as well. Asymptotically, Boyer-Lindquist coordinates are the usual spherical Minkowski coordinates. Thus, the relationship between Carter's equations and the group velocity of the waves that was established in this section allows one to transform between the in and out states on the large local coordinate patch and the asymptotic states at stationary infinity.

In order to give a simple example of how these transformations of asymptotic scattering states would be performed, one must make some standard definitions. If a local vacuum is constructed that is in accordance with the definition in Sec. VI [see (6.5)] then this transformation is manifested as relabeling of the quantum numbers of the in and out states in one frame in terms of the quantum numbers in the other frame. As in (6.5), the local in and out vacuums are associated with local creation and destruction operators by

$$\begin{aligned} &a(p^\mu(x),s)_{\rm in}\,|\,0_{\rm in}\rangle = 0 \ ,\\ &a(p^\mu(x),s)_{\rm out}\,|\,0_{\rm out}\rangle = 0 \ ; \end{aligned} \tag{5.5a}$$

$$\begin{aligned} &b(p^\mu(x),s)_{\rm in}\,|\,0_{\rm in}\rangle = 0 \ ,\\ &b(p^\mu(x),s)_{\rm out}\,|\,0_{\rm out}\rangle = 0 \ , \end{aligned} \tag{5.5b}$$

where $a^\dagger(p^\mu(x),s)_{\rm in}$ creates an in particle state with a four-spinor index s and $p^\mu(x) = (p^0(x),\mathbf{p}(x))$ is the four-momentum of the particle state evaluated "near" the boundary of the local coordinate patch. The radial wave function of the created particle in the region near the boundary of the patch goes like $\exp(-i\int p^\mu(x)dx_\mu)$ as stated in the previous paragraph. Likewise, $b^\dagger(p^\mu(x),s)_{\rm in}$ creates an in particle state with four-spinor index s and four-momentum $p^\mu(x) = (-p^0(x),\mathbf{p}(x))$. The out creation operators act analogously on the out vacuum.

The fact that the destruction operators annihilate the vacuums is a manifestation of choosing local vacuums with no particles in them as is done in Sec. VI. Thus, one does not have the effects of Hawking radiation in this formalism. If for some reason one were interested in including thermal effects, one could proceed along the lines of this discussion with the added implementation of a Bogoliubov transformation on the local Fock space (see Ref. 42).

There are in and out vacuums at stationary infinity with the corresponding creation and destruction operators:

$$\begin{aligned} &\tilde{a}(k,m,w)_{\rm in}\,|\,\tilde{0}_{\rm in}\rangle = 0 \ ,\\ &\tilde{a}(k,m,w)_{\rm out}\,|\,\tilde{0}_{\rm out}\rangle = 0 \ ; \end{aligned} \tag{5.6a}$$

$$\begin{aligned} &\tilde{b}(k,m,w)_{\rm in}\,|\,\tilde{0}_{\rm in}\rangle = 0 \ ,\\ &\tilde{b}(k,m,w)_{\rm out}\,|\,\tilde{0}_{\rm out}\rangle = 0 \ , \end{aligned} \tag{5.6b}$$

where $\tilde{a}^\dagger(k,m,w)_{\rm in}$ creates an in state particle out of the in vacuum at stationary infinity with quantum numbers k, m, and w and $\tilde{b}^\dagger(k,m,w)_{\rm in}$ creates an in state particle with quantum numbers k, $-w$, and $-m$ [see (6.4)]. Similar relations hold for the out states of the out vacuum at stationary infinity.

As an example of the transformation of asymptotic states, consider a single particle in state at stationary infinity

$$\psi_{\rm in} = \tilde{a}^\dagger(k,m,w)_{\rm in}\,|\,\tilde{0}_{\rm in}\rangle \ . \tag{5.7}$$

The transformation to the set of local asymptotic states is given by

$$\psi_{\rm in} \rightarrow \begin{cases} a^\dagger(p^\mu(x),s)_{\rm in}\,|\,0_{\rm in}\rangle, & p^0 > 0 \ ,\\ b^\dagger(-p^0(x),\mathbf{p}(x),s)_{\rm in}\,|\,0_{\rm in}\rangle, & p^0 < 0 \ , \end{cases} \tag{5.8}$$

where $p^\mu(x)$ is given in terms of k, m, and w by (4.7) evaluated near the boundary of the coordinate patch. It should be noted that p_θ can be determined from k, m, and w from (4.10). The interesting aspect of this transformation is that there are modes with values of w, m, and k that imply from the relation (4.7a) values of $p^0 < 0$. These modes propagate locally as negative-energy states. The four-spinor index s in a coordinate patch near the horizon is essentially fixed by the boundary conditions at the horizon that are discussed in the paragraph that follows (2.8). In particular, with the representation of the field that is chosen in Sec. VI one has only the two and three components of the spinor as is described in the aforementioned paragraph following (2.8).

The entire set of asymptotic in states (many particle states) can be constructed like those in (5.7) in the standard way by stringing together many in creation operators (see Ref. 54). The transformation to the local asymptotic in states is totally analogous to (5.8) but requires more effort to write down.

The corresponding transformation of the out states is basically the same with one added complication. One must distinguish between which local out states reach infinity and which are swallowed by the hole. This divides the local out Fock space into two subspaces. It can be seen from this sketchy discussion that the analysis of a particular scattering problem will require a great deal of time and labor to compute.

VI. THE QUANTUM SPINOR FIELD

The free-particle solutions of the Dirac equation are given in general form by (2.1). In this representation, conjugate spinors can be constructed from ψ as[16]

$$\bar{\psi}=\psi^{\dagger}\alpha, \quad \text{where } \alpha=\begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix} . \tag{6.1}$$

The γ matrices in the representation that is used in this paper are given by Iyer and Kumar:[17]

$$\gamma^{\mu}=\begin{bmatrix} 0 & \sigma^{\mu} \\ \tilde{\sigma}^{\mu} & 0 \end{bmatrix},$$

where (6.2)

$$\tilde{\sigma}^{\mu}=-\epsilon\sigma^{\mu T}\epsilon, \quad \epsilon=\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix},$$

and

$$\sigma^{t}=\begin{bmatrix} (r^2+a^2)/2\,|\bar{\rho}\,|^2 & ia\sin\theta/2^{1/2}\bar{\rho}^{*} \\ -ia\sin\theta/2^{1/2}\bar{\rho} & (r^2+a^2)/\Delta \end{bmatrix},$$

$$\sigma^{\theta}=\begin{bmatrix} 0 & -1/2^{1/2}\bar{\rho}^{*} \\ -1/2^{1/2}\bar{\rho} & 0 \end{bmatrix},$$

$$\sigma^{\phi}=\begin{bmatrix} a/2\,|\bar{\rho}\,|^2 & i\csc\theta/2^{1/2}\bar{\rho}^{*} \\ -i\csc\theta/2^{1/2}\bar{\rho} & a/\Delta \end{bmatrix}, \tag{6.3}$$

$$\sigma^{r}=\begin{bmatrix} -\Delta/2\,|\bar{\rho}\,|^2 & 0 \\ 0 & 1 \end{bmatrix}.$$

The Dirac field can be expanded in terms of the basis modes (2.1):

$$\Psi(x)=\sum_{m}\int dw \sum_{k} a(w,m,k)\psi(w,m,k,x) + b^{\dagger}(w,m,k)\psi(-w,-m,k,x) . \tag{6.4}$$

The creation and destruction operators are defined by the following operations on the vacuum:

$$a(w,m,k)\,|\,0\rangle=0, \quad b(w,m,k)\,|\,0\rangle=0 . \tag{6.5}$$

Defined as such, the local vacuum does not contain any particles. Consequently, the following analysis does not deal with thermal propagator effects (see Ref. 42).

It should be noted that the outgoing modes have been neglected in the summation over states in (6.4). Their effects are expected to be small as deduced in the discussion leading to (2.7). The outgoing states are needed to form a complete set of modes. However, the appropriate normalization is such that the amplitudes of these modes are negligibly small. Iyer and Kumar divide these states into two sets: type I and type II (Ref. 18). The solutions that dominate the mode sum in (6.4) are of type I. The type-II solutions are unphysical, corresponding to outgoing waves at the horizon in the remote past. If one were to include the outgoing states in the computation of the propagators, there would be an additional term of the order of the ratio of the Compton wavelength of the wave to the radius of curvature of the horizon. As was stated in Sec. II, this ratio is very small.

It is basically assumed in this paper that the questions of quantum field theory are clearly posed in the Kerr space-time. For a discussion of the ambiguities see the sources previously quoted in this section as well as the review article of DeWitt (see Ref. 43) or the previously quoted book by Birrell and Davies.

VII. CAUSAL PROPAGATORS

So far, the relevance of this paper to scattering theory has been the definition of scattering states and the construction of a local dual momentum space. It has been shown how each of these can be transformed back and forth from the local coordinate patch to the stationary frames at asymptotic infinity. The computation of QED scattering involves one more construction, the intermediate (off-mass-shell) scattering states. Consequently, one is led to the study of Feynman propagators. In this section, in analogy to what was done for the scattering states, the propagators that are computed in the spatial coordinates at asymptotic infinity will be transformed to spatial propagators in a local coordinate system near the event horizon. Following this, it takes a long calculation that is deferred to Appendix C to rewrite the propagators in the local dual momentum space. This is the object that is relevant to scattering calculations.

As expected, the space-time near the horizon is not warped enough to introduce any noticeable curvature effects in the propagator (see Ref. 44). Propagators involve momentum exchanges over space-time intervals that are small on the scale of the curvature. Locally, in the coordinates that are used it is found that the propagator is like a flat-space propagator that has been ultrarelativistically boosted in the radial direction. This manifests itself as $\not{p}\approx p^0\gamma_0-p^r\gamma_r$ in the numerator of the integrand in momentum space (7.19a). Equivalently, it is found that with the choice of vacuum (6.5) global effects do not manifest themselves in these calculations.

The Feynman propagator is defined in terms of the fields (6.4) as a time-ordered vacuum expectation value:

$$S_F(x;x')_{\alpha\beta}=i\langle 0\,|\,T[\Psi(x)\bar{\Psi}(x')]\,|\,0\rangle_{\alpha\beta} . \tag{7.1}$$

This is a bispinor, but the points x and x' will be taken to be very close together. Thus, one has the same Dirac matrices and representation of the field at both points.

The computation requires a well-defined chronology operation. A logical choice near the horizon is given in terms of the local coordinates X_i that were introduced in Sec. IV [i.e., $\Theta(X_0-X'_0)$ implies future directed, etc.].

Before calculating, one needs some relations involving the angular functions. Orthogonality is given by[19]

$$\int [S_k^-(\theta)S_{k'}^{-*}(\theta)+S_k^+(\theta)S_{k'}^{+*}(\theta)]d\Omega=\delta_{kk'} \tag{7.2}$$

and since $S_k^-(\pi-\theta)=S_k^+(\theta)$

$$\int |S_k^+(\theta)|^2 d\Omega=\int |S_k^-(\theta)|^2 d\Omega=\tfrac{1}{2} . \quad (7.3)$$

Completeness of the functions was alluded to in Sec. II:

$$\sum_k [S_k^-(\theta)S_k^{-*}(\theta')+S_k^+(\theta)S_k^{+*}(\theta')]=\delta(\theta-\theta') . \quad (7.4)$$

A summation that appears in the calculation of the propagator is

$$\sum_{k_j} k_j{}^2[S_{k_j}^-(\theta)S_{k_j}^{-*}(\theta')+S_{k_j}^+(\theta)S_{k_j}^{+*}(\theta')]\equiv F(\theta,\theta') . \quad (7.5)$$

Since the differential equation for the angular functions, (2.2), is very complicated, there are no tricks that can be used to evaluate the sum by means of this equation. Instead, the following trick will be used.

Carter has shown that the separation constant is the eigenvalue of an operator $\mathscr{L}$, that is like a generalized angular momentum operator for the Kerr space-time.[20] $\mathscr{L}$ commutes with the Dirac Hamiltonian. Therefore, so does $\mathscr{L}^2$. Hence, eigenstates of the Hamiltonian are also eigenstates of $\mathscr{L}^2$.

The operator $\mathscr{L}^2$ is related to the summation $F(\theta,\theta')$. To see this, recall (6.4). A free Dirac field can be expanded in basis eigenstates of the Hamiltonian as

$$\Psi(r,\theta,\phi,t)=\sum_m \int dw \sum_{k_i} a(w,m,k_i)\psi(w,m,k_i,r,\theta,\phi,t)+b^\dagger(w,m,k_i)\psi(-w,-m,k_i,r,\theta,\phi,t) . \quad (7.6)$$

One can apply the operator $\mathscr{L}^2$ to the free field by moving it through the summation in (7.6), to act on each of the eigenfunctions individually:

$$\mathscr{L}^2\Psi(r,\theta,\phi,t)=\sum_m \int dw \sum_{k_i} k_i{}^2[a(w,m,k_i)\psi(w,m,k_i,r,\theta,\phi,t)+b^\dagger(w,m,k_i)\psi(-w,-m,k_i,r,\theta,\phi,t)] . \quad (7.7)$$

Consider the integral

$$\mathscr{I}=\int \Psi(r,\theta,\phi,t)F(\theta,\theta')d\Omega . \quad (7.8)$$

Using the definitions (7.5) and (7.6), Eq. (7.4) can be used to perform the summation over k_i. Then, after integrating over $d\Omega$ in $\mathscr{I}$, one has

$$\mathscr{I}=\sum_m \int dw \sum_{k_j} k_j{}^2[a(w,m,k_j)\psi(w,m,k_j,r,\theta',\phi,t)+b^\dagger(w,m,k_j)\psi(-w,-m,k_j,r,\theta',\phi,t)] . \quad (7.9a)$$

By (7.7), this is equal to

$$\int \delta(\theta-\theta')\mathscr{L}^2\Psi(r,\theta,\phi,t)d\Omega . \quad (7.9b)$$

One concludes that

$$F(\theta,\theta')=\delta(\theta-\theta')\mathscr{L}^2 . \quad (7.10)$$

$F(\theta,\theta')$ is the analog of a Fourier transform for the angular functions $S_k^\pm(\theta)$ of the operator $\delta(\theta-\theta')\mathscr{L}^2$.

In order to evaluate $F(\theta,\theta')$, one needs only to understand the operator $\mathscr{L}^2$. It is convenient to formulate this problem in the number representation of the Dirac field. Following Bjorken and Drell, a general state of the Dirac field is given by the expression[21]

$$\Phi=\sum_N \sum_{\bar N} c(n_1,\ldots,n_N,\tilde n_1,\ldots,\tilde n_{\bar N})(b_{\bar N}^\dagger)^{\tilde n_{\bar N}}\cdots(b_1^\dagger)^{n_1}(a_N^\dagger)^{n_N}\cdots(a_1^\dagger)^{n_1}|0\rangle ,$$

where

$$\sum_{\bar N}\equiv\sum_{\tilde n_1,\ldots,\tilde n_{\bar N}=0}^{1}$$

and

$$\sum_N\equiv\sum_{n_1,\ldots,n_N=0}^{1} .$$

The wave functions $c(n_1,\ldots,n_N,\tilde n_1,\ldots,\tilde n_{\bar N})$ are the probability amplitudes corresponding to the probability distribution of the population of states:

$$(\Phi,\Phi)=\sum_N \sum_{\bar N} |c(n_1,\ldots,n_N,\tilde n_1,\ldots,\tilde n_{\bar N})|^2 .$$

The expectation value of $\mathscr{L}^2$ is given by

$$(\Phi,\mathscr{L}^2\Phi)=\sum_N \sum_{\bar N} |c|^2\left[\sum_{\alpha=1}^{N} n_\alpha k_\alpha{}^2+\sum_{\beta=1}^{N} n_\beta k_\beta{}^2\right]\equiv k^2 , \quad (7.11a)$$

where k^2 is the classical average of the $k_i{}^2$. Since it is an expectation value, k^2 is an observable of the quantum field. By (7.10), the expectation value of $F(\theta,\theta')$ is

$$(\Phi,F(\theta,\theta')\Phi)=k^2\delta(\theta-\theta') . \quad (7.11b)$$

Equation (7.11b) is indicative of the context in which $F(\theta,\theta')$ appears in the computation of the vacuum expectation value of the time-ordered product (7.1).

The rest of the calculation is straightforward. Define the matrix

$$\tilde{\gamma}^{\,0}_{\alpha\beta}=\begin{bmatrix}0&0&0&0\\0&0&0&1\\1&0&0&0\\0&0&0&0\end{bmatrix}. \qquad (7.12)$$

Let x approach x' in the exponent of $R_2(r-r_+)$, in (2.21) and (3.14). Then the integrals can be linearly approximated as was done in the discussion of the local plane waves in Sec. IV. This is equivalent to demanding that the momentum exchanges in the intermediate states are large compared to the curvature of the space-time. It is a sort of far infrared cutoff to the summation over states. As a consequence, the summations are much easier. As is standard in the computation of causal propagators, a small positive parameter ϵ is introduced with the replacement $w_+\to w_+-i\epsilon$. Then after a calculation that utilizes (7.11) in an intermediate step, the propagator can be computed. The calculation is entirely straightforward but involves the handling of large complicated terms:

$$\begin{aligned}S_F(x;x')_{\alpha\beta}=&\ [2\pi i\delta(\theta-\theta')\Theta(X_0-X_0')/\Delta][\delta(a(t-t')/(r^2+a^2)-(\phi-\phi'))]\\&\times\{\delta_+((t-t')+(r^2+a^2)(r-r')/\Delta)+[(k_0{}^2+m_e{}^2r_+{}^2)(r-r')/2(r_+{}^2+a^2)]\\&\quad\times\Theta((t-t')+(r^2+a^2)(r-r')/\Delta)\}\tilde{\gamma}^{\,0}_{\alpha\beta}\\&-[2\pi i\delta(\theta-\theta')\Theta(X_0'-X_0)/\Delta][\delta(a(t-t')/(r^2+a^2)-(\phi-\phi'))]\\&\times\{\delta_+^*((t-t')+(r^2+a^2)(r-r')/\Delta)-[(k_0{}^2+m_e{}^2r_+{}^2)(r-r')/2(r_+{}^2+a^2)]\\&\quad\times\Theta(-(t-t')-(r^2+a^2)(r-r'))\}\tilde{\gamma}^{\,0}_{\alpha\beta}\,,\end{aligned} \qquad (7.13)$$

where $\delta_+(\lambda)=(1/\pi)\int_0^\infty dx\, e^{ix\lambda}=\delta(\lambda)+1/\pi\lambda$.

The term that involves $k_0{}^2$ comes from doing an integral over dw with $w_+\to w_+-i\epsilon$, that yields a θ function. $k_0{}^2$ is the value of the classical average, k^2 of (7.11), evaluated at the pole of the integral form of the θ function. The pole is at $w=-m\Omega_H$. If $k^2=\mathscr{K}$, then

$$k_0{}^2=\tilde{p}_\theta{}^2+m^2\rho_+{}^4/(r_+{}^2+a^2)^2\sin^2\theta+m_e{}^2a^2\cos^2\theta\,, \qquad (7.14)$$

where $\Omega_H=a/(r_+{}^2+a^2)$ is the angular velocity of the event horizon as observed in the stationary frames at asymptotic infinity. Since (7.13) is a spatial propagator, (4.4) and (4.7) can be used to rewrite $k_0{}^2$ along the lines of the quantum-mechanical correspondence principle:

$$k_0{}^2=-\partial^2/\partial\theta^2-(1/g_{\phi\phi})\partial^2/\partial\phi^2+m_e{}^2a^2\cos^2\theta\,. \qquad (7.15)$$

Using (4.3) and (4.2), the propagator (7.13) can be rewritten in terms of the local Lorentz basis of Sec. IV. Ignoring contributions of order Δ

$$\begin{aligned}S_F(x;x')_{\alpha\beta}\approx&\ \Theta(X_0-X_0')\delta(X_\phi-X_\phi')\delta(X_\theta-X_\theta')\\&\times\{\delta_+[(X_0-X_0')-(X_r-X_r')]+\tfrac{1}{2}(p_\theta{}^2+p_\phi{}^2+m_e{}^2)\,|X_r-X_r'|\,\Theta[(X_0-X_0')-(X_r-X_r')]\}(\rho/\Delta^{1/2})\tilde{\gamma}^{\,0}_{\alpha\beta}\\&-\Theta(X_0'-X_0)\delta(X_\phi-X_\phi')\delta(X_\theta-X_\theta')\\&\times\{\delta_+[(X_0-X_0')-(X_r-X_r')]+\tfrac{1}{2}(p_\theta{}^2+p_\phi{}^2+m_e{}^2)\,|X_r-X_r'|\,\Theta[(X_r-X_r')-(X_0-X_0')]\}(\rho/\Delta^{1/2})\tilde{\gamma}^{\,0}_{\alpha\beta}\,,\end{aligned} \qquad (7.16)$$

where as in (7.15) one uses the quantum-mechanical correspondence:

$$p_\theta{}^2+p_\phi{}^2=-\partial^2/\partial X_\theta{}^2-\partial^2/\partial X_\phi{}^2\,. \qquad (7.17)$$

Notice that in (7.14) and (7.16) the support of the δ function of the propagator lies on the principal null congruences. The δ-function support of a causal propagator should lie on the light cone. However, in this case the light cone has collapsed to a null line in the radial direction by an ultrarelativistic headlight effect. The contribution of the other two spatial dimensions manifests itself in the propagator in a manner similar to the contribution of the radial wave function that was found in Sec. II to not lie on the principal null congruences. Recalling Secs. II and V, it was Carter's fourth constant of motion or equivalently k^2 that produced an effect that caused the geodesics to very slightly deviate from the principal null congruences near the event horizon. The Feynman propagator (7.14) shows that k^2 appears in this expression as a coefficient in front of the θ function. When this is rewritten in local coordinates in (7.16) with the correspondence (7.17), the local expression for the propagator looks like it has δ-function support along the null rays with small corrections in the X_θ and X_ϕ directions. These corrections manifest themselves as the k^2 contribution to the θ function. The appearance of this term in (7.16) looks suggestively like a Taylor-series expansion about the principal null congruences in the quantities p_ϕ and p_θ.

Combining (4.2), (6.3), and (6.2), one has the following approximations for the γ matrices near the event horizon:

$$\gamma^0_{\alpha\beta} \approx (\rho/\Delta^{1/2})\tilde{\gamma}^{\,0}_{\alpha\beta}$$

and

$$\gamma^r \approx (\rho/\Delta^{1/2})\tilde{\gamma}^{\,0}_{\alpha\beta} \tag{7.18}$$

with

$$\gamma^\theta \approx \gamma^\phi \approx 0 \ .$$

The fact that $\gamma^0 \approx \gamma^r$ near horizon is another reflection of the approximate one-dimensional nature of wave propagation in this region. Equations (7.17) and (7.18) can be used to rewrite (7.16) as an integral in the local dual momentum space. This is a very long calculation that is presented in Appendix C. The salient feature is that p_r appears differently in the integral than the other momenta [see (C1)] and p_θ and p_ϕ are just small corrections. The end results (C13) and (C16) yield the following expressions that are accurate to $O(\Delta)$:

$$S_F(x;x)_{\alpha\beta} \approx \int \frac{(-p_0\gamma^0_{\alpha\beta}+p_r\gamma^r_{\alpha\beta})+m_e}{p^2-m_e{}^2} e^{ip\cdot x} d^4p \tag{7.19a}$$

$$\approx \int \frac{\not{p}+m_e}{p^2-m_e{}^2} e^{ip\cdot x} d^4p \ . \tag{7.19b}$$

The numerator in the expression (7.19a) looks different than that usually found in a flat-space Dirac propagator. The occurrence of only one momenta to the exclusion of the others is a result of the fact that all particles appear ultrarelativistically boosted in radial direction in the frame of observation. With the covariant form of the same expression (7.19b) one could transform to another frame by means of an ultrarelativistic radial boost where all of the momenta would appear on the same footing. It is implicit in going from (7.19a) to (7.19b) that the $O(\Delta)$ corrections that have been ignored are present. The corrections to $O(\Delta)$ that were not included in (7.16) (for the sake of compactifying the already unruly mathematical expressions) correspond to the slight difference in the rotational velocity of the ZAMF's and the principal null congruences near the horizon. Technically, these need to be retained in going from (7.19a) to (7.19b). There is also an $O(\Delta)$ correction to the overall amplitude of the Green's function from (2.17).

In conclusion, there are no global effects that distort the locally flat nature of the propagator. In the process of showing this some potentially useful calculational techniques such as an analog of Fourier transform for the Kerr space-time were developed. There also appeared to be a provocative relationship between k^2 and the locally measured total orbital angular momentum squared $p_\phi{}^2+p_\theta{}^2$ in (7.14) and (7.16).

VIII. THE ELECTROMAGNETIC FIELD

The literature on the electromagnetic wave equation in the Kerr space-time is far more extensive than for the massive Dirac equation. One reason for this is that the equations are easier to separate. Consequently, the separation was accomplished almost a decade sooner than it was for the massive Dirac equation. For an excellent detailed review of this problem see Chandrasekhar's book (see Ref. 45).

The present state of the solution to this equation in the literature is the same as for the Dirac equation. The same change of variables is used and the radial functions are the same as (2.5). As was done for the massive Dirac equation, the methods of Secs. II and III will be applied resulting in more accurate analytic solutions near the event horizon. In the process the physical nature of the separation constant will be revealed. The radial equation turns out to be much simpler than in the Dirac case.

The separation of the massless vector wave equation in the Kerr space-time was performed by Chandrasekhar by separating the Newman-Penrose scalars of the field into the form[22]

$$\phi_1(r,\theta,\phi,t) = e^{-iwt}e^{-im\phi}R_1(r)S_1(\theta) \ , \tag{8.1a}$$

$$\phi_{-1}(r,\theta,\phi,t) = e^{-iwt}e^{-im\phi}(B/\bar{\rho}^{*2})R_{-1}(r)S_{-1}(\theta) \ , \tag{8.1b}$$

where B^2 is related to the separation constant of the angular equation E by

$$B^2 = (E + w^2a^2 + 2awm)^2 - 4a^2w^2 - 4awm \ . \tag{8.2}$$

The radial equations are

$$\Delta^2 \frac{d^2}{dr^2}(R_1) + 4\Delta(r-M)\frac{d}{dr}(R_1) + [P^2 - 2i(r-M)P + \Delta(4iwr - k_+)]R_1 = 0 \ , \tag{8.3a}$$

$$\Delta^2 \frac{d^2}{dr^2}(R_{-1}) + [P^2 + 2i(r-M)P - \Delta(4iwr + k_-)]R_{-1} = 0 \ , \tag{8.3b}$$

where, as before, $P = w(r^2+a^2)+ma$, and

$$k_\pm = E + w^2a^2 + 2wam - (1\pm 1) \ . \tag{8.4}$$

The variable coefficients and the potential for the differential equation for R_1 have been expanded about $r = r_+$ in power series in Appendix A. Those for R_{-1} are obtained by complex conjugation.

These equations have the same singular point structure as for the Dirac equation. When $a^2 < M^2$, the singular points are regular. The indicial relation for this case can be solved with the aid of Appendix A:

$$\lambda_1 = -\tfrac{1}{2} \mp \tfrac{1}{2} \mp iw_+/2(M^2-a^2)^{1/2} \ , \tag{8.5a}$$

$$\lambda_{-1} = \tfrac{1}{2} \pm \tfrac{1}{2} \pm iw_+/(M^2-a^2)^{1/2} \ . \tag{8.5b}$$

As with the Dirac equation, the physical ingoing solutions will be considered. R_1 is the larger of the ingoing waves. Using the techniques that were developed in Secs. II and III applied to these simpler differential equations, one has, near the horizon for $a^2 < M^2$,

$$R_1(r-r_+)=\{1+[-\tfrac{1}{2}(M^2-a^2)^{-1/2}-wr_+/w_+](r-r_+)+O((r-r_+)^2)\}\Delta^{-1}$$
$$\times\exp\left[-iw\int_{r_+}^{r}dr(r^2+a^2)/\Delta(r)-ima\int_{r_+}^{r}dr/\Delta(r)+i(k_++2)(r-r_+)/2w_+\right]. \tag{8.6}$$

Following the calculations of Sec. III, one has, for $a^2=M^2$,

$$\lambda=-2aiw-\tfrac{3}{2}\,, \tag{8.7}$$

$$a_1\approx -iw+9i/8w_+\,, \tag{8.8}$$

$$R_1(r-a)\approx (r-a)^{-3/2}[1+9i(r-a)/8w_+]e^{-iw(r-a)}e^{-i2aw\ln(r-a)}$$
$$\times H_0^{(1)}((2wa^2+ma)/(r-a)+\tfrac{1}{2}k_+(r-a)/(2wa^2+ma))\,. \tag{8.9}$$

The discussions of Secs. IV–VII imply that k_+ is approximately Carter's fourth constant of motion in positive-definite form (up to corrections due to spin that are of the order unity; for a discussion of these effects see the remarks in the conclusion). This is a very peculiar result in that Chandrasekhar asserts that[23]

$$k_+{}^2-4w^2a^2-4wma>0\,,$$

which is not necessarily true if $k_+=\mathscr{K}$.

It should be noted that the electromagnetic propagator can be calculated as was done for the Dirac propagator in Sec. VII and Appendix C with similar results. The tensor nature of the propagator is much more complicated than in flat space. The covariant form of the tensorial part is simply the metric tensor $g_{\mu\nu}$. However, the gauge conditions in the Kerr space-time are very complicated (see Ref. 46). They constitute two coupled partial-differential equations. In practice, the choice of a gauge that sacrifices the general covariance of the propagator greatly facilitates a QED calculation. If one specifies a gauge locally to simplify a calculation then one has effectively chosen a boundary condition for these partial-differential equations. In general, due to the curvature of the space-time this gauge condition will manifest itself in an entirely different manner in another distant coordinate patch. For example, one could not impose the Coulomb gauge globally. The transformation of the locally observed propagator to the frames that are stationary at asymptotic infinity will be highly dependent on the choice of gauge. The problem is complicated since this gauge will only be physically meaningful and mathematically simplifying in one of the two frames.

IX. CONCLUSION

The main result of this paper is that the separation constant squared of the Dirac equation in the background of an astrophysical black hole is Carter's fourth constant of motion in positive-definite form. Similarly, the separation constant of the electromagnetic field, as chosen by Chandrasekhar, is also equal to Carter's fourth constant.

These results allow one to calculate well-defined solutions to the radial equations. The value of the separation constant is not surprising. Brill *et al.* separated the scalar wave equation so that the separation constant looks like Carter's fourth constant of motion [not in positive-definite form, for a definition of this see MTW (see Ref. 47)] plus an m^2 from the azimuthal separation (see Ref. 48). This is somewhat expected since the Klein-Gordon equation is similar to the Hamilton-Jacobi equation for particle orbits in the Kerr space-time. It is from this equation that Carter discovered this mysterious fourth constant of motion as a separation constant of the equations (see Ref. 49).

Many attempts have been made to interpret Carter's fourth constant as a generalized angular momentum (see Ref. 55). This is motivated by its limiting value as such for the Schwarzschild space-time ($a=0$). In the calculations in this paper, the coupling of the spin angular momentum to the separation constant is obscured by the smallness of a number like $\frac{1}{2}$ compared to $w^2a^2\sim 10^{50}$. In spite of the care in handling small terms like i/w_+, partly motivated by these considerations, no simple combinatorial relationship can be detected for the massive Dirac equation. This could be expected for two reasons. First, the separation constant appears in the angular equation (2.2) in a manner that is very complicated. Second, the coefficient of a_n in (2.12) contains a $-\frac{1}{2}n$ that was dropped as being negligible in calculating a_n^0 in (2.15) and (2.16). As remarked in the paragraph following (2.17) this introduces corrections of the order i/w_+. These are precisely those that are related to spin coupling. In the electromagnetic case there is the great simplifying relation $A_2=B_1$ in Appendix A. Thus, there will be no $-\frac{1}{2}n$ correction to the coefficient of a_n in (2.11). The solution to the equation is more straightforward. The solution (8.6) implies that $k_+=\mathscr{K}-s(s+1)$, where $s=1$. This is suggestive of the identification of k_+ with the total orbital angular momentum squared and $\mathscr{K}$ with the total angular momentum squared. As a check of the validity of the term i/w_+ in the exponential of (8.6) that gives the spin contribution, one can compare with the solution for $a^2=M^2$. To first order there is agreement. There is a $9i/8w_+$ from a_1 in (8.8) and a $-i/8w_+$ from the asymptotic form of the Hankel function (3.7) in (8.9). The connection of k^2 with the total orbital angular momentum squared was already found locally in the study of the causal Dirac propagator as noted in the final paragraph of Sec. VII.

These results can be compared to the previously existing research. Teukolsky and Press have analyzed the electromagnetic separation constant numerically. The angular differential equation is broken into two parts. One is the operator that has spin-weighted spherical harmonics as its solutions.[24] The other part is a noninfinitesimal perturbation. The separation constant is a function of the eigenvalues l and m of the operator which has spin-weighted spherical harmonics as eigenfunctions. When $wa=0$ the perturbation vanishes and as stated above the solutions of the angular equation are spin-weighted harmonics with eigenvalues $(l-s)(l+s+1)$. In this approximation l can be associated with orbital angular momentum. This is consistent with the results of this paper alluded to in the previous paragraph. However, when $wa \gg 0$ the physical interpretation of l is not clear. This is characteristic of the difficulty in trying to split up the orbital angular momentum of a general-relativistic system in an invariant way. For a discussion of angular momentum in the Kerr space-time see Carter's article (see Ref. 50). As stated in Sec. VII he finds that the separation constants in question are the relevant eigenvalues.

Teukolsky and Press find the separation constant for small values of wa, i.e., $wa<3$. However, as stated earlier, this article deals with $wa \sim 10^{25}$. They cite a method of continuing the solutions to large values of wa. Some of their results are tabulated in Table VII of the Appendix of Chandrasekhar's book.[25] Even if l were known in terms of physically measurable quantities, the results of this paper cannot be compared with those in the table. The dimensionless quantity wa was not taken to be a small parameter in the calculations, which is the relevant approximation in the table.

At the end of Sec. VIII, it was shown that if the separation constant of the electromagnetic wave equation were equal to Carter's fourth constant of motion, then an inequality that has been asserted by Chandrasekhar will be violated by certain sets of quantum numbers for the waves. This is a cause for concern. However, some of the values in the previously alluded to Table VII of Chandrasekhar do not satisfy the inequality. In particular, the column labeled $l=1$ and $m=0$. At the bottom of the column the inequality becomes $2.25\times 10^{-2}>64$.

There are two possibilities. The inequality is incorrect or the table is in error (perhaps a computer error). If the inequality is correct then combined with the results of this paper one has a limitation on the spectrum of allowed values of Carter's fourth constant $\mathscr{K}$ for waves propagating near the event horizon. Equivalently, one has a restriction on the allowed values of generalized angular momentum.

It was stated in the Introduction that the results that are obtained in this paper are an attempt at laying the groundwork for QED scattering calculations on the background of the Kerr solution near the event horizon. The motivation for calculating scatterings is enhanced by some previously existing work. Birrell and Davies describe a well-defined renormalization procedure on a curved space-time background. They discuss S-matrix properties as well (see Ref. 51).

The basic scenario for using the results in this paper for a scattering calculation would be the following. The scattering calculation is performed as in flat space-time in a momentum space that is dual to a local coordinate patch. One must transform the momentum-dependent answer to the stationary frames at asymptotic infinity in two stages. First transform the local momentum to the momentum as observed at infinity. Then one must transform the local scattering states to an asymptotic set of scattering states. Local in and out scattering states can be defined unambiguously at the boundary of a large coordinate patch especially for short-wavelength waves. Using the results of Sec. IV the momentum can be transformed and Sec. V gives a prescription for transforming the scattering states. This procedure can be done unambiguously if the local coordinate patch covers most of the "curvature potential." Price has shown that most of this potential is concentrated in a finite region near the event horizon (see Ref. 52). Consequently, the scenario as described seems tractable.

The final scattering calculation could be thought of in the following sense. Scattering is done as in flat space but the corrected scattering states are used. This is similar to scattering in Coulombic fields where to first approximation one uses free-field wave functions as the asymptotic scattering states. Better results are obtained when the Coulomb wave functions are used as the asymptotic states (at the expense of much longer calculations). This is basically what has been outlined in the previous paragraphs. The corrected wave functions are the Kerr wave functions in a WKB-type approximation [as mentioned in the remarks concerning (2.22)]. This is probably an excellent approximation as the curvature of the space-time and the momentum of the particles varies negligibly in a cycle of a wave that describes realistic scattering states (i.e., momentum not near zero, even long radio wavelengths on the order of many kilometers are fine). The process of making these statements more precise and the long calculations that are involved in computing a scattering process must wait for a future work.

APPENDIX A

The differential equations in this article are of the general form

$$A(r)\frac{d^2}{dr^2}(R)+B(r)\frac{d}{dr}(R)+V(r)R=0\ .$$

Expansions of the variable coefficients $A(r)$ and $B(r)$ and the potential $V(r)$ about the event horizon are given below. $R_1(r-r_+)$ and $R_2(r-r_+)$ satisfy complex-conjugate radial components of the Dirac equation. The expansions for $R_1(r-r_+)$ are

$$A_6=m_e{}^2(r-r_+)^6,\quad A_5=m_e{}^2(4r_+-2r_-)(r-r_+)^5\ ,$$

$$A_4=\{m_e^2[r_+^2+4r_+(r_+-r_-)+(r_+-r_-)^2]+k^2\}(r-r_+)^4\ ,$$

$$A_3=[2m_e^2r_+(r_+-r_-)(3r_+-r_-)+2k^2(r_+-r_-)](r-r_+)^3\ ,$$

$$A_2=(k^2+m_e^2r_+^2)(r_+-r_-)^2(r-r_+)^2\ ;$$

$$B_4=(m_e^2M+ikm_e)(r-r_+)^4,\ \ B_3=\{m_e^2[M(3r_+-r_-)-a^2]+k^2+2ikm_e(r_+-r_-)\}(r-r_+)^3\ ,$$

$$B_2=\{m_e^2[Mr_+(3r_+-2r_-)-a^2(2r_+-r_-)]+k^2(2r_+-r_--M)+ikm_e(r_+-r_-)^2\}(r-r_+)^2\ ,$$

$$B_1=[m_e^2r_+(Mr_+-a^2)+k^2(r_+-M)](r_+-r_-)(r-r_+)\ ;$$

$$\mathrm{Re}(V_6)=m_e^2(w^2-m_e^2)(r-r_+)^6,\ \ \mathrm{Re}(V_5)=m_e^2[6w^2r_+-m_e^2(5r_+-r_-)](r-r_+)^5\ ,$$

$$\mathrm{Re}(V_4)=\{w^2[k^2+m_e^2(17r_+^2+2a^2)]+w(2m_e^2ma+km_e)+4m_e^4a^2-10m_e^4r_+^2-2k^2m_e^2\}(r-r_+)^4\ ,$$

$$\mathrm{Re}(V_3)=\{w^2[4k^2r_++m_e^2(20r_+^3+8r_+a^2)]+w[(3r_+-r_-)km_e+8mar_+]-2k^2m_e^2(3r_+-r_-)-10m_e^4r_+^3\}(r-r_+)^3\ ,$$

$$\mathrm{Re}(V_2)=\{w^2[m_e^2(15r_+^4+12a^2r_+^2+a^4)+k^2(6r_+^2+2a^2)]+w[m_e^2ma(12r_+^2+2a^2)+2mak^2+km_ea^2]$$
$$-4m_e^4r_+^4+4m_e^4r_+^2a^2+k(mam_e-k)-(k^2+m_e^2r_+^2)^2\}(r-r_+)^2\ ,$$

$$\mathrm{Re}(V_1)=\{w^2[2m_e^2r_+(3r_+^2+a^2)(r_+^2+a^2)+4k^2r_+(r_+^2+a^2)]$$
$$+w[2m_e^2r_+ma(4r_+^2+2a^2)+4r_+mak^2+a^2km_e(r_+-r_-)]+2m_e^2m^2r_+a^2$$
$$+[kmm_ea-(k^2+m_e^2r_+^2)^2](r_+-r_-)\}(r-r_+)\ ,$$

$$\mathrm{Re}(V_0)=(k^2+m_e^2r_+^2)[w(r_+^2+a^2)+ma]^2\ ;$$

$$\mathrm{Im}(V_4)=Mwm_e^2(r-r_+)^4,\ \ \mathrm{Im}(V_3)=\{w[m_e^2(4r_+^2-4Mr_++5a^2)-k^2]+2m_e^2ma\}(r-r_+)^3\ ,$$

$$\mathrm{Im}(V_2)=\{w[m_e^2(5r_+a^2-5r_-a^2+6r_+^3-6Mr_+^2)+k^2(2r_--r_+-M)]+m_e^2ma(5r_+-r_--M)\}(r-r_+)^2\ ,$$

$$\mathrm{Im}(V_1)=\{w[2m_e^2(r_+^4+a^2r_+^2-a^4)+2a^2k^2]+amm_e^2(3r_+^2-2a^2)+k^2ma\}(r-r_+)\ ,$$

$$\mathrm{Im}(V_0)=(r_+-M)(m_e^2r_+^2+k^2)[w(r_+^2+a^2)+ma]\ .$$

The potential for the case $a^2=M^2$ is obtained by setting $r_+=r_-=M=a$ in the expansions above.

For the electromagnetic wave equation one has, for R_1 (R_{-1} satisfies the complex-conjugate equation),

$$A_4=(r-r_+)^4,\ \ A_3=2(r_+-r_-)(r-r_+)^3,\ \ A_2=(r_+-r_-)^2(r-r_+)^2\ ;$$

$$B_3=4(r-r_+)^3,\ \ B_2=4(2r_+-r_--M)(r-r_+)^2,\ \ B_1=4(r_+-r_-)(r_+-M)(r-r_+)\ ;$$

$$\mathrm{Re}(V_4)=w^2(r-r_+)^4,\ \ \mathrm{Re}(V_3)=4w^2r_+(r-r_+)^3,\ \ \mathrm{Re}(V_2)=[w^2(6r_+^2+2a^2)+2wma-k_+](r-r_+)^2\ ,$$

$$\mathrm{Re}(V_1)=\{4wr_+[w(r_+^2+a^2)+ma]-k_+(r_+-r_-)\}(r-r_+),\ \ \mathrm{Re}(V_0)=[w(r_+^2+a^2)+ma]^2\ ;$$

$$\mathrm{Im}(V_3)=2iw(r-r_+)^3,\ \ \mathrm{Im}(V_2)=-2iw(2r_--r_+-M)(r-r_+)^2\ ,$$

$$\mathrm{Im}(V_1)=-2i\{w[r_+^2+a^2+2r_+(r_+-M)]+ma\}(r-r_+),\ \ \mathrm{Im}(V_0)=-2i(r_+-M)[w(r_+^2+a^2)+ma]\ .$$

As before, the potential for the case $a^2=M^2$ is given by the substitution $r_+=r_-=a=M$.

APPENDIX B

The general form of the recursion relation is

$$\mathrm{Re}(V_6)a_n+\mathrm{Re}(V_5)a_{n+1}+a_{n+2}[(n+\lambda+2)(n+\lambda+1)A_6+\mathrm{Re}(V_4)+i\,\mathrm{Im}(V_4)]$$
$$+a_{n+3}\{(n+\lambda+3)(n+\lambda+2)A_5+(n+\lambda+3)\mathrm{Re}(B_4)+\mathrm{Re}(V_3)+i[\mathrm{Im}(V_3)+(n+\lambda+3)\mathrm{Im}(B_4)]\}$$
$$+a_{n+4}\{(n+\lambda+4)(n+\lambda+3)A_4+(n+\lambda+4)\mathrm{Re}(B_3)+\mathrm{Re}(V_2)+i[\mathrm{Im}(V_2)+(n+\lambda+4)\mathrm{Im}(B_3)]\}$$
$$+a_{n+5}\{(n+\lambda+5)(n+\lambda+4)A_3+(n+\lambda+5)\mathrm{Re}(B_2)+\mathrm{Re}(V_1)+i[\mathrm{Im}(V_1)+(n+\lambda+5)\mathrm{Im}(B_2)]\}$$
$$+a_{n+6}[(n+\lambda+6)(n+\lambda+5)A_2+(n+\lambda+6)\mathrm{Re}(B_1)+\mathrm{Re}(V_0)+i\,\mathrm{Im}(V_0)]=0\ ,$$

where λ is a solution to the indicial equation.

APPENDIX C

The calculation that takes (7.16) and reexpresses it as an integral in the momentum space that is dual to the local Lorentz frame defined in Sec. IV, (7.19), is long and involved. It requires the actual computation of the integrals in the Feynman propagator and extracting the singular parts. As stated earlier, the results look somewhat different than the standard calculations due to the nearly one-dimensional nature of the trajectories. Consequently, the integrals that are involved in the calculation cannot be evaluated by a literature search.

Consider the distribution in the dual momentum space with the standard flat metric, $\eta^{\mu\nu}=\text{diag}(-1,1,1,1)$:

$$I_1=\int \frac{(p_0-p_r)e^{ip\cdot x}}{p_0{}^2-p_r{}^2-p_\theta{}^2-p_\phi{}^2-m_e{}^2+i\epsilon}d^4p\ , \tag{C1}$$

where ϵ is a small positive number. X_0, X_r, X_θ, and X_ϕ are rectangular coordinates that are instantaneously at rest with respect to a ZAMF near the event horizon, as in Sec. IV.

The one-dimensional nature of the calculation is injected by separating out the "radial" momentum. Let

$$w_p\equiv(p_r{}^2+p_\theta{}^2+p_\phi{}^2+m_e{}^2)^{1/2}\equiv(p_r{}^2+\mathcal{M}^2)^{1/2}\ ,$$

where (C2)

$$\mathcal{M}^2\equiv p_\theta{}^2+p_\phi{}^2+m_e{}^2\ .$$

If $p_0>0$, then

$$p_0{}^2-w_p{}^2+i\epsilon=[p_0-(w_p-i\epsilon)][p_0+(w_p+i\epsilon)] \tag{C3a}$$

and, if $p_0<0$,

$$p_0{}^2-w_p{}^2+i\epsilon=[p_0-(w_p+i\epsilon)][p_0+(w_p-i\epsilon)]\ . \tag{C3b}$$

Decomposing I_1, with the aid of (C2) and (C3),

$$I_1=\int \frac{\Theta(p_0)(p_0-p_r)e^{ip\cdot x}}{[p_0-(w_p-i\epsilon)][p_0+(w_p+i\epsilon)]}d^4p + \int \frac{\Theta(-p_0)(p_0-p_r)e^{ip\cdot x}}{[p_0-(w_p+i\epsilon)][p_0+(w_p-i\epsilon)]}d^4p\ . \tag{C4}$$

The first term corresponds to the $\psi(m,w)\psi^\dagger(m,w)$ terms in the time-ordered product (7.1). The second term is associated with $\psi(-m,-w)\psi^\dagger(-m,-w)$ terms in (7.1). If one requires positive-energy waves $p_0>0$ then the second term corresponds to the propagation of antiparticles in the sense of the Stückelberg-Feynman description. Since, the integration in (C4) is over all p_0 and p_r, this restriction can be met without changing I_1 by letting $p_0\to -p_0$ and $p_r\to -p_r$, in the second term. Then, upon integrating over p_0

$$I_1=\Theta(X_0)\int \frac{(w_p-p_r)}{2w_p+i\epsilon}e^{iw_pX_0+i\mathbf{p}\cdot\mathbf{X}}d^3p + \Theta(-X_0)\int \frac{(-w_p+p_r)}{2w_p+i\epsilon}e^{-iw_pX_0-i\mathbf{p}\cdot\mathbf{X}}d^3p\ . \tag{C5}$$

To evaluate (C5), let $w_p=\mathcal{M}\cosh\varphi$ and $p_r=\mathcal{M}\sinh\varphi$. Then,

$$I_1=-i\Theta(X_0)(\partial/\partial X_0-\partial/\partial X_r)\int \exp\{i[\mathcal{M}(X_0\cosh\varphi+X_r\sinh\varphi)+p_\theta X_\theta+p_\phi X_\phi]\}d\varphi\, dp_\theta dp_\phi$$
$$-i\Theta(-X_0)(\partial/\partial X_0-\partial/\partial X_r)\int \exp\{-i[\mathcal{M}(X_0\cosh\varphi+X_r\sinh\varphi)+p_\theta X_\theta+p_\phi X_\phi]\}d\varphi\, dp_\theta dp_\phi\ . \tag{C6}$$

Then, if one proceeds as Bogoliubov and Shirkov,[26] let $\lambda=X_0{}^2-X_r{}^2$, then there are four cases:

(1a) $X_0>0$ $X_0>X_r$, (1b) $X_0>0$ $X_0<X_r$; (2a) $X_0<0$ $|X_0|>X_r$, (2b) $X_0<0$ $|X_0|<X_r$,

and four simplifying substitutions:

(1a) $X_0=(\lambda)^{1/2}\cosh\varphi_0$, $X_r=(\lambda)^{1/2}\sinh\varphi_0$, (1b) $X_0=(-\lambda)^{1/2}\sinh\varphi_0$, $X_r=(-\lambda)^{1/2}\cosh\varphi_0$;

(2a) $X_0=-(\lambda)^{1/2}\cosh\varphi_0$, $X_r=(\lambda)^{1/2}\sinh\varphi_0$, (2b) $X_0=-(-\lambda)^{1/2}\sinh\varphi_0$, $X_r=(-\lambda)^{1/2}\cosh\varphi_0$.

Inserting these substitutions into (C6)

$$I_1=-i\Theta(X_0)\Theta(\lambda)(\partial/\partial X_0-\partial/\partial X_r)\int d\varphi\, dp_\theta dp_\phi \exp\{i[\mathcal{M}(\lambda)^{1/2}\cosh(\varphi+\varphi_0)+p_\theta X_\theta+p_\phi X_\phi]\}$$
$$-i\Theta(X_0)\Theta(-\lambda)(\partial/\partial X_0-\partial/\partial X_r)\int d\varphi\, dp_\theta dp_\phi \exp\{i[\mathcal{M}(-\lambda)^{1/2}\sinh(\varphi+\varphi_0)+p_\theta X_\theta+p_\phi X_\phi]\}$$
$$-i\Theta(-X_0)\Theta(\lambda)(\partial/\partial X_0-\partial/\partial X_r)\int d\varphi\, dp_\theta dp_\phi \exp\{-i[\mathcal{M}(\lambda)^{1/2}\cosh(\varphi-\varphi_0)+p_\theta X_\theta+p_\phi X_\phi]\}$$
$$-i\Theta(-X_0)\Theta(-\lambda)(\partial/\partial X_0-\partial/\partial X_r)\int d\varphi\, dp_\theta dp_\phi \exp\{-i[\mathcal{M}(-\lambda)^{1/2}\sinh(\varphi-\varphi_0)+p_\theta X_\theta+p_\phi X_\phi]\}\ . \tag{C7}$$

These integrals are representations of Hankel functions.[27] Inserting these functions and executing the differentiations yields

$$I_1=-\tfrac{1}{2}i\Theta(X_0)\Theta(\lambda)(X_0+X_r)(\lambda)^{-1/2}\int \mathcal{M}H_1^{(1)}[\mathcal{M}(\lambda)^{1/2}]e^{i(p_\theta X_\theta+p_\phi X_\phi)}dp_\theta dp_\phi$$
$$+(1/\pi)\Theta(X_0)\Theta(-\lambda)(X_0+X_r)(-\lambda)^{-1/2}\int \mathcal{M}K_1[\mathcal{M}(-\lambda)^{1/2}]e^{i(p_\theta X_\theta+p_\phi X_\phi)}dp_\theta dp_\phi$$
$$-\tfrac{1}{2}i\Theta(-X_0)\Theta(\lambda)(X_0+X_r)(\lambda)^{-1/2}\int \mathcal{M}H_1^{(2)}[\mathcal{M}(\lambda)^{1/2}]e^{-i(p_\theta X_\theta+p_\phi X_\phi)}dp_\theta dp_\phi$$
$$+(1/\pi)\Theta(-X_0)\Theta(-\lambda)(X_0+X_r)(-\lambda)^{-1/2}\int \mathcal{M}K_1[\mathcal{M}(-\lambda)^{1/2}]e^{-i(p_\theta X_\theta+p_\phi X_\phi)}dp_\theta dp_\phi\ . \tag{C8}$$

If (C8) is evaluated near the light cone, then $|\lambda|^{1/2}\to 0$ and $|X_0| \gg |\lambda|^{1/2}$. The Lorentz frame in which the integral is being evaluated is boosted ultrarelativistically in the r direction with respect to the rest frame of the propagating particle waves. Thus, near the null cone $X_0 \approx \pm X_r$. However, I_1 vanishes if $X_0=-X_r$. Consequently, the relevant approximation is $X_0 \approx X_r$. Using this and the results of Bogoliubov and Shirkov, who have evaluated the integrands of (C8) near the light cone, (C8) becomes[28]

$$I_1 \approx -i\Theta(X_0)|X_r| \int e^{i(p_\theta X_\theta + p_\phi X_\phi)}[2\delta(\lambda)-2/\pi i\lambda - \tfrac{1}{2}\mathcal{M}^2\Theta(\lambda)+(i\mathcal{M}^2/\pi)\ln|\tfrac{1}{2}\mathcal{M}(\lambda)^{1/2}|]dp_\theta dp_\phi$$
$$+i\Theta(-X_0)|X_r| \int e^{-i(p_\theta X_\theta + p_\phi X_\phi)}[2\delta(\lambda)+2/\pi i\lambda - \tfrac{1}{2}\mathcal{M}^2\Theta(\lambda)+(i\mathcal{M}^2/\pi)\ln|\tfrac{1}{2}\mathcal{M}(\lambda)^{1/2}|]dp_\theta dp_\phi$$
$$+O((\lambda)^{1/2}\ln|\lambda|)\ . \tag{C9}$$

In light of the fact that $X_0 \approx X_r$, one has, effectively near the light cone,

$$\Theta(X_0)\Theta(\lambda)=\Theta(X_0-X_r)\ , \tag{C10a}$$

$$\Theta(-X_0)\Theta(\lambda)=\Theta(X_r-X_0)\ . \tag{C10b}$$

Recall the identity

$$\delta(X_0{}^2-X_r{}^2)=(\tfrac{1}{2}|X_r|)[\delta(X_0-X_r)+\delta(X_0+X_r)]\ ,$$

with the constraint $X_0 \approx X_r$

$$\delta(X_0{}^2-X_r{}^2)=(\tfrac{1}{2}|X_r|)\delta(X_0-X_r)\ . \tag{C11}$$

Substituting (C10) and (C11) into (C9),

$$I_1 \approx -i\Theta(X_0)\int (\{\delta(X_0-X_r)-1/[\pi i(X_0-X_r)]-\tfrac{1}{2}\Theta(X_0-X_r)|X_r|(m_e{}^2+p_\theta{}^2+p_\phi{}^2)\}e^{i(p_\theta X_\theta+p_\phi X_\phi)}+|X_r|\tilde{I})dp_\theta dp_\phi$$
$$+i\Theta(-X_0)\int (\{\delta(X_0-X_r)+1/[\pi i(X_0-X_r)]-\tfrac{1}{2}\Theta(X_r-X_0)|X_r|(m_e{}^2+p_\theta{}^2+p_\phi{}^2)\}e^{-i(p_\theta X_\theta+p_\phi X_\phi)}$$
$$+|X_r|\tilde{I}^*)dp_\theta dp_\phi + O((\lambda)^{1/2}\ln|\lambda|)\ , \tag{C12}$$

where $\tilde{I}$ is the integrand

$$\tilde{I}=(i/\pi)(m_e{}^2+p_\theta{}^2+p_\phi{}^2)\ln|\tfrac{1}{2}(m_e{}^2+p_\theta{}^2+p_\phi{}^2)^{1/2}(X_0{}^2-X_r{}^2)^{1/2}|e^{i(p_\theta X_\theta+p_\phi X_\phi)}\ .$$

Integrating $\tilde{I}$ in the complex p_θ plane, one finds that the integral of $(d/d\lambda)\tilde{I}$ is zero, since it has no residues. The only occasion that it does not vanish is when λ is zero. One concludes that $\tilde{I}$ is negligible to the integrand in (C12) when one is not measuring lightlike separations, since the pole terms are much larger. Neglecting the contribution from $\tilde{I}$ and using the correspondence (7.18)

$$I_1 \approx -i\Theta(X_0)[\delta(X_0-X_r)+i/\pi(X_0-X_r)-\tfrac{1}{2}\Theta(X_0-X_r)|X_r|(m_e{}^2-\partial^2/\partial X_\phi{}^2-\partial^2/\partial X_\theta{}^2)]\delta(X_\theta)\delta(X_\phi)$$
$$+i\Theta(-X_0)[\delta(X_0-X_r)-i/\pi(X_0-X_r)-\tfrac{1}{2}\Theta(X_r-X_0)|X_r|(m_e{}^2-\partial^2/\partial X_\phi{}^2-\partial^2/\partial X_\theta{}^2)]\delta(X_\theta)\delta(X_\phi)\ . \tag{C13}$$

It remains to show that the contribution due to m_e in the numerator of (7.19) is negligible. Consider the distribution

$$I_2=\int \frac{m_e e^{ip\cdot x}}{p^2-m_e{}^2+i\epsilon}d^4p\ . \tag{C14}$$

Proceeding as in (C2)—(C8)

$$I_2=m_e\Theta(X_0)\int \{\Theta(\lambda)H_0^{(1)}(\mathcal{M}(\lambda)^{1/2})+(2i/\pi)\Theta(-\lambda)K_0[\mathcal{M}(-\lambda)^{1/2}]\}e^{i(p_\theta X_\theta+p_\phi X_\phi)}dp_\theta dp_\phi$$
$$+m_e\Theta(-X_0)\int \{\Theta(\lambda)H_0^{(2)}(\mathcal{M}(\lambda)^{1/2})+(2i/\pi)\Theta(-\lambda)K_0[\mathcal{M}(-\lambda)^{1/2}]\}e^{-i(p_\theta X_\theta+p_\phi X_\phi)}dp_\theta dp_\phi\ . \tag{C15}$$

Then, using the limiting forms of the Bessel functions and taking into account the discontinuity on the null cone as in Bogoliubov and Shirkov, I_2 near the null cone is[29]

$$I_2 \approx m_e\lambda\Theta(X_0)\int [\pi i\delta(\lambda)+1/\lambda-\tfrac{1}{2}i\pi\mathcal{M}^2\Theta(X_0-X_r)-\tfrac{1}{2}\mathcal{M}^2\ln|\mathcal{M}(\lambda)^{1/2}|]e^{i(p_\theta X_\theta+p_\phi X_\phi)}dp_\theta dp_\phi$$
$$+m_e\lambda\Theta(-X_0)\int [\pi i\delta(\lambda)-1/\lambda-\tfrac{1}{2}i\pi\mathcal{M}^2\Theta(X_r-X_0)-\tfrac{1}{2}\mathcal{M}^2\ln|\mathcal{M}(\lambda)^{1/2}|]e^{-i(p_\theta X_\theta+p_\phi X_\phi)}dp_\theta dp_\phi\ . \tag{C16}$$

These are the same terms that appear in the expression (C12) for I_1, but are of the order $m_e\lambda/X_r$ as large. This ratio is approximately $2(X_0-X_r)$/(Compton wavelength of the electron). Near the null cone, this ratio is small as stated earlier due to the large boost that is necessary to achieve p_r of the waves.

Consequently, I_2 is much smaller than I_1. It can be concluded that (C13) is a good approximation to (7.19).

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