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Large-scale anisotropy of the cosmic background radiation in Friedmann universes

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We compute the large-scale anisotropy of the cosmic background radiation due to the gravitational field of clumped matter in open, closed, and flat Friedmann universes. Formulas are derived for the mean-square angular fluctuation of the cosmic background radiation in terms of the two-point correlation function of matter. The results depend importantly on whether or not the matter perturbations locally satisfy the integral constraints which express local conservation of energy and momentum, and we discuss these two possibilities. We examine the behavior as spatial curvature goes to zero.

I. INTRODUCTION

The cosmic background radiation (CBR) from the primeval fireball should be slightly anisotropic from a number of causes. The strong clumping of matter on the scale of galaxies and clusters of galaxies implies a degree of inhomogeneity in the early Universe, which may show up as angular anisotropy of the CBR on angular scales of seconds to minutes of arc. The spectrum of spatial inhomogeneity has an unknown dependence on spatial scale; theoretical considerations in the inflationary universe[1] suggest that the spectrum ought to be scale-free (Harrison-Zel'dovich spectrum), and this possibility is currently popular, but other possibilities also exist. Indeed, if matter perturbations are created locally by causal processes acting on scales much smaller than the present observable Universe, the spectrum of inhomogeneity must fall much more steeply than a Harrison-Zel'dovich spectrum on the largest presently observable scales.

If the spectrum of matter inhomogeneity falls more steeply than a Harrison-Zel'dovich spectrum, then small-scale matter perturbations still cause long-range fluctuations in the gravitational potential and, consequently, gravitational red-shifts. The large-scale angular anisotropy of the CBR may be dominated by these red-shifts as discussed by Sachs and Wolfe,[2] and Peebles.[3,4] As a particular case, if the matter fluctuations are created out of an initially homogeneous universe by physical processes which act only on small scales—for instance, as in the model of galaxy formation by explosive events due to Ostriker and Cowie[5]—then the local matter perturbations will conserve mass and momentum. As a consequence, gravitational fields will have vanishing monopole moment and vanishing dipole moment; in this case the matter perturbations must obey certain "constraints"[6,7] and the large-scale anisotropy of the CBR will be much weaker.

For inflationary-universe models, the constraints do not apply to matter perturbations created by quantum fluctuations during the inflationary area, because the horizon was much bigger during the inflationary era than now (as measured in comoving coordinates). However, even in inflationary-universe models, the spectrum of matter perturbations could be strongly enhanced at small distance scales by nonlinear effects at late times—such as in the Ostriker-Cowie model—over and above the underlying Harrison-Zel'dovich spectrum. If so, the matter distribution observed now on scales out to tens of Mpc will approximately obey the constraints.

In this paper we will discuss the prediction for the large-scale anisotropy of the CBR caused by the gravitational red-shift due to matter inhomogeneity, in closed, open, and spatially flat Friedmann universes ($k=+1,-1,0$). The distribution of clumped matter is conveniently measured by the two-point correlation function $\xi(r)$ of galaxies[8] as a function of spatial separation r; this has been well measured out to ~ 20 Mpc. Similarly the possible anisotropy of the CBR can be specified as the rms temperature difference $\langle(\delta T_1-\delta T_2)\rangle^{1/2}$ of the fluctuation δT in the temperature T of the CBR as a function of θ, the angle between celestial directions 1 and 2 on the sky. We give explicit formulas for $\langle(\delta T_1-\delta T_2)\rangle^{1/2}$ in terms of $\xi(r)$. The actual magnitude of the angular fluctuations in T is very dependent on whether or not $\xi(r)$ describes a distribution of matter which locally obeys constraints. An approximate expression of the results for open and spatially flat universes, for $\xi(r)$ not subject to constraints, were previously given by Peebles.[3,4]

A. Constraints on matter perturbations

First let us explain briefly the physics of the constraints. In special relativity there are ten conserved quantities corresponding to the symmetries of Minkowski spacetime: namely, four components of the energy-momentum vector and six components of the angular momentum tensor. Consequently, arbitrary stress-energy perturbations are forbidden. For example, if the energy density perturbation $\delta\rho$ vanishes initially, and then is perturbed by a local process which obeys causality, then the conservation laws tell us that at a later time

$$\int_{G_{\text{big}}} d^3x\, \delta\rho=0\ , \tag{1}$$

$$\int_{G_{\text{big}}} d^3x\, \mathbf{x}\delta\rho=0\ , \tag{2}$$

where the integration is over a large enough spatial volume G_{big}.

In general relativity, energy-momentum of matter is not conserved. In certain special spacetimes, however, conservation laws still hold; one of us has studied these conservation laws.[6,7] In the particular case of the $k=0$ Friedmann model, Eqs. (1) and (2) still hold, even though they no longer follow from any explicit symmetry of spacetime. [For example, Eq. (1) is a law of conservation of energy, even though spacetime is certainly not static.] Similar laws hold in the $k=\pm1$ Friedmann models, as we discuss in Sec. II.

What does this mean for cosmological observations? The laws (1), (2) (or their counterparts for $k=\pm1$) can be rephrased in terms of the two-point mass correlation function[8] $\xi(r)=\langle\delta\rho_1\delta\rho_2\rangle/\rho^2$, where r is the distance between points 1 and 2. For example, for a random distribution of such localized perturbations in a $k=0$ model, ξ must satisfy[9]

$$\int_{G_{\text{big}}} dr\, r^2\xi(r)=0\ , \tag{3}$$

$$\int_{G_{\text{big}}} dr\, r^4\xi(r)=0\ . \tag{4}$$

These restrictions on ξ imply that the consequent fluctuations in the CBR at large angular scales are smaller in amplitude.

B. Summary of the main results

Matter perturbations cause a large-scale anisotropy in the temperature T of the CBR, due to gravitational redshift. Sections III and IV discuss this effect and give a gauge-invariant derivation of it; the limit of small spatial curvature ($k=0$ limit) is treated.

Section V derives the relation between the mean-square anisotropy of the CBR at large angular scales and ξ, the two-point matter correlation. For the $k=0$ Friedmann universe, our main result for this relation is

$$\frac{\langle(T_1-T_2)^2\rangle}{T^2}=\frac{H_0{}^3}{c^3}\left[J_3\sin\frac{\theta}{2}-\frac{H_0}{4_c}J_4+\frac{H_0{}^2}{48c^2}J_5\left[\sin\frac{\theta}{2}\right]^{-1}\right]\ , \tag{5}$$

where

$$J_n=\int dr\, r^{n-1}\xi(r)\ . \tag{6}$$

Here θ is the angle between directions 1 and 2 in the sky and H_0 is the Hubble constant. Equation (5) is valid for $\theta\geq\theta_\xi$, where $\xi(r)$ is assumed to vanish beyond some distance

$$2L_\xi\equiv(4c/H_0)\sin(\theta_\xi/2)\ .$$

The leading term at large θ is the term proportional to J_3, which gives roughly $\Delta T(\theta)\propto\theta^{1/2}$, as pointed out by Peebles.[3] But when perturbations are created by local, causal processes in a noninflationary universe, the constraints (3) and (4) imply that $J_3=J_5=0$ and the leading term is that in J_4, which gives $\Delta T(\theta)\propto$ const. These constraints also must be satisfied in inflationary[1] universes for local, causal processes which operate after the end of inflation, for instance, phase transitions at nuclear density, or explosive events.[5] However, for processes which happen before or during inflation, the integrals in (1) and (2) or (3) and (4) run over a region of integration G_{big} much larger than the present horizon, and so they do not directly constrain the amplitude of fluctuations in the CBR.

We also derive results corresponding to (5) for $k=\pm1$ Friedmann models in Sec. V. Finally, in Sec. VI we obtain expressions for the multipole moments of the CBR, and discuss the limit of the nearly flat Universe.

II. CONSTRAINTS ON STRESS-ENERGY PERTURBATIONS, AND SMALL-CURVATURE LIMIT

We will study linear perturbations off an exact Friedmann-Robertson-Walker (FRW) spacetime with metric $g_{(0)\mu\nu}$ given by

$$ds^2=-dt^2+A^2(t)d\sigma^2=A^2(\eta)(-d\eta^2+d\sigma^2)\ , \tag{7}$$

where

$$d\sigma^2=\begin{cases}S(d\chi^2+\sin^2\chi\,d\Omega^2), & S>0 \text{ or } k=+1\ ,\\ d\chi^2+\chi^2d\Omega^2, & |S|=\infty \text{ or } k=0\ ,\\ |S|(d\chi^2+\sinh^2\chi\,d\Omega^2), & S<0 \text{ or } k=-1\ ,\end{cases}$$

$$dt=A\,d\eta\ ;$$

here $d\Omega^2=d\theta^2+\sin^2\theta\,d\phi^2$ and units are used such that $c=1=G$. The scale factor $A(\eta)$ is normalized to unity at the present time:

$$A_0\equiv A(\eta_0)=1\ ,$$

where η_0 denotes the present time in the coordinate system (7), corresponding to proper time t_0. The "squared radius of curvature" S plays the role of the more familiar k; $k=0$ corresponds to $|S|=\infty$, and otherwise $k=\mathrm{sgn}S$; S has dimensions (length)^2 in physical units as measured at t_0. The relation between Ω_0 and S is

$$S=\frac{c^2}{H_0{}^2}\frac{1}{1-\Omega_0}\ , \tag{8}$$

where $\Omega_0 \equiv \rho_0/\rho_{\rm crit} \equiv$(present density)/(density necessary to close the Universe). We expect physical quantities to have smooth limits as $|S|\to\infty$, since the geometry changes smoothly, as can be seen from the alternate form of the spatial metric:

$$ds^2=-dt^2+A^2(t)\left[\frac{dr^2}{1-r^2/S}+r^2d\Omega^2\right].$$

The choice (7) of coordinates is a convenient one in which to study small deviations from the spatially flat model, by power series in $1/S$.

The perfect-fluid stress-energy tensor $T_{(0)\mu}{}^{\nu}$ is given by

$$T_{(0)\mu}{}^{\nu}=(\rho+p)U^{\nu}_{(0)}U_{(0)\mu}+pg_{(0)\mu}{}^{\nu}\ .$$

Here ρ is the background density and $\vec{\mathrm{U}}_{(0)}=\partial/\partial t$ is the unperturbed fluid four-velocity.

We review briefly integral constraints on stress-energy perturbations in FRW spacetimes (for discussion about general spacetimes, see Ref. 7). Consider perturbations off the background

$$g_{\mu\nu}=g_{(0)\mu\nu}+h_{\mu\nu},\quad T_{\mu}{}^{\nu}=T_{(0)\mu}{}^{\nu}+\delta T_{\mu}{}^{\nu}\ ,$$

where $(h_{\mu\nu},\delta T_{\mu}{}^{\nu})$ are solutions to the linearized Einstein equations. Let G be a spatial volume in the standard coordinates, Eq. (7), with boundary ∂G and timelike normal $\vec{\mathrm{n}}$. These exist constraint four-vector fields $\vec{\mathrm{V}}$ such that $\delta T^{\mu}{}_{\nu}$ must obey a generalized Gauss's law:

$$\int_G dv\, V^{\mu}\delta T^{\alpha}{}_{\mu}n_{\alpha}=\int_{\partial G} da_l B^l \tag{9}$$

for all $h_{\mu\nu}$ and $\dot{h}_{\mu\nu}$. The boundary term B^l is linear in $h_{\mu\nu}$, and vanishes if $h_{\mu\nu}$ is zero on ∂G. So, for example, suppose $\delta T^{\mu}{}_{\nu}$ is local (has compact support) and vanishes at some initial time. Then causality implies that the boundary term is zero if G is taken large enough.

The FRW spacetimes each have ten integral constraint vectors, in the standard coordinates (7). Six of these are just the spatial Killing vectors. The other four, which will be useful in studying density perturbations, are not Killing vectors. For $|S|=\infty$ FRW spacetimes ($k=0$—flat spatial sections) these are

$$\begin{aligned}\vec{\mathrm{V}}_{(0)}&=\frac{\partial}{\partial t}-\frac{\dot{A}}{A}x^i\frac{\partial}{\partial x^i}\ ,\\ \vec{\mathrm{V}}_{(k)}&=x^k\frac{\partial}{\partial t}+\frac{\dot{A}}{A}(\tfrac{1}{2}\delta^{ki}r^2-x^kx^i)\frac{\partial}{\partial x^i},\quad k=1,2,3\ .\end{aligned} \tag{10}$$

For example, let $\delta\rho$ be a local causal source in a flat, pressureless, FRW universe. Then the integral constraints reduce to the special-relativistic statements (1) and (2) that the monopole and dipole moments of $\delta\rho$ vanish.

In the closed FRW universe the boundary term is always zero if G is the entire $t=$const surface (a three-sphere S^3).

The constraint vectors for finite $S(k=\pm1)$ are

$$\vec{\mathrm{V}}=F\frac{\partial}{\partial t}+S\frac{1}{A}\frac{dA}{dt}D^jF\frac{\partial}{\partial x^j}\ , \tag{11}$$

where D^j is the covariant derivative in the spatial metric $d\sigma^2$. For $S>0$, F is any one of four second-order spherical harmonics on S^3 (see Sec. VI and Appendix A):

$$F=Q^{2lm}=\begin{cases}\cos\chi,\sin\chi Y_{1m}(\Omega), & S>0\ ,\\ 1,\chi Y_{1m}(\Omega), & |S|=\infty\ ,\\ \cosh\chi,\sinh\chi Y_{1m}(\Omega), & S<0\ ,\end{cases} \tag{12}$$

where $m=-1,0,+1$ in (12).

The F satisfy the differential equation

$$\left[\Delta+\frac{3}{S}\right]F=0\ , \tag{13}$$

where $\Delta\equiv D_iD^i$.

When $p=0$ we can always choose a gauge which is synchronous and comoving with the irrotational part of the flow $(\delta U^i{}_{|i}=0)$ (Ref. 10). Substituting the constraining vectors (11) into the integral condition (9) gives, for a closed universe ($S>0$),

$$\int dv\,\delta\rho Q^{2lm}=0\ . \tag{14}$$

Equation (14) is also true for the open universe ($S\le0$) if $\delta\rho$ is a local causal source.

Below we will look at first-order curvature effects for slightly open or closed universes. Let $\delta\rho$ be a local source which vanishes beyond some length scale L:

$$\delta\rho(x)\equiv0\ \ \text{for}\ \ \chi\sqrt{|S|}>L\ . \tag{15}$$

Let $l=\chi\sqrt{|S|}$; l is the geodetic distance to the origin. Then neglecting terms of $O(l^4/S^2)$, Eq. (14) becomes

$$\begin{aligned}&\int d^3l\,\delta\rho\left[1-\frac{5}{6}\frac{l^2}{S^2}\right]=0\ \ (\text{all } S)\ ,\\ &\int d^3l\,\delta\rho l Y_{1m}(\Omega)=0\ .\end{aligned} \tag{16}$$

In the limit as $|S|\to\infty$ these do reduce to the $k=0$ constraints for local sources. It makes sense that one must consider local sources to take a large-$|S|$ limit since we have not included the boundary terms.

In Secs. V and VI we will apply the constraints to the computation of the anisotropy in the microwave background.

III. GAUGE-INVARIANT EXPRESSION FOR $\delta T/T$

Perturbations in the metric and matter will cause fluctuations in the observed temperature. This can be computed as follows. A photon is emitted at $E=(\eta_E,\mathbf{x}_E)$ and received at $O=(\eta_O,\mathbf{x}_O)$. We will assume that the observed microwave-background photons were emitted from the last scattering surface when hydrogen recombined. We will make the approximation that recombination happened at a single temperature $T_{\rm rec}\simeq4000$ K, which is a red-shift $z_{\rm rec}\simeq1.5\times10^3$.

One contribution to δT is from fluctuations in the photon field on the last scattering surface, and another contribution is the Doppler shift due to the peculiar velocities of the source and observer.[11-14] A third contribution to δT is the Sachs-Wolfe effect: the photon four-momentum is perturbed as it propagates on the perturbed null geodesic.[2,3,11,15] The derivative of the general gauge-invariant expression for δT is very simple (see also Ref. 16). We follow the approach of Sachs and Wolfe.

Let $\vec{k}=\vec{k}_{(0)}+\delta\vec{k}$ be the four-momentum of the photon and $\vec{u}=\vec{u}_{(0)}+\delta\vec{u}$ be the four-velocity of a local observer. In the FRW background

$$\vec{k}_{(0)}=\frac{1}{A^2}\left[\frac{\partial}{\partial_\eta}-\hat{n}\right],$$

where $\hat{n}$ is a spatial unit vector,

$$1=\hat{n}\cdot\hat{n}\equiv n^i n^j \sigma_{ij}(\mathbf{x})$$

and

$$\vec{u}_{(0)}=\frac{1}{A}\frac{\partial}{\partial\eta}\ . \tag{17}$$

Here $\sigma_{ij}=g_{ij}/A^2$ [cf. Eq. (7)]. The normalization of four-velocity implies that $\delta u^0=0$.

Then the red-shift of photons emitted at recombination is

$$1+z=\frac{T_{\rm rec}}{T_O}=\frac{(\vec{k}\cdot\vec{u})_E}{(\vec{k}\cdot\vec{u})_O}$$

$$=\frac{A(\eta_O)}{A(\eta_{\rm rec})}\left[1+\frac{\delta T_{\rm SW}}{T}+\frac{\delta T_{\rm Dop}}{T}\right], \tag{18}$$

where

$$\frac{\delta T_{\rm SW}}{T}=A^2\delta k^0(O)-A^2\delta k^0(E)\ ,$$
$$\frac{\delta T_{\rm Dop}}{T}=n^j\frac{\delta u_j}{A}(E)-n^j\frac{\delta u_j}{A}(O)\ ; \tag{19}$$

here T_O is the temperature at the observer.

In the physical, perturbed universe, the surface $T=T_{\rm rec}$ is not a surface of constant η. Rather $\eta_{\rm rec}$ is a function of $\mathbf{x}$ defined by

$$T_{\rm rec}=T(\mathbf{x},\eta_{\rm rec}(\mathbf{x}))\ ,$$

and this gives a further correction term in (18). Let

$$T(\mathbf{x},\eta)=T_{(0)}(\eta)+\delta T(\mathbf{x},\eta)$$

and

$$\eta_{\rm rec}(\mathbf{x})=\eta_{(0)\rm rec}+\Delta\eta(\mathbf{x})\ ,$$

where $T_{(0)}(\eta)$ is the temperature in the unperturbed universe,

$$\frac{T_{(0)}(\eta_1)}{T_{(0)}(\eta_2)}=\frac{A(\eta_2)}{A(\eta_1)}\ ,$$

and $\eta_{(0)\rm rec}$ is the time at which recombination would occur in an unperturbed universe, $T_{(0)}(\eta_{(0)\rm rec})=T_{\rm rec}$. Combining these gives

$$A(\eta_{\rm rec}(\mathbf{x}))=A(\eta_{(0)\rm rec})\left[1+\frac{\delta T(\mathbf{x},\eta_{\rm rec})}{T}\right]. \tag{20}$$

Let ρ_γ be the photon density. Then

$$\delta T/T(\mathbf{x},\eta_{\rm rec})=\tfrac{1}{4}\delta\rho_\gamma/\rho(\mathbf{x},\eta_{\rm rec})\ ,$$

which is equal to $\delta\rho_{\rm baryon}/3\rho$ for adiabatic perturbations. Substituting in (18) gives the final expression for $1+z$. Since the red-shift is an observable this is gauge invariant. We write the result in terms of the temperature perturbation at the observer δT_O, where $T_{(0)O}+\delta T_O=T_{\rm rec}/(1+z)$:

$$\frac{\delta T_O}{T_O}=\frac{1}{4}\frac{\delta\rho_\gamma}{\rho}(\mathbf{x},\eta_{\rm rec})-\frac{\delta T_{\rm SW}}{T}-\frac{\delta T_{\rm Dop}}{T}\ . \tag{21}$$

Of course it remains to compute these quantities in terms of given initial data. Sachs and Wolfe[2] computed $\delta T_{\rm SW}$ for a flat FRW universe. One of us has previously worked out the extension for arbitrary background geometries (Ref. 6; see also Refs. 15 and 16). The result, valid for any gauge, is

$$\frac{\delta T_{\rm SW}}{T}=\tfrac{1}{2}\int_0^{\Delta\eta}A^4(\bar{h}_{\mu\nu,\eta}k_{(0)}{}^\mu k_{(0)}{}^\nu -2\bar{h}_{0\beta,\alpha}k_{(0)}{}^\beta k_{(0)}{}^\alpha)_{(0)}dw\ , \tag{22}$$

where $\bar{h}_{\mu\nu}\equiv h_{\mu\nu}/A^2$. The integrand is to be evaluated on the zeroth-order path. In the coordinate system (7) this path, parametrized by w, is

$$\eta=\eta_O-w,\ \ \chi=\frac{w}{|S|^{1/2}},\ \ \theta=\theta_O\ ,$$
$$\phi=\phi_O;\ \ 0\le w\le\eta_O-\eta_{\rm rec}\ . \tag{23}$$

Next we write down solutions for $h_{\mu\nu}$ and $\delta\rho$. We assume that $p=0$ (as in Refs. 2, 6, and 15). The solution of the Einstein equations for $A(\eta)$ is then

$$A(\eta)=\begin{cases}(1-\cos\eta)/(1-\cos\eta_O), & S>0\ ,\\ (\eta/\eta_O)^2, & |S|=\infty\ ,\\ (\cosh\eta-1)/(\cosh\eta_O-1), & S<0\ ,\end{cases}$$

and one can always choose a gauge which is synchronous and comoving with the scalar part of the velocity field.[6,10] In this gauge the integral constraints take the simple form (14).

Further, we shall only include scalar modes in h_{ij} (as in Refs. 6, 11, 12, and 16). This is a good approximation at late times;[6] physically, the vector and tensor modes are red-shifted away. Since $\delta T_{\rm Dop}$ is due only to rotational modes in our gauge we will neglect this term. (The Doppler term due to the peculiar velocities of the source and observer will show up as part of $\delta T_{\rm SW}$.) On large angular scales, $\delta T_{\rm SW}$ dominates $\delta\rho_\gamma(\mathbf{x},\eta_E)$. Therefore, we have $\delta T_O/T_O\simeq\delta T_{\rm SW}/T$.

Lifshitz and Khalatnikov[10] computed the evolution of perturbations in FRW spacetimes in terms of an eigenfunction decompositions. With the above choice of gauge then $p=0$ solutions can be written in terms of a gravitational potential $f(\eta)\Phi(\mathbf{x})$:

$$\frac{\delta\rho}{\rho}=-\tfrac{1}{2}f(\eta)\left[\Delta+\frac{3}{S}\right]\Phi(\mathbf{x})\,,$$

$$\frac{d}{d\eta}\bar{h}^{\,j}_{\,i}=\dot{f}\left[D_iD^j+\frac{1}{S}\delta^j_i\right]\Phi\,,$$

where

$$\ddot{f}+\frac{\dot{A}}{A}\dot{f}-\frac{3}{2}\left[\left[\frac{\dot{A}}{A}\right]^2+\frac{1}{S}\right]f=0\,. \tag{24}$$

Here $\dot{f}\equiv df/d\eta$. The growing mode solutions for f are[10,17]

$$f(\eta)=\begin{cases} b\left[\dfrac{3\sin\eta(\eta-\sin\eta)}{(1-\cos\eta)^2}-2\right], & S>0\,,\\ b\eta^2, & |S|=\infty\,,\\ b\left[\dfrac{3\sinh\eta(\sinh\eta-\eta)}{(\cosh\eta-1)^2}-2\right] & S<0\,,\end{cases} \tag{25}$$

where in each case the constant b is chosen so that f is normalized by $f(\eta_O)=1$. The large-$|S|$ limit is

$$f(\eta)=2q_0\left[\frac{\eta}{\eta_{00}}\right]^2\left\{1+\frac{{\eta_{00}}^2}{S}\frac{1}{2q_0}\left[1-\frac{7}{6}\left[\frac{\eta}{\eta_{00}}\right]^2\right]+\cdots\right\}, \tag{26}$$

where $\eta_{00}\equiv 2/H_0=$the present time in a flat universe; and q_0 is the deceleration parameter, given by $q_0=\Omega_0/2$ or $(2q_0-1){H_0}^2=1/S$. Therefore, the expression for $\delta T/T$ is

$$\frac{\delta T_O}{T_O}=\frac{1}{2}(\dot{f}\mathbf{n}\cdot\nabla\Phi+\ddot{f}\Phi)^O_E-\frac{1}{2}\int_0^{\Delta\eta}dw\left[\dddot{f}+\frac{1}{S}\dot{f}\right]\Phi\,. \tag{27}$$

It is straightforward to transform $\delta T/T$ and $\delta\rho/\rho$ into the gauge-invariant variables of Bardeen.[18] Let $v_0^{nlm}(\eta)$ denote the nlmth harmonics of the gauge-invariant velocity perturbation. (The harmonics Q^n are discussed in Sec. VI and Appendix A.) Then

$$-\tfrac{1}{2}\dot{f}\Phi(\mathbf{x})=\sum\frac{v_s^n(\eta)}{n^2-1}Q^n(\mathbf{x})\equiv V_s\,,$$

and we see the δT_0 depends only on the velocity V_s and $\dot{V}_s$.

For ease in discussion we will divide δT_0 into parts. Let

$$\begin{aligned}\frac{\delta T_{\text{Dop},O}}{T}&=\tfrac{1}{2}\dot{f}\mathbf{n}\cdot\nabla\Phi(O)\,,\\ \frac{\delta T_{\text{Dop},E}}{T}&=-\tfrac{1}{2}\dot{f}\mathbf{n}\cdot\nabla\Phi(E)\,,\\ \frac{\delta T_{\text{pot}}}{T}&=-\tfrac{1}{2}\ddot{f}\Phi(E)\,,\\ \frac{\delta T_{\text{path}}}{T}&=-\frac{1}{2}\int dw\left[\dddot{f}+\frac{1}{S}\dot{f}\right]\Phi\\ &=\frac{1}{S}\frac{25}{3}\int dw\frac{(\eta_0-w)}{{\eta_{00}}^2}\Phi+O(S^{-2})\,.\end{aligned} \tag{28}$$

The potential term at the observer is the same for all directions in the sky and has been dropped. The first two terms have been called "Doppler" because they depend on $\dot{f}\overline{\nabla}\Phi$. In some other gauge they would have appeared in the $n^i\delta u_i$ term.

Perhaps the most interesting point is that the integrand in δT_{path} for the growing mode goes to zero as $|S|\to\infty$, recovering Sachs and Wolfe's expression. Therefore in a flat universe, δT_0 depends only on $\ddot{f}\Phi$ and $\dot{f}\overline{\nabla}\Phi$ at the source and observer.

On the other hand, in a spatially curved universe, the metric fluctuations along the entire path of the photon contribute to δT_0. Comparing the magnitude of δT_{path} to the potential term,

$$\frac{|\delta T_{\text{path}}|}{|\delta T_{\text{pot}}|}\sim\frac{{\eta_{00}}^2}{|S|}=4\,|2q_0-1|\,.$$

The only observed anisotropy in the microwave is a dipole moment, which is generally attributed to Earth's peculiar velocity (Ref. 19), $|v|\lesssim 310\pm 40$ km/sec. This could be thought of as $\delta T_{\text{Dop},O}$, but one must remember that Earth's peculiar velocity at the present time is a nonlinear effect.

IV. SOLUTIONS FOR Φ

The temperature δT_{pot} is proportional to the gravitational potential Φ of the growing mode. In this section we find the solutions for Φ, and derive simple approximate expressions for the Φ due to a local source $\delta\rho$. Split $\delta\rho$ into its space-dependent and time-dependent factors:

$$\frac{\delta\rho}{\rho}(\mathbf{x},\eta)=f(\eta)P(\mathbf{x})\,. \tag{29}$$

Then from (24)

$$\left[\Delta+\frac{3}{S}\right]\Phi(\mathbf{x})=-2P(\mathbf{x})\,. \tag{30}$$

We will find the Green's function $G(\mathbf{x},\mathbf{x}')$ such that the solution to (30) is

$$\Phi(\mathbf{x})=-2\int dv'G(\mathbf{x},\mathbf{x}')P(\mathbf{x}')\,. \tag{31}$$

In flat space there is a simple multipole expansion for $\Phi(\mathbf{x})$ if the source is local, which is valid at points $\mathbf{x}$ outside the source. In Ref. 4 this expansion was calculated for a $k=0$ FRW universe. For $k=0$ the integral constraints imply that the monopole and dipole moments of $\delta\rho$ vanish. This means that the magnitude of Φ and hence δT_{pot} is reduced by a factor of order $\epsilon^2 \ll 1$, where ϵ is the multipole expansion parameter. If the Universe has been FRW since the big bang, then $\epsilon^2 \sim (1+z_p)^{-1}$, where z_p is the red-shift when the Universe became matter dominated.

In this section we will see that the same is true for a local source in a "slightly" open or closed FRW universe, and compute the first-order effects of the curvature. Φ is calculated by finding the Green's function G for (30), expanding G in a far-field and small-curvature limit and imposing the integral constraints.

A. Exact expressions for the Green's function

We give G for the three-space S^3 ($S>0$), for projective three-space P^3 ($S>0$), for flat three-space E^3 ($|S|=\infty$), and for the pseudosphere or hyperbolic space H^3 ($S<0$). P^3 is the three-sphere with antipodal points identified—this changes the boundary conditions. Care must be used for S^3, since in this case the operator has four zero modes Q^{2lm}; cf. (12) and (13). Details are contained in Appendix C. Expressed in the coordinate system (χ,θ,ϕ) of Eq. (7), the Green's functions are

$$S^3\colon\ G_S(\mathbf{x},\mathbf{x}')=-\frac{1}{\sqrt{S}\,4\pi}\left[\frac{\cos 2\psi}{\sin\psi}\left[1-\frac{\psi}{\pi}\right]-\frac{1}{2\pi}\cos\psi\right], \tag{32}$$

$$P^3\colon\ G_P(\mathbf{x},\mathbf{x}')=-\frac{1}{\sqrt{S}\,4\pi}\frac{\cos 2\psi}{\sin\psi}, \tag{33}$$

$$E^3\colon\ G_E(\mathbf{x},\mathbf{x}')=-\frac{1}{4\pi l}, \tag{34}$$

$$H^3\colon\ G_H(\mathbf{x},\mathbf{x}')=-\frac{1}{\sqrt{-S}\,4\pi}\left[\frac{\cosh 2\alpha}{\sinh\alpha}-2\cosh\alpha\right], \tag{35}$$

where

$$\cos\psi=\cos\chi\cos\chi'+\sin\chi\sin\chi'\cos\gamma \quad (S>0), \tag{36}$$

$$l=l^2+l'^2-2ll'\cos\gamma \quad (|S|=\infty), \tag{37}$$

$$\cosh\alpha=\cosh\chi\cosh\chi' -\sinh\chi\sinh\chi'\cos\gamma \quad (S<0), \tag{38}$$

and where γ is in turn given by

$$\cos\gamma=\cos\theta\cos\theta'+\sin\theta\sin\theta'\cos(\phi-\phi').$$

In the foregoing equations, $\mathbf{x}$ has coordinates (χ,θ,ϕ), and $\mathbf{x}'$ has (χ',θ',ϕ), in the coordinates of (7).

The Green's functions (34) and (35) for flat and open universes solve (30) for any source P. However, the Green's functions (32) and (33) for closed universes solve (30) only for sources P which satisfy the constraints (14) (with $\delta\rho\equiv P$). Finding a Green's function not subject to this constraint is impossible because of the four zero modes Q^{2lm}. Appendix C discusses this point in more detail. It is also possible to write G as an infinite sum of eigenfunctions (Appendix B).

B. Far-field, large-radius-of-curvature expansion

Given any source $P(\mathbf{x})$, Eqs. (32)–(35) give the exact solution for Φ. However, we can learn some useful information about Φ by constructing a far-field, large-radius-of-curvature expansion.

Assume that $P(\mathbf{x})$ is a local source at $\mathbf{y}$, with length scale L. Let $l_{\mathbf{xy}}$ be the geodesic distance between $\mathbf{x}$ and $\mathbf{y}$,

$$l_{\mathbf{xy}}=\begin{cases}\sqrt{S}\,\psi, & S>0,\\ l, & |S|=\infty,\\ \sqrt{-S}\,\alpha, & S<0,\end{cases} \tag{39}$$

where ψ is defined in (36), l in (37), and α in (38). Expand $dv'G(\mathbf{x},\mathbf{x}')$ in powers of $l'/\sqrt{|S|}$ and $l/\sqrt{|S|}$, i.e., radius of curvature large compared to length scales of interest. Next expand in powers of $\epsilon\equiv l'/l$ (usual far field approximation). Then the integral constraints, expanded for local sources (16), are used. The leading terms for Φ are

$$\Phi(\mathbf{x},\mathbf{y})=\frac{1}{l^3}Y_{2m}(\Omega)Q_{2m}\left[1+\frac{l^2}{2S}\right] +O\left[\epsilon^2,\frac{l^3}{|S|^{3/2}},\epsilon^2\frac{l^3}{|S|^{3/2}}\right]\frac{1}{l}, \tag{40}$$

where

$$Q_{2m}=\tfrac{2}{5}\int d^3z\,P(z)z^2Y_{2m}$$

and

$$\epsilon\leq\frac{L}{l}.$$

The first nonzero curvature correction is of order

$$l^2/|S|\lesssim|(q_0-\tfrac{1}{2})|.$$

The functional dependence on angle Ω of this term is exactly the same as the flat-space term, a result of using the integral constraints. The curvature term of order $S^{-1/2}$ vanishes by the constraints.

When $|S|\to\infty$ in Eq. (40) we recover the $k=0$ expression[7] for Φ, valid for local sources. The important feature of this expression is that Φ starts at the quadrupole term. For S large but finite the effect of the constraints is the same—the magnitude of the gravitational potential due to local sources is suppressed:

$$\frac{|\delta T_{\text{with IC}}|}{|\delta T_{\text{without}}|}\sim\epsilon^2(1+2|2q_0-1|^{1/2}), \tag{41}$$

and $\epsilon\ll 1$ whenever the multipole expansion is valid. Here $\delta T_{\text{with IC}}$ is the angular fluctuation of the CBR caused by sources obeying the constraints, while $\delta T_{\text{without}}$ is that caused by sources which do not obey the constraints. If the Universe has been FRW since the big bang,

$$\epsilon^2=(L/l)^2 \lesssim (1+z_p)^{-1} \lesssim 10^{-4} .$$

The first difference between S^3 and P^3 enters at the next order, and is equal to

$$Q/(S^{3/2}3\pi^2), \quad Q\equiv 2\int d^3z\, z^2 P .$$

However, this term is independent of position and so only contributes an unobservable monopole moment to δT. That is, the difference between a P^3 and an S^3 universe is unobservable to order $l^4/S^2 \lesssim (2q_0-1)^2$.

We can compare the magnitudes of different terms in δT_0 (28). If $\delta\rho$ is a local source, then (26) and (40) imply

$$\frac{|\delta T_{\text{Dop},E}|}{|\delta T_{\text{pot}}|} \sim \epsilon \sim \frac{1}{\sqrt{1+z_E}} \ll 1 . \quad (42)$$

On the other hand, if $\delta\rho$ is an extended source with wave number $n/\sqrt{|S|}$ (see Sec. VI),

$$\frac{|\delta T_{\text{Dop},E}|}{|\delta T_{\text{pot}}|} \sim \eta_E \frac{n}{\sqrt{|S|}} . \quad (43)$$

This is greater than one for scales which have entered their horizon.

As for the other terms in (28), δT_{path} and the first-order curvature correction to δT_{pot} are both $O(l^2/|S|)$.

V. AUTOCORRELATION OF δT ON LARGE ANGULAR SCALES

We now want to calculate the mean-square fluctuation

$$\tau(\theta) \equiv \frac{\langle (T_1-T_2)^2\rangle}{T^2} \quad (44)$$

of the CBR, where θ is the angle between points 1 and 2 on the sky. It is better to work with τ rather than the closely related two-point correlation function

$$\tau_{\text{corr}}(\theta)=\frac{\langle \delta T_1 \delta T_2\rangle}{T^2}$$

of the CBR, because τ_{corr} blows up in general as $|S| \to \infty$, while τ remains finite. The formal relation between the two is

$$\tau(\theta)=2\tau_{\text{corr}}(\theta)-2\tau_{\text{corr}}(0) .$$

The problem as $|S|\to\infty$ is that $\tau_{\text{corr}}(0)$, the variance of δT averaged over all possible observers in a universe, blows up in the $k=0$ Friedmann universe if perturbations are not local, because the long-range gravitational potential itself blows up in first-order perturbation theory.

Through Eq. (27), τ is related to the long-range gravitational potential Φ. Neglecting the Doppler terms [cf. Ref. 2 and Eq. (42)] we have

$$\frac{\delta T}{T}=\int_O^E dw\, W(w)\Phi(\mathbf{x}) ,$$

where

$$W(w)=-\tfrac{1}{2}\ddot f\delta(\eta-\eta_E)-\tfrac{1}{2}\dddot f-\frac{1}{2S}\dot f \quad (45)$$

and where the integral runs along the null geodesic $(\eta(w),\mathbf{x}(w))$ from the observation event $(\eta_O,\mathbf{x}_O)$ (with $\eta_O\equiv\eta_0$) to the emission event $(\eta_E,\mathbf{x}_E)$. Here $f(\eta)$ is the growing mode solution (25) in an open universe, normalized as $f(\eta_0)=1$.

In turn Φ is related to P, the density perturbation of matter at η_0, through Eq. (31):

$$\frac{\delta T}{T}=-2\int_O^E dw\, W(t)\int dv_y G(\mathbf{x},\mathbf{y})P(\mathbf{y}) .$$

We then have for two points 1 and 2 on the sky separated by an angle θ:

$$\begin{aligned}\tau(\theta)&=\left\langle \left[\frac{\delta T_1}{T}-\frac{\delta T_2}{T}\right]^2\right\rangle\\
&=\int_O^E dw\int_O^E dw' W(w)W(w')[\Phi(\mathbf{x}_1)-\Phi(\mathbf{x}_2)][\Phi(\mathbf{x}_1')-\Phi(\mathbf{x}_2')]\\
&=4\int_O^E dw\int_O^E dw' W(w)W(w')\int dv_y\int dv_{y'}[G(\mathbf{x}_1,\mathbf{y})-G(\mathbf{x}_2,\mathbf{y})][G(\mathbf{x}_1',\mathbf{y}')-G(\mathbf{x}_2',\mathbf{y}')]\langle[P(\mathbf{y})P(\mathbf{y}')]\rangle\\
&=4\int_O^E dw\int_O^E dw' W(w)W(w')\int dv_y\int dv_{y'}[G(\mathbf{x}_1,\mathbf{y})-G(\mathbf{x}_2,\mathbf{y})][G(\mathbf{x}_1',\mathbf{y}')-G(\mathbf{x}_2',\mathbf{y}')]\xi(l_{yy'}) ,\end{aligned} \quad (46)$$

where we have used Eq. (29) evaluated at η_0 to obtain ξ in the last line, and $l_{yy'}$ is the spatial geodetic distance between points $\mathbf{y}$ and $\mathbf{y}'$ in the time-independent metric $d\sigma^2$ of (7), as defined in Eq. (39) above. The points $\mathbf{x}_1(w)$ and $\mathbf{x}_1(w')$ run along a null geodesic in direction 1, while the $\mathbf{x}_2(w)$ and $\mathbf{x}_2(w')$ run along a null geodesic in direction 2, starting at O and running toward the past. In Eq. (46), $\xi(l)$ is the normalized two-point correlation function of matter at η_0,

$$\xi(l_{yy'})=\langle \delta\rho_y \delta\rho_{y'}\rangle/\rho^2$$

expressed in terms of spatial distance l today; ξ is assumed translation invariant and isotropic.

A. Flat universe ($|S|=\infty$)

First, we will proceed with the $k=0$ case of (46). Then the integrals in dw reduce to the end-point contribution

$$-\ddot f_E\Phi/2=-H_0{}^2\Phi/4 \ \text{ at } \eta_E ,$$

and five of the six integrals in $\mathbf{y}$ and $\mathbf{z}$ can readily be done.

We obtain a formula which gives for the flat Friedmann universe the relation between the two-point correlation function $\xi(l)$ of matter in spatial distance l at the present time, and the mean square fluctuation $\tau(\theta)$ of the CBR at angle θ on the sky:

$$\tau(\theta)=\tfrac{1}{4}H_0{}^4\int_0^\infty l'^2 dl'\xi(l')X(l,l') , \quad (47)$$

where the kernel $X(l,l')$ is

$$X(l,l')=\begin{cases} l-l'+l'^2/3l, & l'\le l ,\\ l^2/3l', & l'\ge l ,\end{cases} \quad (48)$$

and where

$$l=\frac{4}{H_0}\sin\frac{\theta}{2} . \quad (49)$$

For example, assume that $\xi(l)$ vanishes for $l>2L_\xi$. Then on large angular scales, $\theta\ge\theta_\xi$, where $\sin(\theta_\xi/2)\equiv H_0L_\xi/2$, Eq. (47) for $\tau(\theta)$ takes the form (5).

If the perturbation $\delta\rho$ obeys the integral constraint equations (1) and (2) [or (14)] without boundary term,[6,7] as it must if the cause of perturbations is a local, causal process in a noninflationary universe, then[10] ξ must satisfy Eqs. (3) and (4) and consequently $J_3\equiv 0\equiv J_5$. In this case the large-angle behavior of τ is $\tau\propto$ const rather than roughly $\tau^{1/2}\propto\theta^{1/2}$ (cf. Ref. 3):

$$\frac{\langle (T_1-T_2)^2\rangle}{T^2}=-\frac{H_0{}^4}{c^4}J_4 \quad \theta\ge\theta_\xi . \quad (50)$$

The reason that $\tau(\theta)$ is constant is that $\tau_{\rm corr}(\theta)$ is zero at large angle:

$$\frac{\langle \delta T_1\delta T_2\rangle}{T^2}\equiv 0, \quad \theta\ge\theta_\xi . \quad (51)$$

The value of τ is simply the sum of the variances in each beam here. Furthermore, the magnitude of δT is suppressed for sources which obey the constraints (3) and (4):

$$\frac{\tau^{1/2}_{\text{with constraints}}(\theta)}{\tau^{1/2}_{\text{without}}(\theta)}\sim\frac{1}{(1+z_E)^{1/4}} . \quad (52)$$

B. Open universe ($S<0$)

Turning to the case $k=-1$, (46) becomes

$$\tau(\theta)=2\int_O^E dw\int_O^E dw' W(w)W(w')[C(\mathbf{x}_1\mathbf{x}_1')-C(\mathbf{x}_1\mathbf{x}_2')] , \quad (53)$$

where C is a convolution of Green's functions with the two-point correlation of matter:

$$C(\mathbf{x},\mathbf{x}')\equiv 4\int dv_{\mathbf{y}}\int dv_{\mathbf{y}'}G(\mathbf{x},\mathbf{y})G(\mathbf{x}',\mathbf{y}')\xi(l_{\mathbf{y}\mathbf{y}'}) . \quad (54)$$

Equation (54) is hard to integrate directly, but we can obtain an explicit expression for C by differentiating (54) successively to get rid of the integrals and Green's functions on the right, so as to get a fourth-order differential equation in C, which can then be solved.

Let $L_{\mathbf{x}}\equiv(\Delta_{\mathbf{x}}+3/S)$ be the linear operator which appears in (30). Operating on (54) with $L_{\mathbf{x}}$ and using (30) and (31) gets rid of one Green's function and one spatial integral:

$$L_{\mathbf{x}}C(\mathbf{x},\mathbf{x}')=4\int dv_{\mathbf{y}'}G(\mathbf{x}',\mathbf{y}')\xi(l_{\mathbf{x}\mathbf{y}'}) . \quad (55)$$

Now C is invariant under motions of hyperbolic space; i.e., if $\mathcal{R}$ is any motion (isometry) of hyperbolic space, then

$$C(\mathbf{x},\mathbf{x}')=C(\mathcal{R}\mathbf{x},\mathcal{R}\mathbf{x}') .$$

In particular, $C(\mathbf{x},\mathbf{x}')$ is symmetric in its arguments; this fact can be easily seen from the definition (54) of C, but it also follows by taking $\mathcal{R}$ to be a particular motion which interchanges $\mathbf{x}$ and $\mathbf{x}'$. It is less obvious that $L_{\mathbf{x}}C(\mathbf{x},\mathbf{x}')$ is also symmetric. Again, let $\mathcal{R}$ be a motion of hyperbolic space which interchanges $\mathbf{x}$ and $\mathbf{x}'$. Then

$$\begin{aligned} L_{\mathbf{x}}C(\mathbf{x},\mathbf{x}')&=L_{\mathcal{R}\mathbf{x}}C(\mathcal{R}\mathbf{x},\mathcal{R}\mathbf{x}')\\ &=L_{\mathbf{x}'}C(\mathbf{x}',\mathbf{x}) ;\end{aligned}$$

we have used the fact that the operator L is also invariant under motions. Thus $L_{\mathbf{x}}C(\mathbf{x},\mathbf{x}')$ is symmetric. In fact, $C(\mathbf{x},\mathbf{x}')$ and $L_{\mathbf{x}}C(\mathbf{x},\mathbf{x}')$ are functions only of $l_{\mathbf{x}\mathbf{x}'}$. Using the symmetry of (55),

$$L_{\mathbf{x}}C(\mathbf{x},\mathbf{x}')=4\int dv_{\mathbf{y}'}G(\mathbf{x},\mathbf{y}')\xi(l_{\mathbf{x}'\mathbf{y}'})$$

and operating on this again with $L_{\mathbf{x}}$, the second Green's function and the second integral are also eliminated:

$$L_{\mathbf{x}}L_{\mathbf{x}}C(\mathbf{x},\mathbf{x}')=4\xi(l_{\mathbf{x}\mathbf{x}'}) . \quad (56)$$

In one-dimensional form this equation becomes

$$\frac{1}{\sinh\alpha}\left[\frac{\partial^2}{\partial\alpha^2}-4\right]^2(\sinh\alpha C)=4S^2\xi(\alpha) . \quad (57)$$

Here α is related to the geodetic distance $l_{\mathbf{x}\mathbf{x}'}$ between $\mathbf{x}$ and $\mathbf{x}'$ by $l_{\mathbf{x}\mathbf{x}'}=|S|^{1/2}\alpha$ [cf. (38) and (39)]. Equation (57) can readily be solved for C by constructing the Green's function $H_H(\alpha,\alpha')$ for the fourth-order operator on the left. Then the solution is

$$C(\alpha)=4S^2\int_0^\infty \sinh^2\alpha' d\alpha' H_H(\alpha,\alpha')\xi(\alpha') , \quad (58)$$

where the Green's function H_H on hyperbolic space is defined by

$$H_H(\alpha,\alpha')=\begin{cases} \dfrac{e^{-2\alpha}}{16\sinh\alpha\sinh\alpha'}[(2\alpha+1)\sinh 2\alpha'-2\alpha'\cosh 2\alpha'], & \alpha'\le\alpha ,\\[2ex] \dfrac{e^{-2\alpha'}}{16\sinh\alpha\sinh\alpha'}[(2\alpha'+1)\sinh 2\alpha-2\alpha\cosh 2\alpha], & \alpha'\ge\alpha .\end{cases} \quad (59)$$

[$H_H(\alpha,\alpha')$ is symmetric in its arguments because the fourth-order operator in (57) is self-adjoint.]

Then Eq. (53) gives the formula for the relation between τ and ξ in open Friedmann universes:

$$\tau(\theta)=8S^2\int_O^E dw\int_O^E dw' W(w)W(w')\int_0^\infty \sinh^2\alpha' d\alpha'[H_H(\alpha_{11},\alpha')-H_H(\alpha_{12},\alpha')]\xi(\alpha')\ , \tag{60}$$

where $|S|^{1/2}\alpha_{11}\equiv l_{\mathbf{x}_1\mathbf{x}_1'}$, and $|S|^{1/2}\alpha_{12}\equiv l_{\mathbf{x}_1\mathbf{x}_2'}$. The points $\mathbf{x}_1(w)$ and $\mathbf{x}_1(w')$ run along a null geodesic in direction 1, while the $\mathbf{x}_2(w)$ and $\mathbf{x}_2(w')$ run along a null geodesic in direction 2, starting at O and running toward the past; cf. (23). W is defined in Eq. (45). Equation (60) is somewhat more complicated than Eq. (47) for flat universes in that it involves a twofold integration, but it is still conveniently usable.

When $\delta\rho$ obeys the integral constraints (14) without boundary term in an open universe, then ξ must satisfy

$$0=\int_0^\infty \sinh^2\alpha\, d\alpha \cosh\alpha\,\xi(\alpha) \tag{61}$$

which becomes (3) in the flat limit $S\rightarrow-\infty$. Apparently (4), which also holds in the flat limit, has no generalization to $k\neq0$ models. (Of course one could choose a gauge, consistent with a synchronous gauge, such that $\int dv\,\delta\rho=0$; in this gauge ξ would satisfy $\int dv\,\xi=0$. However, we emphasize that the integral constraints and the consequent restrictions on ξ that we have discussed here are gauge invariant.)

In the open universe, $\tau(\theta)$ is more complicated that in the spatially flat universe, partly because of the integrations over the null geodetic path. For nearly flat open universes, however, we can ignore the integrations, which are $O(r^2/|S|)$, and keep only the end-point contribution at $\eta=\eta_E$ in (45).

We further assume that $\xi(\alpha)\equiv0$ for $\alpha\geq\alpha_\xi$, for some radius α_ξ. Then on sufficiently large angular scales, $\theta\geq\theta_\xi$, (60) reduces to

$$\tau(\theta)=\frac{S^2\ddot{f}^2}{4}\int_0^{\alpha_\xi}\sinh^2\alpha' d\alpha'\xi(\alpha')\left[\frac{2\alpha' e^{-2\alpha'}}{\sinh\alpha'}-\frac{(2\alpha+1)e^{-2\alpha}\cosh\alpha'}{\sinh\alpha}+\frac{e^{-2\alpha}\alpha'\cosh2\alpha'}{\sinh\alpha\sinh\alpha'}\right]. \tag{62}$$

Here α is defined using (38) from θ by $\sinh(\alpha/2)=\sinh\eta_0\sin(\theta/2)$. If the sources are causal and hence satisfy the constraints (61), the first term in the integral vanishes. To take the large $|S|$ limit, let $\alpha=l/|S|^{1/2}$ and $\alpha'=l'/|S|^{1/2}$. We assume $\xi(\alpha')$ has a limit $\xi(l')$. Then

$$\tau(\theta)=\frac{H_0{}^2}{4}\int l'^2 dl'\xi(l')\left[l-l'+\frac{l'^2}{3l}\right]+O\left[\frac{l}{\sqrt{|S|}}\right], \tag{63}$$

which reduces to (5) as $|S|\rightarrow\infty$.

Unless ξ satisfies the constraints, $\langle\delta T_1{}^2\rangle$ and $\langle\delta T_1\delta T_2\rangle$ individually diverge as $|S|\rightarrow\infty$. Their difference (63) has a finite limit for any source. As mentioned above, this is the reason for using $\tau(\theta)$ rather than $\tau_{\rm corr}(\theta)$.

C. Closed universe ($S>0$)

The case $k=+1$ may be handled substantially the same way as the case $k=-1$. We present the calculation for spatial topology S^3; the result for P^3 then follows. The four zero modes of the operator L cause some complications, because the Green's function G of (32) now obeys

$$L_{\mathbf{x}}G(\mathbf{x},\mathbf{x}')=\delta(\mathbf{x},\mathbf{x}')-\delta^{||}(\mathbf{x},\mathbf{x}')$$

as shown in Appendix C, Eq. (C8).

Equation (46) still implies Eqs. (53) and (54). Operating once on (54) with $L_{\mathbf{x}}$ yields

$$L_{\mathbf{x}}C(\mathbf{x},\mathbf{x}')=4\int dv_{\mathbf{y}'}G(\mathbf{x}',\mathbf{y}')\xi(l_{\mathbf{x}\mathbf{y}'})-4\int dv_{\mathbf{y}}\int dv_{\mathbf{y}'}\delta^{||}(\mathbf{x},\mathbf{y})G(\mathbf{x}',\mathbf{y})\xi(l_{\mathbf{y}\mathbf{y}'})$$

and a second operation with $L_{\mathbf{x}}$ annihilates the second integral and gives, using as before the symmetry of the first term,

$$L_{\mathbf{x}}L_{\mathbf{x}}C(\mathbf{x},\mathbf{x}')=4\xi(l_{\mathbf{x}\mathbf{x}'})-4\int dv_{\mathbf{y}'}\delta^{||}(\mathbf{x},\mathbf{y}')\xi(l_{\mathbf{x}'\mathbf{y}'})\ . \tag{64}$$

Operating still a third time with $L_{\mathbf{x}}$ finally gets rid of all the integrals:

$$L_{\mathbf{x}}L_{\mathbf{x}}L_{\mathbf{x}}C(\mathbf{x},\mathbf{x}')=4L_{\mathbf{x}}\xi(l_{\mathbf{x}\mathbf{x}'})\ . \tag{65}$$

In one-dimensional form this equation becomes

$$\frac{1}{\sin\psi}\left[\frac{\partial^2}{\partial\psi^2}+4\right]^3(\sin\psi C)=4S^2\frac{1}{\sin\psi}\left[\frac{\partial^2}{\partial\psi^2}+4\right][\sin\psi\,\xi(\psi)]\ , \tag{66}$$

where $S^{1/2}\psi \equiv l_{\mathbf{x}\mathbf{x}'}$ [cf. (36) and (39)]. The Green's function of the operator on the left in (64) can be constructed; starting from the homogeneous solutions to (66) simplifies the work. We then obtain the expression for C as

$$C(\psi)=4S^2\int_0^{\pi}\sin^2\psi' d\psi' H_S(\psi,\psi')\xi(\psi') , \quad (67)$$

where

$$H_S(\psi,\psi')=\begin{cases}\dfrac{1}{16\pi\sin\psi\sin\psi'}\Bigg[2(\pi-\psi)\psi'\cos2\psi\cos2\psi'-(\pi-\psi)\cos2\psi\sin2\psi'+\psi'\sin2\psi\cos2\psi' \\ \qquad +\left[\psi'^2+(\pi-\psi)^2-\dfrac{3}{8}-\dfrac{\pi^2}{3}\right]\sin2\psi\sin2\psi'\Bigg], \quad \psi'\le\psi , \\ \dfrac{1}{16\pi\sin\psi\sin\psi'}\Bigg[2(\pi-\psi')\psi\cos2\psi\cos2\psi'-(\pi-\psi')\cos2\psi'\sin2\psi+\psi\sin2\psi'\cos2\psi \\ \qquad +\left[\psi^2+(\pi-\psi')^2-\dfrac{3}{8}-\dfrac{\pi^2}{3}\right]\sin2\psi\sin2\psi'\Bigg], \quad \psi'\ge\psi .\end{cases}$$

Here H_S has been constructed to be orthogonal to the zero modes on both its arguments.

We therefore obtain a formula for the relation between τ and ξ in closed Friedmann universes:

$$\tau(\theta)=8S^2\int_O^E dw\int_O^E dw' W(w)W(w')\int_0^{\pi}\sin^2\psi' d\psi'[H_S(\psi_{11},\psi')-H_S(\psi_{12},\psi')]\xi(\psi') , \quad (68)$$

where $S^{1/2}\psi_{11}\equiv l_{\mathbf{x}_1\mathbf{x}_1'}$, and $S^{1/2}\psi_{12}\equiv l_{\mathbf{x}_1\mathbf{x}_2'}$. Other notation is as in (60).

The perturbation $\delta\rho$ must always satisfy the integral constraints (14) in a closed S^3 Friedmann model. It follows[16,18] that ξ must satisfy

$$0=\int_0^{\pi}\sin^2\psi d\psi\cos\psi\xi(\psi) . \quad (69)$$

Again, the generalization of (4) seems to be lacking.

Formula (68) applies also to the closed Friedmann model with topology P^3. For present purposes, the P^3 model can be regarded as a special case of the S^3 model in which $\delta\rho$ is even under the antipodal map; that is, $\delta\rho$ has the same value at any point $\mathbf{x}$ as at the antipodal point to $\mathbf{x}$. Then ξ is also even, $\xi(\pi-\psi)=\xi(\psi)$, and Eq. (68) applies directly. Only the even parts of H_S contribute to the integral. There are no integral constraints in P^3; (69) vanishes identically because $\cos\psi$ is odd while $\xi(\psi)$ is even under the antipodal map $\psi\leftrightarrow\pi-\psi$.

The result (68) for closed universes goes over smoothly to (5) as $S\to\infty$.

D. Numerical results

We have evaluated the main results (47), (60), and (68) numerically for some particular examples. Following Peebles,[3,4] we assumed that ξ vanishes on large-distance scales. As discussed in Sec. I, this assumption is invalid in pure inflationary models of the Universe, in which the matter perturbation spectrum is a Harrison-Zel'dovich spectrum. Results for such a spectrum will be substantially different.[3] This assumption may be approximately valid even in inflationary models if nonlinear processes have strongly enhanced the spectrum at small scales.

Then ξ can be represented by a three-dimensional δ function at the origin $l=0$:

$$\xi(l)=J_3 l^{-2}\delta(l) , \quad (70)$$

where J_3 is used as a normalization constant; cf. (6). Then $J_4=J_5=0$, and we assume that $J_3\neq0$. For $\Omega_0=1(\,|S|=\infty\,)$ we then have[3]

$$\tau^{1/2}(\theta)=(J_3H_0{}^3c^{-3})^{1/2}\left[\sin\frac{\theta}{2}\right]^{1/2} .$$

The observational value of J_3 is not very well determined, but if it does not vanish, $(J_3H_0{}^3/c^3)^{1/2}\sim2\times10^{-4}$ is a reasonable guess.[4]

For open universes, we evaluated (60) for a number of values of Ω_0 in the range $0.01\le\Omega_0\le1$, and for $0.01\le\sin(\theta/2)\le1$. The results for $\tau^{1/2}(\theta)$ are displayed in Figs. 1 and 2. There are two main regimes apparent in these results. First, as Ω_0 decreases over the range $1\ge\Omega_0\gtrsim0.5$, $\tau^{1/2}(\theta)$ decreases very rapidly at the largest θ; see Fig. 1. This decrease is due to the change of the long-range gravitational point potential Φ from $\Phi\propto1/l$ at $\Omega_0=1$ to $\Phi\propto\exp(-l/|S|^{1/2})/l$ for $\Omega_0<1$. This change means that the gravitational potential at the horizon size $L_H=2c/H_0$ is $\Phi\propto\exp[-2(1-\Omega_0)^{1/2}]$, a very rapidly decreasing function of Ω_0 near $\Omega_0=1$.

The second regime begins at $\Omega_0\approx0.5$ and continues as $\Omega_0\to0$; see Fig. 2. Here, $\tau^{1/2}(\theta)$ becomes flat at large θ, due to the fact that the long-range gravitational potential has died away entirely, and the two beams are completely uncorrelated. At these large θ, $\tau_{\text{corr}}(\theta)\approx0$ and $\tau(\theta)$ is merely the sum of the variances in each beam. At small θ, $\tau^{1/2}(\theta)$ increases gradually as Ω_0 decreases; this is due to the fact that ξ is measured at t_0, while the red-shift depends on ξ for t all the way back to t_E. Because perturbations grow more slowly as the Universe becomes more open, and eventually freeze in amplitude, a given amplitude of perturbation today implies a larger and larger amplitude in the past, the more open the Universe is. Therefore ξ was larger at early times in an open universe. Similar results were calculated by Peebles[4] using a different method; our results agree with his.

Wilson[20] computed the temperature fluctuations of the CBR in an open universe with $\Omega_0=0.1$, for various assumptions about the spectrum of matter fluctuations, and found similar results. In particular he pointed out that the two beams are uncorrelated at large angles.

For closed universes, we can choose either S^3 or P^3 topology. For topology S^3, we are forbidden to use (70) as a model for ξ, because it does not satisfy the constraint (69). A point mass perturbation is forbidden to exist in a spherical universe, just as a single point charge is. Instead we adopt $\xi \propto \delta^1$, as in Appendix C, Eq. (C8):

$$\xi(l)=J_3\left[l^{-2}\delta(l)-\frac{8}{\pi}\cos\left[\frac{l}{\sqrt{S}}\right]\right], \tag{71}$$

which satisfies (69). The question of whether this is the appropriate choice depends on the physics of the early Universe, and is beyond the scope of this paper. In P^3 there are no constraints and we used (70); the δ function at the origin also implies a δ function at the antipode in P^3.

We evaluated $\tau^{1/2}(\theta)$ for values of Ω_0 in the range $1\leq\Omega_0\leq 3$; the results are displayed in Fig. 3 for S^3, and Fig. 4 for P^3. The magnitude of $\tau^{1/2}(\theta)$ is larger as Ω_0 increases from 1; this is because the magnitude of the long-range gravitational potential becomes greater. There is a strong dip apparent near $\theta=180°$ [$\sin(\theta/2)=1$] as Ω_0 becomes large; this is because we are almost "seeing around the Universe," and the two beams are viewing regions of the recombination surface which are physically close to each other. In the S^3 Universe two beams separated by $\theta=180°$ on the sky view the same point is $\Omega_0=\infty$, i.e., if the Universe has just ceased its expansion and is about to recollapse; therefore as $\Omega_0\to\infty$, the end-point contribution in the integrals along the null rays of (68) vanishes.

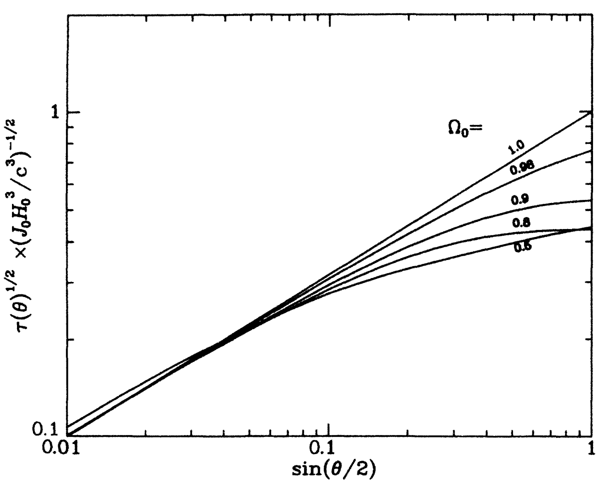

FIG. 1. Angular fluctuations of the CBR for open Friedmann universes. Plotted is rms fractional temperature fluctuation $\tau^{1/2}(\theta)=\langle(T_1-T_2)^2\rangle^{1/2}/T$ as a function of angle θ between beams 1 and 2. Separate curves correspond to various values of the cosmological density parameter Ω_0, $0.5\leq\Omega_0\leq 1.0$, as labeled. The two-point correlation function ξ of matter is assumed to vanish on large scales, and to possess a nonzero moment J_3, as in Eq. (70). See Sec. V D.

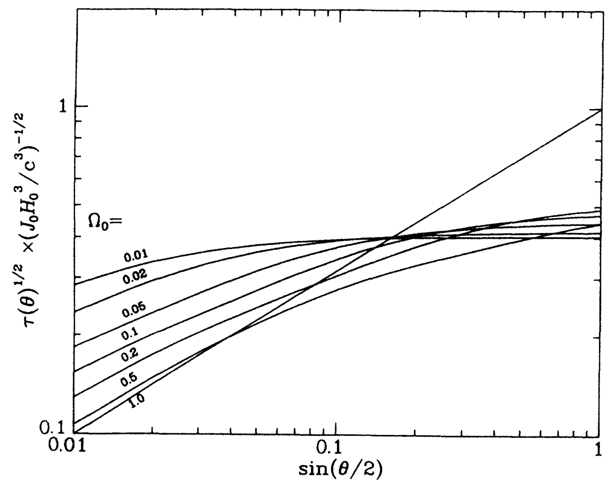

FIG. 2. Angular fluctuations of the CBR for open universes. Same as Fig. 1, except that the range of Ω_0 is $0.01\leq\Omega_0\leq 1.0$, as labeled.

In the P^3 universe these two beams view the same point if $\Omega_0=2$. Indeed the notch at $\theta=180°$ in Fig. 4 is strong at $\Omega_0=2$; it would be infinitely strong if it were not for the contribution from all along the null ray. The magnitude of $\tau^{1/2}(\theta)$ is greater in P^3 than in S^3 because the gravitational potential is stronger, because it is concentrated in half the spatial volume.

VI. MULTIPOLE MOMENTS OF δT BY AN EIGENFUNCTION EXPANSION AND LARGE-S LIMIT

When studying perturbations in a $k=0$ FRW universe it is often useful to work in Fourier space. In this section we discuss the corresponding eigenfunctions formalism for the $k=\pm 1$ cases, for example, the relation between the power spectrum and the transform of ξ. We will compute the expectation value of the multipole moments of δT, and take the large-S limit.

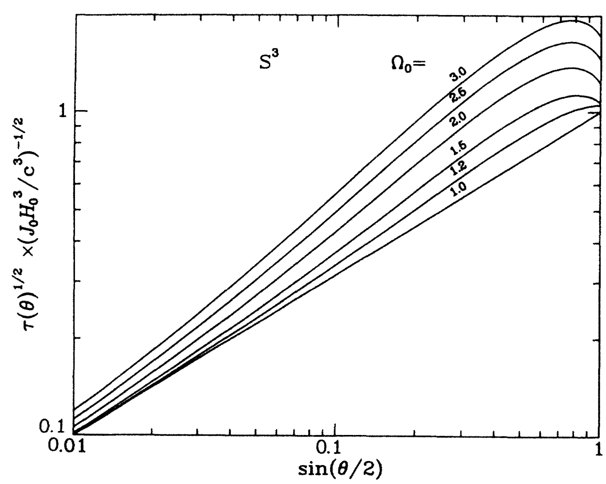

FIG. 3. Angular fluctuations of the CBR for closed Friedmann universes with spherical topology S^3. Notation is same as for Fig. 1; the range of Ω_0 is $1.0\leq\Omega_0\leq 3.0$, as labeled. The two-point correlation function ξ is as in Eq. (71).

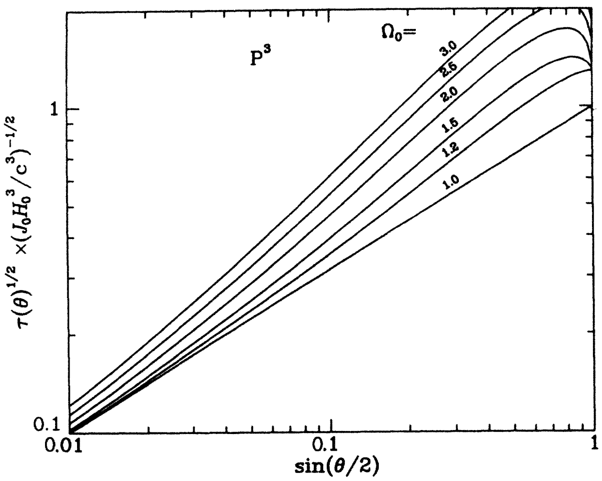

FIG. 4. Angular fluctuations of the CBR for closed Friedmann universes with projective-space topology P^3. Notation is same as for Fig. 1; the range of Ω_0 is $1.0 \le \Omega_0 \le 3.0$, as labeled. The two-point correlation function ξ is as in Eq. (70).

A. Eigenfunction expansion and the power spectrum

The eigenfunctions[10,21] are solutions to the Helmholtz equation on S^3 or H^3:

$$D_i D^i Q = -\frac{(n^2-k)}{|S|} Q \ . \qquad (72)$$

The solutions are the spherical (pseudospherical) harmonics on $S^3(H^3)$:

$$Q^{nlm}(\mathbf{x}) = Y_{lm}(\Omega)\Pi_{nl}(\chi) \ ,$$

$$n=1,2,3,\ldots, \quad l=0,\ldots,n-1 \ \text{ for } S>0 \ ,$$

$$n \text{ a continuous variable for } S<0 \ .$$

The Y_{lm} are the usual spherical harmonics on S^2. Some relevant properties of the Q^n are summarized in Appendix B.

Let

$$P(\mathbf{x}) = \sum c_{nlm} Q^{nlm} \ . \qquad (73)$$

(When n is a continuous variable, $\sum_n$ means $\int dn$.) When $S>0$, the functions F which appear in the integral constraints are just the $n=2$ spherical harmonics. Therefore the constraints are equivalent to

$$c_{2lm}=0 \quad (S>0) \ . \qquad (74)$$

Note that there is no analogous statement for the negatively curved case ($S<0$). The constraint functions for $S<0$ [Eq. (12)] are unbounded eigenfunctions of Δ, whereas the Q^{nlm} are bounded.

In terms of the eigenfunction expansions, the integral constraints are equivalent to the statement that (30) can be solved for Φ if and only if the source satisfies (74). This is a familiar statement about the existence of solutions to inhomogeneous equations $L\Phi=s$ when the linear operator L has zero eigenmodes.

The solution to (24) is

$$\Phi(\mathbf{x}) = \sum \frac{2c_{nlm}Q^{nlm}}{(n^2-4k)/|S|} \quad (n \ne 2 \text{ for } S>0) \ . \qquad (75)$$

The absence of the $n=2$ mode is noted in Refs. 10 and 18.

In flat space the power spectrum $|P_k|^2$ is the Fourier transform of the correlation function ξ:

$$\xi(\mathbf{x}_1,\mathbf{x}_2) = \langle P(\mathbf{x}_1)P(\mathbf{x}_2)\rangle \ . \qquad (76)$$

There is a similar statement which is true on S^3. Assume that ξ depends only on the geodesic distance between the two points. This means that ξ depends only on the angle ψ between the points, (36), and can be expanded in the $l=0$ harmonics. Let

$$\xi(\psi) = \sum e_n \Pi_{n0}(\psi) \ . \qquad (77)$$

Using the angle addition theorem for S^3, we find (see Appendix C)

$$\langle c_{nlm} c^*_{NLM} \rangle = e_n \delta_{Nn}\delta_{lL}\delta_{mM} \ . \qquad (78)$$

For example, in a Harrison-Zel'dovich spectrum the amplitude of $\langle |\delta\rho^n/\rho|^2\rangle^{1/2}$ at horizon crossing for the scale n is equal to ϵ_H independent of n. Comparing to Ref. 16 one finds

$$e_n = [4\pi\epsilon_H / f(n_H)]^2 \qquad (79)$$

$f(n_H)$ denotes f evaluated at the horizon crossing time for the scale n.

B. Multipole moments of δT

Observations of the microwave anisotropy are reported in terms of multipole moments of δT. Substitute the expansion (75) for Φ into Eq. (27) for δT_0 and let

$$\frac{\delta T_0}{T_0} = \sum a_{lm} Y_{lm}(\Omega) \ . \qquad (80)$$

Now, $\delta T_{\text{Dop},O}$ contributes only to the dipole moment a_{10}, so let $a_{10} = a_{1\text{Dop}} + a_{1\text{grav}}$,

$$a_{1\text{Dop}} = \tfrac{1}{2}\dot{f}(\eta_0')\,|\nabla\Phi(0)|\,\left[\frac{4\pi}{3}\right]^{1/2} \ .$$

Of course this dipole term includes only linear effects. For example, the nonlinear dynamics of Earth moving in the Galaxy are not included. Taking the expectation values of δT_0 gives

$$\langle |a_{lm}|^2\rangle = \sum_{n\ge l+1} \frac{\langle |c_{nlm}|^2\rangle}{(n^2-4k)^2/S^2}[B_{nl}(\Delta\eta)]^2 \ ,$$

$$l=2,3,4,\ldots, \qquad (81)$$

where

$$B_{nl}(\Delta\eta) = -(\dot{f}\Pi_{nl}' + \ddot{f}\Pi_{nl})\,|_E - \int_0^{\Delta\eta} dw \left[\left[\dddot{f} + \frac{1}{S}\dot{f}\right]\Pi_{nl}\right]_{(0)}$$

and

$$\langle |c_{nlm}|^2\rangle = e_n \ \text{ for } S>0 \ .$$

Recall that at E, $\eta=\eta_E$, $\chi=\Delta\eta/\sqrt{S}$. Expression (81) also holds for $a_{1\text{grav}}$. In this case the sum starts at $n=3$, since $e_2=0$.

In the special case of a Harrison-Zel'dovich spectrum, (81) agrees with Ref. 16. Wilson[20] defines an equation similar to (81) for the multipole expansion in an open universe, and computes the lowest moments for the case $n=\sqrt{S}$.

For Gaussian fluctuations, $\langle\,|a_{lm}|^2\rangle$ is just the standard deviation of the random variables a_{lm}. For example, from observations[22–24] $\langle\,|a_{2m}|^2\rangle \lesssim 5\times10^{-8}$.

It is interesting to note that for a closed universe Eq. (81) and the observed limit on the quadrupole moment may give an upper bound on $\langle\,|a_{1\text{grav}}|^2\rangle$. The dependence of the multipole moments on the matter distribution is contained in the e_n, and these are identical for $l=1$ and $l=2$, since $e_2\equiv0$. Therefore if one could find a bound M such that $B_{nl}\le MB_{n2}$ for all n, then we would have

$$\langle\,|a_{1\text{grav}}|^2\rangle \le M^2 5\times10^{-8} .$$

This is currently being investigated numerically by Schaefer.

C. Large-S limit

In taking a large-S limit, we must specify what is bring held constant. Suppose $\delta\rho$ is the sum or randomly scattered local perturbations. We will consider a sequence of spheres with increasing radius such that the number density of local perturbations remains constant. This means that the norm of the eigenfunctions $V^{-1}\int dv\,|Q|^2$ is independent of S. As $|S|\to\infty$, $n^2/|S|\to q^2$, a continuous variable.

The eigenfunctions are solutions to the usual flat-space Helmholtz equation:

$$\nabla^2 Q=-q^2 Q ,$$
$$Q_{lm}(q)=j_l Y_{lm}(qr) , \tag{82}$$

where the j_l are spherical Bessel functions.

From Appendix B, $\Pi_{nl}(r/\sqrt{|S|})\to j_l(qr)$, which implies

$$B_{nl}\to B_l(q)=-\frac{2}{\eta_0}\left[\frac{\eta E}{\eta_0}j_l'(q\Delta\eta)+j_l(q\Delta\eta)\right] ,$$
$$\langle\,|a_{lm}|^2\rangle\to\int dq\frac{E(q)}{q^4}B_l(q)^2 , \tag{83}$$

where $E(q)$ are the coefficients for the expansion of ξ in spherical Bessel functions. For details see Appendix B.

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APPENDIX A: SPHERICAL HARMONICS ON THE THREE-SPHERE, AND LIMIT AS $|S|\to\infty$

In this section we review some properties of the eigenfunctions of the Helmholtz equation on S^3 and H^3,

$$D_iD^iQ=-\frac{(n^2-k)}{S}Q , \tag{A1}$$

and take the limit of the eigenfunctions as $|S|\to\infty$.

The case $S>0$ is treated first. The solutions for $S>0$ are[10,21,25]

$$Q^{nlm}(\mathbf{x})=\Pi_{nl}(\chi)Y_{lm}(\Omega) ,$$
$$n=1,2,3,\ldots,\quad l=0,1,\ldots,n-1 ,$$

where

$$\Pi_{nl}(\chi)=\frac{\sin^l\chi\, d^{l+1}(\cos n\chi)}{M_{nl}\,d(\cos\chi)^{l+1}}$$
$$=M_{nl}\left[\frac{\pi}{2\sin\chi}\right]^{1/2}P_{n-1/2}^{-1/2-l}(\cos\chi) ,$$

and (A2)

$$M_{nl}=[(n^2-1)\cdots(n^2-l^2)]^{1/2} .$$

The Y_{lm} are the usual spherical harmonics of S^2 and the P^α_β are the Legendre polynomials. We choose the normalization

$$\frac{2}{\pi}\int_0^\pi dx\,\sin^2\chi\,\Pi_{nl}\Pi_{n'l}=\frac{\delta_{nn'}}{n^2} . \tag{A3}$$

The Π_{ln} satisfy the equation

$$L_l(\cos\chi,\sin\chi)\Pi_{ln}\equiv\left[\frac{d^2}{d\chi^2}+\frac{2\cos\chi}{\sin\chi}-\frac{l(l+1)}{\sin^2\chi}\right]\Pi_{ln}$$
$$=-(n^2-1)\Pi_{ln} . \tag{A4}$$

For $l=0$

$$\Pi_{n0}(\chi)=\frac{\sin n\chi}{n\sin\chi} . \tag{A5}$$

In Appendix C, the Green's function was found for the linear operator $L\equiv(D_iD^i+3/S)$ on S^3. The Green's function can also be written as a sum of eigenfunctions divided by their eigenvalues. However, L has four zero eigenmodes, the Q^{2lm}. These modes must be deleted from the sum for G, and

$$G(\mathbf{x},\mathbf{x}')=-\frac{2}{\pi}\sum_{nlm,n\neq2}Q^{nlm}(\mathbf{x})Q^{nlm}(\mathbf{x}')\frac{n}{(n^2-4)/S} . \tag{A6}$$

Clearly $\int dv\,GQ^{2lm}=0$; i.e., $G\in\mathscr{S}^\perp$; cf. Eq. (C7) below.

The solutions for $S<0$ are[10,25]

$$Q^{nlm}(\mathbf{x})=P_{ql}(\chi)Y_{lm}(\Omega),\quad q\geq 0\ . \tag{A7}$$

The P_{nl} can be obtained[15] from the Π_{nl}

$$P_{ql}(\chi)=\Pi_{iq,l}(i\chi)\ . \tag{A8}$$

This can be checked by substituting $\chi\to i\chi$ and $n\to in$ in (A4), which yields

$$L_l(\cosh\chi,\sinh\chi)\Pi_{in,l}(i\chi)=-(n^2+1)\Pi_{in,l}\ , \tag{A9}$$

which is the equation that defines $P_{nl}(\chi)$.

Therefore (A2) is true with $n\to iq$. The normalization becomes

$$\frac{2}{\pi}\int_0^\infty dx\,\sinh^2\chi P_{ql}P_{q'l}=\frac{\delta(q-q')}{q^2}\ . \tag{A10}$$

For $l=0$

$$P_{q0}(\chi)=\frac{\sin q\chi}{q\sinh\chi}\ . \tag{A11}$$

Lastly, consider the limit of the eigenfunctions as $|S|\to\infty$. Then $n^2/|S|\to k^2$, a continuous variable. Equation (A1) becomes the flat-space Helmholtz equation with eigenfunctions $Q_{lm}(k)=Y_{lm}j_l(kr)$, where $j_l(kr)$ are the spherical Bessel functions. Indeed, $\Pi_{n0}(r/\sqrt{S})$ and

$$P_{n0}(r/\sqrt{-S})\to\sin kr/kr=j_0(kr)\ .$$

The normalization integrals (A3) and (10) also have the correct limit.

As an example take the continuum limit of the eigenfunction expansion of $P(\mathbf{x})$ [Eq. (B1)]:

$$P(\mathbf{x})=\sum_{lm}Y_{lm}(\Omega)\sum_n\frac{1}{\sqrt{S}}(\sqrt{S}\,c_{nlm})\Pi_{nl}\left[\frac{r}{\sqrt{S}}\right]$$

$$\to\sum_{lm}Y_{lm}\int dq\,C_{lm}(q)j_l(qr)\quad\text{as}\quad S\to\infty\ ,$$

where

$$C_{lm}(q)=\lim\sqrt{S}\,c_{nlm}$$
$$=\frac{2}{\pi}q^2\int d^3r\,Y^*_{lm}j_l(qr)P(r)\ . \tag{A12}$$

APPENDIX B: RELATION OF THE POWER SPECTRUM TO THE CORRELATION FUNCTION ON S^3

Let

$$P(\mathbf{x})=\sum c_{nlm}Q^{nln}$$

and

$$\xi(\mathbf{x}_1,\mathbf{x}_2)=\xi(\psi)=\sum e_n\Pi_{n0}(\psi)\ , \tag{B1}$$

where ψ is the angle on the three-sphere between $\mathbf{x}_1$ and $\mathbf{x}_2$ (36). Note that the geodesic distance between $\mathbf{x}_1$ and $\mathbf{x}_2$ is $\sqrt{S}\,\psi$. We wish to compute

$$\langle c_{nlm}c^*_{NLM}\rangle=\left[\frac{2}{\pi}\right]^2 n^2\int dv_1dv_2 Q^{nlm}_{(\mathbf{x}_1)}Q^{NLM*}_{(\mathbf{x}_2)}\xi(\mathbf{x}_1,\mathbf{x}_2)\ . \tag{B2}$$

Substitute the expansion (B1) for ξ and the explicit forms of the Q^{nlm} (see Appendix A) into (B2). Then use the angle addition theorem for the three-sphere[21] (note that a different normalization is used here than in Ref. 21):

$$\sum_{l=0}^{n-1}\sum_{m=-l}^{l}\Pi_{nl}(\chi_1)\Pi_{nl}(\chi_2)Y_{lm}(\Omega_1)Y^*_{lm}(\Omega_2)=\Pi_{n0}(\psi)\ . \tag{B3}$$

Using this theorem in the expansion for ξ, and the orthogonality properties of the eigenfunctions (see Appendix A), it follows that

$$\langle c_{nlm}c^*_{NLM}\rangle=e_n\delta_{nN}\delta_{lL}\delta_{mM}\ . \tag{B4}$$

APPENDIX C: THE GREEN'S FUNCTION FOR $\Delta+3/S$

Let $L=\Delta+3/S$. We wish to find $G(\mathbf{x},\mathbf{x}')$ such that the solution to (30) is given by (31). First consider $S>0$.

(i) Projective three-space (P^3). P^3 is a three-sphere with antipodal points identified. The closed P^3 cosmological model can be regarded as an S^3 model in which all physical quantities are symmetric under inversion.

If the source point is at the origin, then the Green's function G_P is a function of χ only. G_P must satisfy

$$LG_P(\chi)=0,\quad 0<\chi\leq\pi/2\ , \tag{C1}$$

$$G_P(\chi)\to\frac{-1}{\sqrt{S}\,4\pi\chi}\quad\text{as}\ \chi\to 0\ , \tag{C2}$$

$$G_P(\chi)=G_P(\pi-\chi),\quad\text{inversion symmetry}\ . \tag{C3}$$

Noting that

$$LG_P(\chi)=\frac{1}{\sin\chi}\left[\frac{\partial^2}{\partial^2\chi}+4\right](G_P\sin\chi)\ , \tag{C4}$$

the solution is easily found to be

$$G_P(\chi)=\frac{-1}{\sqrt{S}\,4\pi}\frac{\cos 2\chi}{\sin\chi}\ . \tag{C5}$$

For an arbitrary source point $\mathbf{x}'$, χ is replaced by the angle ψ between $\mathbf{x}$ and $\mathbf{x}'$

$$G_P(\mathbf{x},\mathbf{x}')=\frac{-1}{\sqrt{S}\,4\pi}\frac{\cos 2\psi}{\sin\psi}\ , \tag{C6}$$

where $\cos\psi$ is defined in (36).

(ii) Three-sphere (S^3). The operator L has four zero eigenmodes. These are the functions Q^{2lm} [Eq. (16)] which appear in the integral constraints. Therefore, (30) can be solved if and only if the source P satisfies

$$\int dv\,PQ^{2lm}=0\ . \tag{C7}$$

If P satisfies (C7) we will say $P\in\mathscr{S}^\perp$.

In particular, the three-dimensional δ function $\delta(\mathbf{x},\mathbf{x}')$ on S^3 does not belong to $\mathscr{S}^\perp$ and (30) cannot be solved for it as source. Indeed, denoting the complete set of eigenfunctions of L by Q^{nlm},

$$\delta(\mathbf{x},\mathbf{x}')=\frac{2}{\pi}\sum_{nlm}Q^{nlm}(\mathbf{x})Q^{nlm}(\mathbf{x}')$$
$$\equiv\delta^{\perp}(\mathbf{x},\mathbf{x}')+\delta^{||}(\mathbf{x},\mathbf{x}')\ ,$$

where $\delta^{||}$ is precisely the sum over the $n=2$ eigenfunctions, the zero modes:

$$\delta^{||}(\mathbf{x},\mathbf{x}')=\frac{2}{\pi^2}\cos\psi\ ,$$

where ψ is given by (36). However, the equation

$$LG_S(\mathbf{x},\mathbf{x}')=\delta(\mathbf{x},\mathbf{x}')-\delta^{||}(\mathbf{x},\mathbf{x}')=\delta^{\perp}(\mathbf{x},\mathbf{x}') \tag{C8}$$

can be solved. G_S is still not specified completely, since if G_S is a solution, so is G_S+H, where H is any (continuous) solution of the homogeneous equation. H will be fixed by requiring that $G_S\in\mathscr{S}^{\perp}$.

The domain of G_S also fixes different boundary conditions. On S^3 we want a function that has a singularity at $\chi=0$ and nowhere else. This cannot be constructed just from the homogeneous solutions to (30), $\cos\chi$ and $\cos2\chi/\sin\chi$. However, we can find G_S which satisfies (C8) on S^3 as follows. Operate on (C8) with L to annihilate the $\delta^{||}$ on the right. Then G_S must satisfy

$$\begin{aligned}&LLG_S(\chi)=0,\ \ 0<\chi\leq\pi\ ,\\ &G_S\to\frac{-1}{\sqrt{S}\ 4\pi\chi}\ \ \text{as } \chi\to0\ ,\\ &G_S\in\mathscr{S}^{\perp}\ .\end{aligned} \tag{C9}$$

The solution for general source point $\mathbf{x}'$ is

$$G_S(\mathbf{x},\mathbf{x}')=\frac{-1}{\sqrt{S}\ 4\pi}\left[\left(1-\frac{\psi}{\pi}\right)\frac{\cos2\psi}{\sin\psi}-\frac{1}{2\pi}\cos\psi\right]; \tag{C10}$$

ψ is defined in (36).

Note that the last term in G_S is $\cos\psi$, which is a solution to the homogeneous equation. So for any $s\in\mathscr{S}^{\perp}$, $\int dv\, s\cos\psi=0$. Therefore if one substitutes G_S into (31) to solve for Φ, the $\cos\psi$ term can just as well be omitted from G_S.

G_S can also be expressed as an infinite sum of eigenfunctions (see Appendix A).

(iii) Hyperbolic space (or pseudosphere, H^3). The Green's function on H^3 was given by D'Eath[26] as

$$G_H(\mathbf{x},\mathbf{x}')=-\frac{1}{4\pi\sqrt{-S}}\left[\frac{\cosh2\alpha}{\sinh\alpha}-2\cosh\alpha\right],$$

where α is defined in (38).

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